Catrgories and functors

Notations. We write $\mathcal{S}et$, $\mathcal{T}op$, $\mathcal{A}b$, $\mathcal{G}rp$, $\mathcal{C}mr$, $\mathcal{M}od_K$, $\mathcal{V}ec_{\Bbbk} = \mathcal{M}od_{\Bbbk}$, $\mathcal{A}ss_{\Bbbk}$, A-$\mathcal{M}od$, $\mathcal{M}od$-A for the categories of sets, topological spaces, abelian groups, all groups, commutative rings[1], modules over commutative ring K, vector spaces and associative algebras over a field $\Bbbk$, left and right modules over algebra A respectively. Categories of functors $\mathcal{C} \to \mathcal{D}$ and presheaves[2] $\mathcal{C}^{\text{opp}} \to \mathcal{D}$ are denoted by $\mathcal{F}un(\mathcal{C}, \mathcal{D})$ and $p\mathcal{S}h(\mathcal{C}, \mathcal{D})$.

SHA1◇1. Let Δ_{big} be the category of all finite ordered sets with order preserving maps as the morphisms and $\Delta \subset \Delta_{\text{big}}$ be its full small subcategory formed by sets $[n] \overset{\text{def}}{=} \{0, 1, \dots, n\}$, $n \geqslant 0$, ordered usually. Show that **а)** Δ and Δ_{big} are equivalent **б)** algebra $\mathbb{Z}[\Delta]$ is generated by the identity arrows $e_n = \text{Id}_{[n]}$, the inclusions $\partial_n^{(i)} : [n-1] \hookrightarrow [n]$, $0 \leqslant i \leqslant n$, $i \notin \partial_n^{(i)}\big([n-1]\big)$, and surjections $s_n^{(i)} : [n] \twoheadrightarrow [n-1]$, $0 \leqslant i \leqslant n-1$, $(i+1) \mapsto i$. **в*)** Find generators for the ideal of relations between these generating arrows.

SHA1◇2. For a given $X \in \text{Ob}\,\mathcal{C}$ let a functor $h^X : Y \mapsto \text{Hom}(X, Y)$ and a presheaf $h_X : Y \mapsto \text{Hom}(Y, X)$ take an arrow $\varphi : Y_1 \to Y_2$ respectively to the left and right multiplications by this arrow:

$$\varphi_* : \text{Hom}(X, Y_1) \to \text{Hom}(X, Y_2),\ \psi \mapsto \varphi \circ \psi \quad \text{and} \quad \text{Hom}(Y_2, X) \to \text{Hom}(Y_1, X),\ \psi \mapsto \psi \circ \varphi .$$

Show that prescriptions $X \mapsto h^X$ and $X \mapsto h_X$ define a pre-sheaf $h^* : \mathcal{C}^{\text{opp}} \to \mathcal{F}un(\mathcal{C}, \mathcal{S}et)$ and a functor $h_* : \mathcal{C} \to p\mathcal{S}h(\mathcal{C}, \mathcal{S}et)$ respectively.

SHA1◇3. Show that functor $h^X : \mathcal{A}b \to \mathcal{A}b$ takes an exact sequence $0 \to A \to B \to C \to 0$ to an exact sequence $0 \to \text{Hom}(X, A) \to \text{Hom}(X, B) \to \text{Hom}(X, C)$ whose rightmost arrow may be non-surjective. Formulate and prove dual property of functor $h_X : \mathcal{A}b \to \mathcal{A}b$.

SHA1◇4. Describe products and coproducts in **а)** $\mathcal{S}et$ **б)** $\mathcal{T}op$ **в)** $\mathcal{M}od_K$ **г)** $\mathcal{G}rp$ **д)** $\mathcal{C}mr$.

SHA1◇5. Fix prime $p \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $A_n = \mathbb{Z}/(p^n)$. For $m > n$ write $\psi_{nm} : A_m \twoheadrightarrow A_n$ for the factorization mapping and $\varphi_{mn} : A_n \hookrightarrow A_m$ for the embedding $[1] \mapsto [p^{m-n}]$. In category $\mathcal{A}b$ describe **а)** $\varprojlim A_n$ along ψ_{mn} **б)** $\text{colim}\, A_n$ along φ_{mn}.

SHA1◇6. Let $B_n = \mathbb{Z}/(n)$. For $n | m$ write $\psi_{nm} : B_m \twoheadrightarrow B_n$ and $\varphi_{mn} : B_n \hookrightarrow B_m$ for the factorization mapping and the embedding $[1] \mapsto [m/n]$ respectively. In category $\mathcal{A}b$ describe **а)** $\varprojlim B_n$ along ψ_{nm} **б)** $\text{colim}\, B_n$ along φ_{mn}.

SHA1◇7. Prove that a functor $G : \mathcal{D} \to \mathcal{C}$ admits a left adjoint functor F iff for each $X \in \text{Ob}\,\mathcal{C}$ a functor $h_G^X : Y \mapsto \text{Hom}_{\mathcal{C}}(X, G(Y))$ is corepresentable, and in this case $F(X)$ corepresents h_G^X. Formulate and prove the dual criteria for the existence of right adjoint functor G to a given functor $F : \mathcal{C} \to \mathcal{D}$.

SHA1◇8. Show that any left adjoint functor commutes with colimits and any right adjoint functor commutes with limits[3].

SHA1◇9. For an arbitrary extension $S \subset R$ of associative algebras with units construct left and right adjoint functors to the restriction functor $\text{res}_S^R : R\text{-}\mathcal{M}od \to S\text{-}\mathcal{M}od$.

SHA1◇10. Given a topological space $\mathcal{X}$, write $S(\mathcal{X}) : \Delta^{\text{opp}} \to \mathcal{S}et$ for the simplicial set that takes $[n] \in \text{Ob}\,\Delta$ to $S_n(\mathcal{X}) \overset{\text{def}}{=} \text{Hom}_{\mathcal{T}op}(\Delta^n, \mathcal{X})$, where $\Delta^n \subset \mathbb{R}^{n+1}$ is the standard regular n-dimensional simplex, and takes an order preserving arrow $\varphi : [n] \to [m]$ to the right multiplication mapping $f \mapsto f \circ |\varphi|$, where $|\varphi| : \Delta^n \to \Delta^m$ stays for the affine mapping acting on the vertices as φ. Show that the functor $S : \mathcal{T}op \to p\mathcal{S}h(\Delta)$ is right adjoint to the geometric realization functor $p\mathcal{S}h(\Delta) \to \mathcal{T}op$.

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[1]with unity and homomorphisms sendinding unity to unity

[2]i.e. contravariant functors

[3]functor $F : \mathcal{C} \to \mathcal{D}$ *commutes with* (co) *limits*, if for each $L \in \text{Ob}\,\mathcal{C}$ and any diagram $\Phi : \mathcal{N} \to \mathcal{C}$ the condition «L is the (co) limit of Φ in $\mathcal{C}$» implies the condition «$F(L)$ is the (co) limit of $F \circ \Phi$ in $\mathcal{D}$

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