Reminder on complexes and their (co)homologies

SHA3◇1. For complexes A, B of (left) modules over a ring with unit, let $\mathrm{Hom}_{\mathrm{DG}}(A,B) \overset{\text{def}}{=} \bigoplus_i \mathrm{Hom}^i(A,B)$, where $\mathrm{Hom}^i(A,B) \overset{\text{def}}{=} \bigoplus_\nu \mathrm{Hom}\left(A^\nu, B^{\nu+i}\right)$, be the complex with differential $d : \psi \mapsto d_B\psi - (-1)^{|\psi|}\psi d_A$, where $\psi \in \mathrm{Hom}^{|\psi|}(A,B)$ is homogeneous of degree $|\psi|$. Let $\mathrm{Hom}(A,B) \overset{\text{def}}{=} \ker d \cap \mathrm{Hom}^0_{\mathrm{DG}}(A,B)$ and $\mathrm{Hom}_{\mathcal{H}o}(A,B) \overset{\text{def}}{=} H^0\left(\mathrm{Hom}_{\mathrm{DG}}(A,B)\right) = \mathrm{Hom}(A,B)/\operatorname{im} d$. Prove that **a)** complexes with the Hom-spaces as the morphisms form an abelian category[1] **b)** complexes with the $\mathrm{Hom}_{\mathrm{DG}}$-complexes as the morphisms form a DG-category[2] **c)** complexes with the $\mathrm{Hom}_{\mathcal{H}o}$-spaces as the morphisms form a category[3] **d)** every $\varphi \in \mathrm{Hom}(A,B)$ induces the well defined homomorphism of graded modules $\varphi_* : \bigoplus_i H^i(A) \to \bigoplus_i H^i(B)$ **e)** if $\varphi = \psi$ in $\mathrm{Hom}_{\mathcal{H}o}(A,B)$, then $\varphi_* = \psi_*$ **f)** the cohomology functors $A \mapsto \bigoplus_i H^i(A)$ commute with filtered colimits in $\mathcal{C}om$.

SHA3◇2. For every complex K in the category $\mathcal{C}om$, construct a functorial in K decreasing filtration of K by subcomplexes $\cdots \supset F^i \supset F^{i+1} \supset \cdots$ such that every factor complex $G^i = F^i/F^{i+1}$ has at most one nonzero cohomology module, which is situated at degree i and is canonically isomorphic to **a)** K^i **b)** $H^i(K)$.

SHA3◇3. Given an abelian category $\mathcal{A}$ and abelian group A, a function $\alpha : \mathrm{Ob}\,\mathcal{A} \to A$ is called *additive*, if for every exact triple $0 \to K \to L \to M \to 0$ in $\mathcal{A}$, the equality $\alpha(L) = \alpha(K) + \alpha(M)$ holds in A. For every additive function α and a bounded[4] complex K, prove the *Euler formula* $\sum(-1)^i\alpha\left(K^i\right) = \sum(-1)^i\alpha\left(H^i(K)\right)$.

SHA3◇4. For an exact triple $0 \to K \xrightarrow{\varphi} L \xrightarrow{\psi} M \to 0$ in $\mathcal{C}om$ construct a functorial long exact sequence of cohomology modules $\cdots \to H^i(K) \xrightarrow{\varphi_*} H^i(L) \xrightarrow{\psi_*} H^i(M) \xrightarrow{\delta} H^{i+1}(K) \xrightarrow{\varphi_*} H^{i+1}(L) \to \cdots$.

SHA3◇5. Let V be a vector space of finite dimension over a field $\Bbbk$ with $\operatorname{char}\Bbbk = 0$. For a nonzero covector $\xi \in V^*$ compute cohomologies of the complexes ΛV and ΛV^* with the differentials[5]

$$\partial_\xi : \Lambda^m V \to \Lambda^{m-1}V, \quad \eta \mapsto \partial_\xi\eta \quad \text{и} \quad \xi : \Lambda^m V^* \to \Lambda^{m+1}V^*, \quad \omega \mapsto \xi \wedge \omega .$$

SHA3◇6. Fix a basis in a vector space $V = \Bbbk^n$ and write x_i and ξ_i for the images of the ith basis vector in the algebras SV and ΛV respectively. For the endomorphisms d and ∂ of the vector space $SV \otimes \Lambda V = \bigoplus_{k,m} S^kV \otimes \Lambda^m V$ defined by the prescriptions

$$\partial = \sum x_i \otimes \frac{\partial}{\partial \xi_i} : S^kV \otimes \Lambda^m V \to S^{k+1}V \otimes \Lambda^{m-1}V, \quad f \otimes \omega \mapsto \sum_{i=1}^n x_i \cdot f \otimes \frac{\partial \omega}{\partial \xi_i},$$

$$d = \sum \frac{\partial}{\partial x_i} \otimes \xi_i : S^kV \otimes \Lambda^m V \to S^{k-1}V \otimes \Lambda^{m+1}V, \quad f \otimes \omega \mapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} \otimes \xi_i \wedge \omega,$$

a) prove that they do not depend on the choice of basis in V and have $\partial^2 = 0 = d^2$

b) compute the s-commutator $\partial d + d\partial$ and the homology spaces of the both differentials.

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[1] That is, an *exact category* in the sense of Task 2½ such that the Hom-functor takes values in $\mathcal{A}b$, the composition is distributive with respect to the addition of morphisms, and every pair of objects has both the product and the coproduct isomorphic one to the other. It is called the *category of complexes* and denoted by $\mathcal{C}om$.

[2] This means that the differentials of $\mathrm{Hom}_{\mathrm{DG}}$-complexes interact with the composition of morphisms via the *Leibniz rule*: $d(\varphi\psi) = (d\varphi)\psi + (-1)^{|\varphi|}\varphi(d\psi)$. This category is called the DG-*category of complexes* and denoted by $\mathcal{C}om_{\mathrm{DG}}$.

[3] It is called the *homotopy category of complexes* and denoted by $\mathcal{H}o$.

[4] A complex K is called *bounded above* (resp. *below*), if $K^i = 0$ for all $i \gg 0$ (resp. for all $i \ll 0$). A complex is called *bounded*, if it is bounded both above and below.

[5] The operator ∂_ξ acts on $\omega \in \Lambda^m V$ as follows. Write $\omega' \in V^{\otimes(m-1)}$ for the contraction in the first tensor factor between ξ and the unique alternating tensor $\widetilde{\omega} \in V^{\otimes m}$ mapped to ω under the projection $V^{\otimes m} \twoheadrightarrow \Lambda^m V$. Then $\partial_\xi\omega$ is defined as the image of ω' under the projection $V^{\otimes(m-1)} \twoheadrightarrow \Lambda^{m-1}V$ multiplied by $m = \deg\omega$. In terms of dual bases e_i and ξ_i in V^* and V, $\partial_{\xi_i} = \frac{\partial}{\partial e_i}$.

Individual report card of ______________________________________ .
(write your name and surname)

Task 3 (6.02.2018)

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