Algebraic Geometry
start up course

by A. L. Gorodentsev[1]

This is a *geometric* introduction into algebraic geometry. The main goal of these lectures is to show a collection of some basic geometric figures underlying technical constructions of modern *algebraic* geometry. I hope that after this course students will be able to translate things from the continuous (infinitely rich but mostly intuitive) world of figures to the restrictive (in fact — finite, at most potentially countable) langauge of formulae and will be in a position for reading advanced books in algebraic geometry[2].

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[2]There are three of them I recommend especially — *Intersection Theory* of W. Fulton and two monographs of V. Danilov: *Algebraic Manifolds And Schemes* and *Cohomologies Of Algebraic Manifolds* published in the books «Algebraic Geometry – 1, 2» of Springer's yellow series known as «Russian Math. Encyclopedia»

Contents

Affine and projective spaces, polynomials, projections, and projective linear isomorphisms.

<u>POLYNOMIALS.</u> Let V be n-dimensional vector space over an arbitrary field k, $(e_1, e_2, \dots, e_n)$ be a basis in V. The *dual space* V^* consists of all k-linear functions $V \longrightarrow k$; its *dual base* $(x_1, x_2, \dots, x_n)$ is formed by the coordinate functions in V w. r. t. $\{e_i\}$, that is $x_i(e_j) = 1$, if $i = j$, and $x_i(e_j) = 0$ otherwise. We write $S^dV^* \subset k[x_1, x_2, \dots, x_n]$ for a subspace of all homogeneous polynomials of degree d. So, $k[x_1, x_2, \dots, x_n] = \bigoplus\limits_{d\geqslant 0} S^dV^*$ and $S^rV^* \cdot S^sV^* \subset S^dV^{r+s}$, i. e. $k[x_1, x_2, \dots, x_n]$ is a *graded* k-algebra with the graded components S^dV^*. There is *the evaluation map* $k[x_1, x_2, \dots, x_n] \xrightarrow{\text{ev}} \{k\text{-valued functions on } V\}$, which sends a polynomial $f(x_1, x_2, \dots, x_n)$ to the function $p = (p_1, p_2, \dots, p_n) \longmapsto f(p) = f(p_1, p_2, \dots, p_n)$. The image of ev is called *a space of algebraic functions* on V.

Exercice 1.1. Show that this subspace and all the subspaces $\text{ev}(S^dV^*)$ do not depend on a base choice in V.

<u>Lemma 1.2</u>. *The evaluation map is injective*[1] *iff k is infinite.*

Proof. If k is finite, say consists of q elements, then the space of k-valued functions on V consists of q^{q^n} elements and is finite as well. Since the polynomial algebra is infinite, $\ker \text{ev} \neq 0$. The inverse argumentation uses induction on n. Non zero polynomial $f(x)$ in one variable can't have more than $\deg f$ roots. So, $f(x) \equiv 0$ as soon as $f(p) = 0$ for infinitely many $p \in k$. A polynomial in n variables $f(x_1, x_2, \dots, x_n) = \sum\limits_{\nu=0}^{d} \varphi_\nu(x_1, x_2, \dots, x_{n-1}) \cdot x_n^{d-\nu}$ may be considered as a polynomial in one variable x_n with coefficients from $k[x_1, x_2, \dots, x_{n-1}]$. Evaluating all φ_ν at any point $(p_1, p_2, \dots, p_{n-1}) \in k^{n-1}$, we get a polynomial in x_n with constant coefficients. It equals zero by the above reason. So, all φ_ν are identically zero functions. By the the inductive assumption they should be the zero polynomials as well. □

<u>AFFINE SPACE $\mathbb{A}(V)$</u> is just a set of all vectors in V. A point presenting the zero vector is called *an origin*; we denote it by O. All other points may be pictured as the ends of non zero vectors drawn from the origin. We write $\mathbb{A}^n$ instead of $\mathbb{A}(V)$ for n-dimensional space whose feeling is not essential. A set $X \subset \mathbb{A}(V)$ is called *a closed affine algebraic set* if it coincides with a set of all solutions of some (possibly, infinite) system of polynomial equations on V.

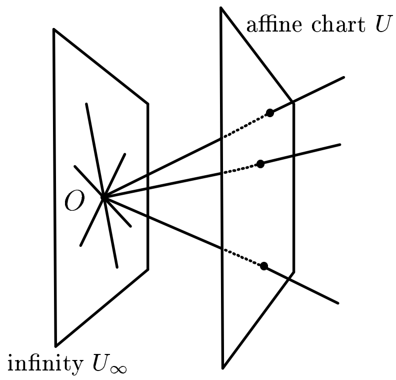

Fig. 1.1. The projective world.

<u>PROJECTIVE SPACE[2] $\mathbb{P}(V)$</u> is a set of all 1-dimensional vector subspaces in V, that is, a set of all lines through the origin in $\mathbb{A}(V)$. If $\dim V = (n+1)$, we write $\mathbb{P}_n$ instead of $\mathbb{P}(V)$ and call it *n-dimensional projective space*. The rays going from the origin produce "the usual" points as soon as they meet some screen inside $\mathbb{A}(V)$, that is, a codimension 1 affine hyperplane $U \subset \mathbb{A}(V)$ that does not contain the origin — like ion fig. 1.1. Such a hyperplane is called *an affine chart* on $\mathbb{P}(V)$. Of course, no such a chart does cover the whole of $\mathbb{P}_n$: the difference $U_\infty \overset{\text{def}}{=} \mathbb{P}_n \setminus U$ consists of all lines laying in a parallel copy of U passing through the origin. It may be identified with $\mathbb{P}_{n-1} \subset \mathbb{P}(V)$ and is called *an infinity hyperplane* for the chart U. So, $\mathbb{P}_n$ is decomposed into disjoint union of "affine cells": $\mathbb{P}_n = U \sqcup U_\infty = \mathbb{A}^n \sqcup \mathbb{P}_{n-1} = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{P}_{n-2} = \cdots = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \dots \sqcup \mathbb{A}^0$

Exercice 1.3*. If you have some experience in CW-topology, try to calculate integer homology groups of the real and complex projective spaces using the above cell decomposition.

<u>GLOBAL HOMOGENEOUS COORDINATES.</u> Let $\dim V = (n+1)$ and $v = (x_0, x_1, \dots, x_n)$, $w = (y_0, y_1, \dots, y_n)$ be two vectors presented by their coordinates w. r. t. a given fixed basis $\{e_0, e_1, \dots, e_n\} \subset V$. Then v and w give the same point $p \in \mathbb{P}_n$ iff they are proportional, i. e. $x_\nu = \lambda y_\nu\ \forall \nu$ for some $\lambda \neq 0$. So, only the ratios $x_\mu : x_\nu$ are well defined for the point $p \in \mathbb{P}_n$. These ratios are called *homogeneous coordinates* on $\mathbb{P}_n$ (w. r. t. the base $\{e_0, e_1, \dots, e_n\} \subset V$). We write $p = (x_0 : x_1 : \dots : x_n)$ for a point p presented (up to proportionality) by $v = (x_0, x_1, \dots, x_n)$. For any homogeneous polynomial $f \in S^d(V^*)$, the zero set $(f)_0 \overset{\text{def}}{=} \{v \in V \mid f(v) = 0\}$

[1] i. e. different polynomials always give the different polynomial functions on V

[2] in this section we always take $\dim V = n + 1$

is well defined subset in $\mathbb{P}(V)$, because of $f(v) = 0 \iff f(\lambda v) = \lambda^d f(v) = 0$. Such the subsets and their intersections are called *closed projective algebraic sets*. For example, the equation

$$x_0^2 + x_1^2 = x_2^2 \tag{1}$$

presents the curve $C \subset \mathbb{P}_2$; if char $(k) \neq 2$, it is known as *non degenerate plane conic*.

LOCAL AFFINE COORDINATES. Let $U \subset \mathbb{A}(V)$ be an affine chart on $\mathbb{P}_n = \mathbb{P}(V)$, say, given by the equation $\alpha(x) = 1$, where $\alpha \in V^*$. Take any n linear forms $\xi_1, \xi_2, \ldots, \xi_n \in V^*$ whose restrictions $\xi_i|_U$ are linearly independent[3]. Then for any $v \in V$ we have $v/\alpha(v) \in U$ and can use n functions $x_i(p) = \xi_i(v/\alpha(v))$, $1 \leqslant i \leqslant n$ as *local* affine coordinates inside $U \subset \mathbb{P}_n$ for the point $p \in U$ corresponding to v. These coordinates depend on both U and a choice of ξ_i's. Note, that the sentences: «$\alpha(p) \to 0$», «$x_i(p) \to \infty$ for some i», and «$p \to U_\infty = \mathbb{P}_n \setminus U$» do really mean the same: «p is running away from U to infinity».

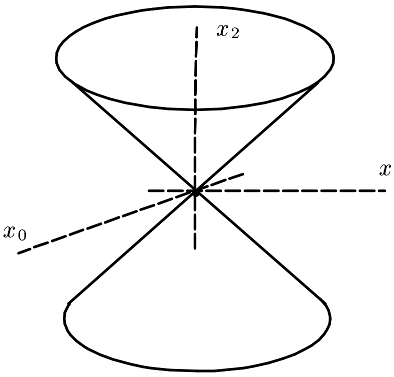

Fig. 1.2. The cone.

Example 1.4: affine conics. Consider the non degenerate conic (1) over $k = \mathbb{R}$ in some affine charts. Let U_0 be given by the equation $\{x_0 = 1\}$. Then $t_1 = x_1|_{U_0} = x_1/x_0$ and $t_2 = x_2|_{U_0} = x_2/x_0$ can be taken as local affine coordinates on U_0. Dividing the both sides of (1) by x_2^2, we get for $C \cap U_0$ the equation $t_2^2 - t_1^2 = 1$, that is, $C \cap U_0$ is a hyperbola. Similarly, in a chart $U_2 = \{x_0 = 1\}$ with local affine coordinates $t_0 = x_0/x_2$, $t_1 = x_1/x_2$ we get the equation $t_0^2 + t_1^2 = 1$, which shows that $C \cap U_2$ is a circle. Finally, consider a chart U given by $x_2 - x_1 = 1$ with local affine coordinates $t = x_0|_U = \dfrac{x_0}{x_2 - x_1}$ and $u = (x_2 + x_1)|_U = \dfrac{x_2 + x_1}{x_2 - x_1}$. After dividing by $(x_2 - x_1)^2$ and some eliminations, we see that $C \cap U$ is the parabola $u = t^2$.

Exercice 1.5. The affine cone $x_0^2 + x_1^2 = x_2^2$ in $\mathbb{A}^3$ is drawn on fig. 1.2. Picture there each of 3 previous affine charts and outline their intersections with the cone.

HINT. A good idea is to draw 3 separated pictures.

PROJECTIVE CLOSURE. Any closed affine algebraic set $X \subset \mathbb{A}^n$ is always an affine piece of appropriate closed projective algebraic set $\widetilde{X} \subset \mathbb{P}_n$ called *a projective closure of* X. If X is given by polynomial equations $\{ f_\nu(t_1, t_2, \ldots, t_n) = 0 \}$, we substitute $t_i = x_i/x_0$ and multiply the ν-th equation by $x_0^{\deg f_\nu}$; then the resulting equations $\widetilde{f}_\nu(x_0, x_1, \ldots, x_n) = 0$ are homogeneous and define a closed projective algebraic set $\widetilde{X} \subset \mathbb{P}_n$ such that $\widetilde{X} \cap U_0 = X$, where U_0 is the affine chart given by $x_0 = 1$ (it is identified with original $\mathbb{A}_n \supset X$ by taking $t_i = x_i|_{U_0}$).

STANDARD AFFINE COVERING AND GLUING RULES. $\mathbb{P}_n$ is covered by $(n+1)$ affine charts $U_\nu \subset \mathbb{A}^{n+1}$ given by $\{x_\nu = 1\}$. This covering is called *standard*. For each $\nu = 0, 1, \ldots, n$ we take n forms

$$t_i^{(\nu)} = x_i|_{U_\nu} = \frac{x_i}{x_\nu}\,, \quad \text{where } 0 \leqslant i \leqslant n\,,\ i \neq \nu$$

as standard local affine coordinates on U_ν. Topologically, $\mathbb{P}_n$ is constructed from $(n+1)$ distinct copies of $\mathbb{A}_n$ denoted as $U_0, U_1, \ldots, U_n$ by gluing them together along the actual intersections $U_\mu \cap U_\nu \subset \mathbb{P}_n$. Namely, we identify a point of U_ν with a point of U_μ iff they correspond to the same point of $\mathbb{P}_n$. In homogeneous coordinates, the intersection $U_\mu \cap U_\nu$ consists of all x such that both x_μ and x_ν are non zero. This locus is presented inside U_μ and U_ν by inequalities $t_\nu^{(\mu)} \neq 0$ and $t_\mu^{(\nu)} \neq 0$ respectively and a point $t^{(\mu)} \in U_\mu$ is glued with a point $t^{(\nu)} \in U_\nu$ iff

$$t_\nu^{(\mu)} = 1/t_\mu^{(\nu)} \qquad \text{and} \qquad t_i^{(\mu)} = t_i^{(\nu)}/t_\mu^{(\nu)} \quad \text{for} \quad i \neq \mu, \nu$$

Right hand sides of these equations are called *the transition functions* from $t^{(\nu)}$ to $t^{(\mu)}$ over $U_\mu \cap U_\nu$. For example, $\mathbb{P}_1$ can be produced from two copies of $\mathbb{A}^1$ by identifying for all $t \neq 0$ the point t in one of them with the point $1/t$ in the other.

Exercice 1.6*. If you have some experience in smooth topology, prove that real and complex projective lines are analytic manifolds isomorphic to the circle S^1 (in real case) and to the Riemann sphere S^2 (in complex case).

PROJECTIVE SUBSPACES. A closed projective algebraic subset is called *a projective subspace* if it can be given by a system of *linear* homogeneous equations. Any projective subspace $L \subset \mathbb{P}(V)$ has a form $L = \mathbb{P}(W)$, where

[3] this means that $n+1$ forms $\alpha, \xi_1, \xi_2, \ldots, \xi_n$ form a base of V^*

$W \subset V$ is a vector subspace. Note that 0-dimensional projective subspaces[4] coincide with the points. Since $\operatorname{codim}_{\mathbb{P}(V)}\mathbb{P}(W) = \operatorname{codim}_V W$, we have $L_1 \cap L_2 \neq \varnothing$ for any two projective subspaces L_1 and L_2 such that $\operatorname{codim} L_1 + \operatorname{codim} L_2 \leqslant n$. For example, any two lines on $\mathbb{P}_2$ have non empty intersection[5].

Two projective subspaces L_1 and L_2 in $\mathbb{P}_n$ are called *complementary* to each other, if

$$L_1 \cap L_2 = \varnothing \qquad \text{and} \qquad \dim L_1 + \dim L_2 = n - 1\,.$$

For example, any two skew lines in 3-dimensional space are complementary.

Exercice 1.7. Show that $\mathbb{P}(U)$ and $\mathbb{P}(W)$ are complementary in $\mathbb{P}(V)$ iff $V = U \oplus W$.

Lemma 1.8. *If $L_1, L_2 \subset \mathbb{P}(V)$ are two complementary linear subspaces, then any point $p \in \mathbb{P}(V) \setminus (L_1 \cup L_2)$ lies on a unique line crossing the both subspaces.*

Proof. We have $V = U_1 \oplus U_2$, where $\mathbb{P}(U_i) = L_i$. So, any $v \in V$ has a unique decomposition $v = u_1 + u_2$ with $u_i \in U_i$. If $v \notin U_1 \cup U_2$, then both u_i, u_2 are non zero and there is a unique 2-dimensional subspace that contains v and has non zero intersections with both U_i: it coincides with a linear span of u_1, u_2. □

Projections. For any two complementary projective subspaces $L_1, L_2 \subset \mathbb{P}_n$, a *projection* from L_1 onto L_2 is a map $\pi^{L_1}_{L_2} : (\mathbb{P}_n \setminus L_1) \longrightarrow L_2$ that sends any point $q \in L_2$ to itself and any point $p \in \mathbb{P}_n \setminus (L_1 \sqcup L_2)$ to $\ell \cap L_2$, where ℓ is the unique line passing through p and crossing both L_1 and L_2 (like in Lemma 1.8). In homogeneous coordinates $(x_0 : x_1 : \ \ldots\ : x_n)$ such that L_1 is coordinated by $(x_0 : x_1 : \ \ldots\ : x_m)$ and L_2 is coordinated by $(x_{m+1} : x_{m+2} : \ \ldots\ : x_n)$, the projection $\pi^{L_1}_{L_2}$ is nothing but taking $x_\nu = 0$ for $0 \leqslant \nu \leqslant (m+1)$.

Example 1.9: projecting a conic onto a line. Let $\pi^p_\ell : Q \longrightarrow \ell$ be a projection of the conic $Q = \{x_0^2 + x_1^2 = x_2^2\} \subset \mathbb{P}_2$ onto the line $\ell = \{x_0 = 0\}$ from the point $p = (1:0:1) \in Q$. Inside the standard affine chart U_2, where $x_2 = 1$, it looks like on fig. 1.3. It is bijective, because the pencil of all lines passing through p is bijectively parameterized by the points $t \in \ell$ and any such a line (pt) intersects Q exactly in one more point $q = q(t)$ in addition to p except for the the tangent line at p, which is given by $x_0 = x_2$ and crosses ℓ at the point[6] $t = (0:1:0)$ that corresponds to $q(t) = p$ itself. Moreover, this bijection is *birational*, i. e. the corresponding $(q_0 : q_1 : q_2) \in Q$ and $(0 : t_1 : t_2) \in L$ are *rational* algebraic functions of each other. Namely, $(t_1 : t_2) = (\,q_1 : (q_2 - q_0)\,)$ and $(q_0 : q_1 : q_2) = (\,(t_1^2 - t_2^2) : 2\,t_1 t_2 : (t_1^2 + t_2^2)\,)$.

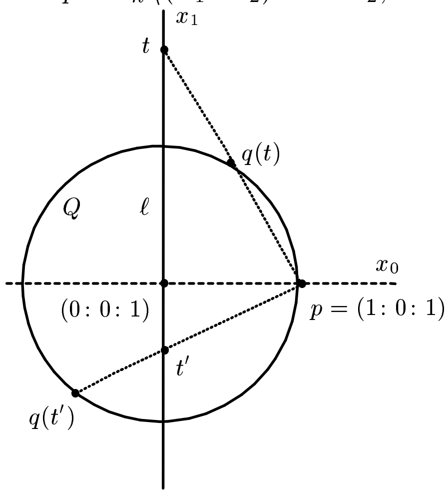

Fig. 1.3. Projecting a conic.

Exercice 1.10. Check these formulas and note that the second of them gives the full list of the Pythagorean triples[7] $(q_0 : q_1 : q_2)$ when (t_1, t_2) runs through $\mathbb{N} \times \mathbb{N}$.

Matrix notations for linear maps. Let $\operatorname{Hom}(U, W)$ be the space of all linear maps from n-dimensional vector space U to m-dimensional vector space W. Denote by $Mat_{m\times n}(k)$ the space of matrices with m rows, n columns, and entries in the field k. Any pair of basises $\{u_1, u_2, \ldots, u_n\} \subset U$ and $\{w_1, w_2, \ldots, w_m\} \subset W$ presents an isomorphism $\operatorname{Hom}(U, W) \xrightarrow{\sim} Mat_{m\times n}(k)$ that sends an operator $U \xrightarrow{\alpha} W$ to a matrix $A = (a_{ij})$ whoose j-th column consists of m coordinates of the vector $\alpha(u_j) \in W$ w. r. t. the basis $\{w_1, w_2, \ldots, w_m\}$, i. e.

$$\alpha(u_j) = \sum_{\nu=1}^{m} a_{\nu j}\, w_\nu$$

or shortly $(\alpha(u_1),\, \alpha(u_2),\, \ldots\,,\, \alpha(u_n)) = (w_1, w_2, \ldots, w_m) \cdot A$. Let ${}^t z$ and ${}^t y$ be the transposed[8] coordinate rows for $u = (z_1, z_2, \ldots, z_m) \in U$ and $w = \alpha(u) = (y_1, y_2, \ldots, y_n) \in W$. Then ${}^t z = A\,{}^t y$, because of

$$\begin{aligned}(w_1, w_2, \ldots, w_m) \cdot {}^t z = w = \alpha(u) = \alpha((u_1, u_2, \ldots, u_n) \cdot {}^t y) = \\ = (\alpha(u_1),\, \alpha(u_2),\, \ldots\,,\, \alpha(u_n)) \cdot {}^t y = (w_1, w_2, \ldots, w_m) \cdot (\,A\,{}^t y\,)\end{aligned}$$

Linear projective transformations. If $\dim U = \dim W = (n+1)$, then any linear isomorphism $U \xrightarrow{\alpha} W$ induces the bijection $\mathbb{P}(U) \xrightarrow{\overline{\alpha}} \mathbb{P}(W)$, which is called *the projective linear transformation* or *the linear isomorphism*. A point set $\{p_1, p_2, \ldots, p_m\} \subset \mathbb{P}_n$ is called *linearly general*, if any $(n+1)$ of p_i don't lay together in

[4] that is, $\mathbb{P}_0 = \mathbb{P}(k^1)$
[5] in terms of $\mathbb{A}^3$ this means that any two planes containing the origin are intersected along a line
[6] this intersection point lays at the infinity on fig. 1.3
[7] i. e. the right triangles with integer side lengths
[8] i. e. arranged in columns

any hyperplane $\mathbb{P}_{n-1} \subset \mathbb{P}_n$. Equivalently, the points $\{p_i\}$ are linearly general in $\mathbb{P}_n = \mathbb{P}(V)$ iff $(n+1)$ vectors representing any $(n+1)$ of them always form a basis of V.

Lemma 1.11. *For any two linearly general collections of $(n+2)$ points $\{p_0, p_1, \dots, p_{n+1}\} \in \mathbb{P}(U)$ and $\{q_0, q_1, \dots, q_{n+1}\} \in \mathbb{P}(V)$, where $\dim U = \dim V = (n+1)$, there exists a unique up to proportionality linear isomorphism $V \xrightarrow{\alpha} W$ such that $\overline{\alpha}(p_i) = q_i$ $\forall i$. In particular, two matrices give the same projective linear transformation iff they are proportional.*

Proof. Fix some vectors u_i and w_i representing the points p_i and q_i. By the linear generality, we can take $\{u_1, u_2, \dots, u_{n+1}\}$ and $\{w_1, w_2, \dots, w_{n+1}\}$ as the basises in U and W and identify a map $\alpha \in \mathrm{Hom}(U, W)$ by the square matrix in these basises. Then, $\overline{\alpha}(p_i) = q_i$ for $1 \leqslant i \leqslant (n+1)$ iff the matrix A of α is diagonal, say with $(d_1, d_2, \dots, d_{n+1})$ on the main diagonal. Now consider the first vectors u_0 and w_0. Again by the linear generality, all the coordinates of $u_0 = (x_1, x_2, \dots, x_{n+1})$ w.r.t. the basis $\{u_i\}_{1 \leqslant i \leqslant (n+1)}$ and all the coordinates of $w_0 = (y_1, y_2, \dots, y_{n+1})$ w.r.t. the basis $\{w_i\}_{1 \leqslant i \leqslant (n+1)}$ are non zero. Since the coordinates of $\alpha(u_0)$ are proportional to the ones of w_0, we have $y_i : y_j = (d_i x_i) : (d_j x_j)$ $\forall i, j$. Hence, all d_i are uniquely recovered from just one of them, say d_1, as $d_i = d_1 \cdot (y_i x_1) : (x_i y_1)$. □

Exercice 1.12. Let ℓ_1 and ℓ_2 be two lines on $\mathbb{P}_2$. Fix any point p outside $\ell_1 \cup \ell_2$ and consider the a $\ell_1 \xrightarrow{\gamma_p} \ell_2$ that sends $t \in \ell_1$ to the crossing point of ℓ_2 and the line (tp). Check that γ_p is the linear isomorphism.

Linear projective group. All linear isomorphisms $V \longrightarrow V$ form a group called *the general linear group* of V and denoted by $GL(V)$. This group acts on $\mathbb{P}(V)$. By Lemma 1.11, the kernel of this action coincides with the subgroup of all scalar dilatations $H \subset GL(V)$. Hence, the group of all projective linear automorphisms of $\mathbb{P}(V)$ equals to the factor group $GL(V)/H$, which is denoted by $PGL(V)$ and called *the projective linear group*. As soon as some basis $\{e_0, e_1, \dots, e_n\} \subset V$ is fixed, we can identify $GL(V)$ with the group $GL_{n+1}(k) \subset Mat_{n+1}(k)$ of all non degenerated square matrices. Under this identification the dilatations go to the scalar matrices and $PGL(V)$ turns into the group $PGL_{n+1}(k) \overset{\text{def}}{=} GL_{n+1}(k)/\{\text{scalar diagonal matrices } \lambda E\}$ of all non degenerate square matrices considered up to proportionality.

Example 1.13: linear-fractional group. The group $PGL_2(k)$ consists of all square matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which have $ad - bc \neq 0$ and are considered up to proportionality. It acts on $\mathbb{P}_1$ via $(x_0 : x_1) \overset{A}{\longmapsto} ((ax_0 + bx_1) : (cx_0 + dx_1))$. In terms of the standard affine chart $U_0 \simeq \mathbb{A}_1$ with the affine coordinate $t = x_1/x_0$, this action takes $t \longmapsto (ct + d)/(at + b)$

Exercice 1.14. Check that $(AB)(t) = A(B(t))$.

Exercice 1.15. For any 3 different points $p, q, r \in U_0$ write down the unique linear-fractional transformation α such that $\alpha(p) = 0$, $\alpha(q) = 1$, and $\alpha(r) = \infty$.

Hint. It has to be of the form $t \overset{\alpha}{\longmapsto} \vartheta \cdot (t-p)/(t-r)$, where $\vartheta \in k$.

Proposition 1.16. *Let $\mathbb{P}_1 \xrightarrow{\varphi} \mathbb{P}_1$ be a bijection over an algebraically closed field. If everywhere except for a finite number of points φ is given by the formula $\varphi(x_0 : x_1) = (f_0(x_0, x_1) : f_1(x_0, x_1))$, where f_i are rational algebraic functions, then $\varphi \in PGL_2(k)$.*

Proof. To give a well defined map, the functions f_i have to be homogeneous of the same degree. Moreover, multiplying by denominators and eliminating common factors we can suppose that f_i are *coprime polynomials*. Since φ is bijective, each $\vartheta = (\vartheta_0 : \vartheta_1) \in \mathbb{P}_1$ (except for a finite set) has precisely one preimage, that is the homogeneous equation

$$\vartheta_1 \cdot f_0(x_0, x_1) - \vartheta_0 \cdot f_1(x_0, x_1) = 0 \tag{2}$$

has just one root up to proportionality, i.e. is pure d-th power of some linear form in $(x_0 : x_1)$, where $d = \deg f_1 = \deg f_2$. Now note, that all degree d homogeneous polynomials in $(x_0 : x_1)$ considered up to a scalar factor form the projective space $\mathbb{P}_d = \mathbb{P}(S^d U^*)$, where U is 2-dimensional vector subspace underlying $\mathbb{P}_1$ in question. When ϑ varies through $\mathbb{P}_1$, the equations (2) draw a line in this $\mathbb{P}_d$, whereas pure d-th powers of linear forms form a curve called the *Veronese curve* of degree d. It is easy to see (see the Exercice 1.17 below) that over an infinite field the Veronese curve of degree $d \geqslant 2$ is *twisted*, that is any $(k+1)$ its distinct points do not lay in a common $(k-1)$-dimensional subspace for all $k = 2, 3, \dots, d$. In particular, the line (2) can intersect the Veronese curve in at most two points, if $d \geqslant 2$. So, $d = 1$, i.e. φ is a projective linear isomorphism. □

Exercice 1.17. The Veronese curve of degree d is defined as an image of the Veronese map

$$\mathbb{P}_1 = \mathbb{P}(U^*) \xrightarrow{v_d} \mathbb{P}_d = \mathbb{P}\left(S^d(U^*)\right) \tag{3}$$

which takes a linear form $\varphi \in U^*$ to its d-th power $\psi^d \in S^d(U^*)$. Show that if the ground field contains more than d points, then any $d+1$ distinct points of the Veronese curve are linearly general.

Hint. Write $\varphi \in U^*$ as $\alpha_0 x_0 + \alpha_1 x_1$ and $f \in S^d(U^*)$ as $\sum\limits_{\nu=0}^{d} a_\nu \cdot \binom{d}{\nu} x_0^{d-\nu} x_1^\nu$ and use $(\alpha_0 : \alpha_1)$, $(a_0 : a_1 : \dots : a_d)$ as homogeneous coordinates on $\mathbb{P}_1$ and $\mathbb{P}_d$ in question. Then an intersection of the Veronese curve with $(d-1)$-dimensional hyperplane given by linear equation $\sum A_\nu a_\nu = 0$ consists of $(\alpha_0 : \alpha_1)$ satisfying degree d equation $\sum A_\nu \cdot \alpha_0^{d-\nu} \alpha_1^\nu = 0$.

Projective quadrics[1].

QUADRATIC AND BILINEAR FORMS. The zero set $Q = (q)_0 \subset \mathbb{P}(V)$ of a non zero quadratic form $q \in S^2V^*$ is called *a projective quadric*. If $2 \neq 0$ in k, then in homogeneous coordinates any quadratic form can be written as

$$q(x) = \sum_{i,j} a_{ij}\, x_i x_j = x \cdot A \cdot {}^t\!x \ ,$$

where $x = (x_0, x_1, \dots, x_n)$ is the coordinate row, ${}^t\!x$ is its transposed column version, and $A = (a_{ij})$ is a *symmetric* matrix over k, called *a Gram matrix* of q. Hence, there exists a unique bilinear form $\widetilde{q}(u, w)$ on $V \times V$ such that $q(x) = \widetilde{q}(x, x)$. It can be calculated by the following pairwise equivalent ways

$$\widetilde{q}(x,y) = \sum_{ij} a_{ij}\, x_i y_j = x \cdot A \cdot {}^t\!y = \frac{1}{2}\sum_i y_i \frac{\partial q(x)}{\partial x_i} = \frac{1}{2}\Big(q(x+y) - q(x) - q(y)\Big) = \frac{1}{4}\Big(q(x+y) - q(x-y)\Big)$$

The form $\widehat{q}$ is called *a polarization* of q. It gives a kind of scalar product on V and the Gram matrix elements are nothing but the scalar products $a_{ij} = \widetilde{q}(e_i, e_j)$, of basic vectors $\{e_i\}$.

In an other base, say by $(e'_0, e'_1, \dots, e'_n) = (e_0, e_1, \dots, e_n) \cdot C$, the new Gram matrix A' is expressed trogh the old Gram matrix A as

$$A' = {}^t C \cdot A \cdot C \ .$$

So, *the Gram determinant* $\det q \stackrel{\text{def}}{=} \det A \in k/k^2$, which is considered up to multiplications by the squares from k, doesn't depend on the choice of base. A quadric is called *non singular* if $\det q \neq 0$. Otherwise, it is called *singular*. Two quadrics are called *isomorphic* or *projectively equivalent* if they are produced from each other by a projective automorphism (i. e. by a base change).

Theorem 2.1. ([2]) *Any quadratic form on V has a diagonal Gram matrix in some base.*

Proof. If $q \equiv 0$, then its Gram matrix is diagonal. If not, then $q(v) = \widetilde{q}(v, v) \neq 0$ for some $v \in V$. Take it as the first vector of the base being constructed. Note that $V = L \oplus L^{\perp}$, where L is one dimensional subspace spaned by v and $L^{\perp} = \{\, w \in V \mid \widetilde{q}(v, w) = 0 \,\}$ is its orthogonal. Indeed, any $u \in V$ is uniquely decomposed as $u = \lambda v + v^{\perp}$ with $v^{\perp} \in L^{\perp}$. Namely, $\lambda = \widetilde{q}(v, u)/\widetilde{q}(v, v)$ and $v^{\perp} = u - (\widetilde{q}(v, u)/\widetilde{q}(v, v)) \cdot v$. Now, repeat these arguments to $L^{\perp}$ to find the next base vector, and so on. □

Corollary 2.2. *Over an algebraically closed field any quadric may be written as $\sum x_i^2 = 0$ in some coordinate system; in particular all non singular quadrics are projectively equivalent.*

Proof. Diagonal elements of the Gram matrix turns into units after the base change $e_i \longmapsto e_i/\sqrt{q(e_i)}$. □

Example 2.3: quadric on $\mathbb{P}_1$, by the Theorem 2.1, can be written in an appropriate coordinates either as $a\,x_0^2 + b\,x_1^2 = 0$ or as $a\,x_0^2 = 0$. The second is singular (i. e. has $\det q = 0$) and is called *a double point*. It consists of just one point $(0 : 1)$, which has «a multiplicity 2» in any reasonable sense. The first quadric is non singular and either is empty or consists of two distinct points. More precisely, $Q = \varnothing$ iff $-\det q \equiv -b/a \pmod{k^2}$ is not a square in k; in particular, this is impossible over an algebraically closed field. If $-\det q = \delta^2$ is a square, then $Q = \{\, (-\delta/a : 1)\, , (\delta/a : 1) \,\}$.

QUADRIC AND LINE. By the above example, for any quadric Q and a line ℓ there are exactly 4 possibilities: either $\ell \subset Q$, or $\ell \cap Q$ consists of 2 distinct points, or $\ell \cap Q$ is a double point, or $\ell \cap Q = \varnothing$. Moreover, the last case is impossible, if k is algebraically closed. A line is called *a tangent line*, if it lays on Q or crosses Q via a double point.

CORRELATIONS. Any quadratic form q on V induces the linear map $V \xrightarrow{\widehat{q}} V^*$ that sends a vector $v \in V$ to the linear form

$$\widehat{q}(v) \ : \ w \longmapsto \widetilde{q}(w, v)$$

The map $\widehat{q}$ is called *a correlation* (or *a polarity*) of q. Its matrix written in dual bases $\{e_i\} \subset V$, $\{x_i\} \subset V^*$ coincides with the Gram matrix A. In particular, q is non singular iff $\widehat{q}$ is an isomorphism. The space

$$\ker(q) \stackrel{\text{def}}{=} \ker \widehat{q} = \{\, v \in V \mid \widetilde{q}(w, v) = 0 \ \ \forall\, w \in V \,\}$$

[1] WARNING: in this lecture we always suppose that $\operatorname{char} k \neq 2$

[2] It is known as the *Lagrange theorem*

is called *a kernel* of q. Its projectivization $Sing Q \stackrel{\text{def}}{=} \mathbb{P}(\ker q) \subset \mathbb{P}(V)$ is called *a vertex space* of $Q = (q)_0$ and $\operatorname{codim}_{\mathbb{P}(V)} Sing Q$ is called *a corank* of Q.

Theorem 2.4. *The intersection $Q' = L \cap Q$ is non singular for any projective subspace $L \subset \mathbb{P}(V)$ complementary to $Sing Q$; moreover, Q is the cone over Q' with the vertex space $Sing Q$, i. e. Q is the union of all lines crossing both Q' and $Sing Q$.*

Proof. Take any direct decomposition $V = \ker q \oplus U$ and let $L = \mathbb{P}(U)$. If $u \in U$ satisfy $\widetilde{q}(u, u') = 0 \quad \forall u' \in U$, then automatically $\widetilde{q}(u, v) = 0 \quad \forall v \in V$ and $u = 0$, because of $\ker q \cap U = 0$. Since $Q' = Q \cap L$ is given by the restriction $q|_U$, it is non singular. Further, for any line $\ell = \mathbb{P}(W)$ such that $\dim W \cap \ker Q = 1$ we have $\dim W \cap U = 1$ and $\operatorname{cork} q|_U \geqslant 1$. So, if $\ell \cap Sing Q = \{p\}$ is just one point, then $\ell \cap L \neq \varnothing$ and either $\ell \subset Q$ or $\ell \cap Q\{p\}$. That's all we need. □

Corollary 2.5. *A quadric $Q \subset \mathbb{P}_n$ over an algebraically closed field is uniquely up to an isomorphism defined by its corank; it may be equal to* $0, 1, \dots, n$.

Proof. Corank is the number of diagonal zeros in the diagonal Gram matrix. □

Tangent space T_pQ to a quadric Q at a point $p \in Q$ is defined as the union of all tangent lines passing through p.

Lemma 2.6. *Let p and p' be distinct points and $p \in Q = (q)_0$. The line $\ell = (p\,p')$ is tangent to Q iff $\widetilde{q}(p, p') = 0$, i. e. iff p and p' are orthogonal with respect to polarization of q.*

Proof. Take some vectors u, u' representing p and p'. Then $\ell = \mathbb{P}(U)$. The restriction $q|_U$ has the Gram matrix

$$\begin{pmatrix} \widetilde{q}(u,u) & \widetilde{q}(u,u') \\ \widetilde{q}(u',u) & \widetilde{q}(u',u') \end{pmatrix} .$$

It is singular iff $\widetilde{q}(u, u') = 0$, because of $\widetilde{q}(u, u) = 0$ by the lemma assumption. □

Corollary 2.7. $p \in Sing Q \iff T_pQ$ *is the whole space* $\iff \dfrac{\partial q}{\partial x_i}(p) = 0 \ \forall i$. □

Corollary 2.8. *If $p \in (Q \setminus Sing Q)$, then $T_pQ = \{x \in \mathbb{P}_n \mid \widetilde{q}(p, x) = 0\}$ is a hyperplane of codimension 1.* □

Corollary 2.9. *Let $p \notin Q$ and a hyperplane $C \subset \mathbb{P}_n$ be given by the equation $\widetilde{q}(p, x) = 0$ in x. Then $Q \cap L$ consists of all points where Q is touched by the tangent lines coming from p.* □

Polar mappings. The spaces $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ are called *dual* and denoted by $\mathbb{P}_n$ and $\mathbb{P}_n^\times$ when V is not essential. Since any codimension 1 subspace $U \subset V$ is defined by linear form $\xi \in V^*$, which is unique up to proportionality, $\mathbb{P}_n^\times$ is just *the space of hyperplanes* in $\mathbb{P}_n$ and vice versa. If $Q = (q)_0 \subset \mathbb{P}(V)$ is non singular, then the linear isomorphism $\mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V^*)$ induced by the correlation $\widehat{q}$ is called *a polarity* of Q. It sends a point $p \in \mathbb{P}_n$ to the hyperplane $L \subset \mathbb{P}_n$ given by the equation $\widetilde{q}(p, x) = 0$ like in the previous Corollary. L is called *a polar* of p and p is called *a pole* of L with respect to q. So, Q is just the set of all points laying on their own polars. Note that some non singular quadratic forms q can produce *empty* quadrics Q over an non closed field; but their polar mappings will be always visible!

Exercice 2.10. Show that p lays on the polar of q iff q lays on the polar of p (for any pair of distinct points and any polarity).

Exercice 2.11. Consider a circle in the real Euclidean affine plane $\mathbb{R}^2$. How to draw the polar of a point that lays: a) outside b) inside this circle? Describe geometrically the polarity defined by the «imaginary» circle «given» in $\mathbb{R}^2$ by the equation $x^2 + y^2 = -1$.

Proposition 2.12. *Two polarities coincide iff the corresponding quadratic forms are proportional.*

Proof. This follows from Lemma 1.11 □

Corollary 2.13. *Over an algebraically closed field two quadrics coincide iff their quadratic equations are proportional.*

Proof. Let $Q = Q'$. We can suppose that the quadrics are non singular, because of their equations are not changed under direct summation with the kernel $\ker q = \ker q'$. Non singular case is covered by the above Proposition. □

The space of quadrics. All the polarities on $\mathbb{P}_n = \mathbb{P}(V)$ are one-to-one parameterized by the points of the projective space

$$\mathbb{P}_{\frac{n(n+3)}{2}} = \mathbb{P}(S^2V^*) ,$$

which will be referred as *a space of quadrics*.

Complex projective conics. A quadric on the projective plane is called *a projective conic*. A projective conic over $\mathbb{C}$, up to an isomorphism, coincides either with *a double line* $x_0 = 0$, which has corank 2, or with *a reducible conic*[3] $x_0^2 + x_1^2 = 0$, which has corank 1, or with the non singular conic $x_0^2 + x_1^2 + x_2^2 = 0$. The space of all conics in $\mathbb{P}_2 = \mathbb{P}(V)$ is $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$.

[3] i. e. a pair of crossing lines

Example 2.14: standard model for non singular conic. Let U be 2-dimensional vector space. The quadratic *Veronese map* (Example 5.14), which sends a linear form ξ to its square ξ^2, gives the embedding $\mathbb{P}(U^*) = \mathbb{P}_1 \overset{v}{\hookrightarrow} \mathbb{P}_2 = \mathbb{P}(S^2U^*)$. Let us identify $\mathbb{P}(S^2U^*)$ with the space of quadrics on $\mathbb{P}_1 = \mathbb{P}(U)$. Then the Veronese embedding is a bijection between the points of $\mathbb{P}_1$ and the singular quadrics on $\mathbb{P}_1$, that is with the double points. So, the Veronese conic $Q_V = \{\, q \in S^2U^* \mid \det q = 0 \,\}$ is the quadric; it is called *a Veronese quadric*. Fixing a base (x_0, x_1) for U^* and induced base $\{x_0^2,\, 2\,x_0x_1,\, x_1^2\}$ for S^2U^*, we can coordinate a quadric $q(x) = q_0\, x_0^2 + 2\, q_1\, x_0x_1 + q_2\, x_1^2 \in S^2U^*$ by $(q_0 : q_1 : q_2)$ and a linear form $\xi(x) = t_0x_0 + t_1x_1 \in U^*$ by $(t_0 : t_1)$. Then Veronese sends $(t_0x_0 + t_1x_1) \longmapsto (t_0x_0 + t_1x_1)^2 = t_0^2\, x_0^2 + 2\, t_0t_1\, x_0x_1 + t_1^2\, x_1^2$. So, Q_V has the equation $\det(q) = q_0q_2 - q_1^2 = 0$ and admits the quadratic parameterization $(q_0 : q_1 : q_2) = (t_0^2 : t_0t_1 : t_1^2)$. Note that any non singular conic $Q \subset \mathbb{P}_2$ can be identified with Q_V by an appropriate base choice, in particular, any smooth Q admits a quadratic parameterization by the points of $\mathbb{P}_1$.

Proposition 2.15. *Two distinct non singular conics cann't have more then 4 common points.*

Proof. Parameterize the first conic by $x = v(t_0, t_1)$ via Veroneze. If the second conic is given by the quadratic equation $q(x) = 0$, then the t-coordinates of the intersection points are given by the 4-th degree equation $q(v(t)) = 0$. □

Proposition 2.16. *Any 5 points in $\mathbb{P}_2$ lay on some conic. If no 3 of the points are collinear, then this conic is unique and non singular.*

Proof. Since the equation $q(p) = 0$ is linear in q for a given p, the conics passing through $p \in \mathbb{P}_2$ form the hyperplane in $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$. Any 5 hyperplanes in $\mathbb{P}_5$ have non zero intersection, so, some conic exists. If it is singular, it consists of one or two lines. Hence at least 3 of 5 points are collinear. By the previous proposition, non singular conic is unique. □

COMPLEX PROJECTIVE QUADRICS ON $\mathbb{P}_3$ up to an isomorphism are: *a double plane* $x_0^2 = 0$; *a reducible quadric* $x_0^2 + x_1^2 = 0$, which is a pair of crossing planes (or a cone with a line vertex over a pair of distinct points on an complementary line); *a simple cone* $x_0^2 + x_1^2 + x_2^2 = 0$, which is a cone with one point vertex over a non singular plane conic; and a non singular quadric $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$. The projective space of all quadrics on $\mathbb{P}_3$ has dimension $3 \cdot 6 : 2 = 9$. In particular *any 9 points in $\mathbb{P}_3$ lay on some quadric.*

Claim 2.17. *Any 3 lines on $\mathbb{P}_3$ lay on some quadric.*

Proof. Take 3 distinct points on each line and draw a quadric through these 9 points. □

Note that if 3 lines are pairwise skew, then a quadric containing these lines must be non singular. If 2 of the lines are skew, then such a quadric is ether reducible or non singular (but nether a cone nor a double plane).

SEGRE QUADRIC IN $\mathbb{P}_3$. Let U_-, U_+ be 2-dimensional spaces and $W = \mathrm{Hom}(U_-, U_+)$ be the space of all linear maps. Then $\mathbb{P}_3 = \mathbb{P}(W)$ consists of non zero linear maps considered up to proportionality. Non zero but degenerate maps form a quadric $Q_S \subset \mathbb{P}_3$, which is called *a Segre quadric*. In the matrix notations, Q_S is presented by the determinantal equation

$$\det \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} = \alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10} = 0\ . \tag{4}$$

This quadric is bijectively parameterized by the points of $\mathbb{P}_1 \times \mathbb{P}_1 = \mathbb{P}(U_-^*) \times \mathbb{P}(U_+)$ as follows. *A Segre embedding*

$$\mathbb{P}(U_-^*) \times \mathbb{P}(U_+) \overset{s}{\hookrightarrow} \mathbb{P}(\mathrm{Hom}(U_-, U_+))$$

sends a pair $(\xi, v) \in U_-^* \times U_+$ to rank 1 operator $U_- \xrightarrow{\xi\otimes v} U_+$ defined by the rule $U_- \ni u \longmapsto \xi(u)\, v \in U_+$. So, $\ker(\xi \otimes v)$ coincides with the zero subspace of the linear form ξ and $\mathrm{im}\,(\xi \otimes v)$ is spanned by v. Since any non zero degenerate operator $U_- \xrightarrow{\varphi} U_+$ has 1-dimensional kernel and 1-dimensional image, it may be written as $\varphi = \xi \otimes v$ for some $\xi \in U_-^*$, $v \in U_+$, which are up to proportionality uniquely defined by φ. So, the Segre embedding maps $\mathbb{P}_1 \times \mathbb{P}_1$ bijectively onto Q_S.

In terms of coordinates, if $\xi = (\xi_0 : \xi_1) \in U_-^*$ and $v = (t_0 : t_1) \in U_+$, then the operator $\xi \otimes v$ sends the first basic vector $(1, 0) \in U_-$ to $\xi_0 \cdot v$ and the second basic vector $(0, 1) \in U_-$ to $\xi_1 \cdot v$. Hence, $\xi \otimes v$ has a the matrix $\begin{pmatrix} \xi_0t_0 & \xi_1t_0 \\ \xi_0t_1 & \xi_1t_1 \end{pmatrix}$. So, the Segre embedding gives a parametrization for the quadric (4) by the formulas

$$\alpha_{00} = \xi_0t_0\ , \quad \alpha_{01} = \xi_1t_0\ , \quad \alpha_{10} = \xi_0t_1\ , \quad \alpha_{11} = \xi_1t_1\ .$$

where $(\,(\xi_0 : \xi_1)\,,\,(t_0 : t_1)\,) \in \mathbb{P}_1 \times \mathbb{P}_1$. In other words, any rank one 2×2 - matrix (a_{ij}) has proportional rows and proportional columns and their ratios

$$([\text{row }1)] : [\text{row }2]) \overset{\text{def}}{=} (t_0 : t_1) \qquad \text{and} \qquad ([\text{col }1)] : [\text{col }2]) \overset{\text{def}}{=} (\xi_0 : \xi_1)\ .$$

define the matrix up to a scalar factor.

Two rulings for non singular quadric on $\mathbb{P}_3$. Note that $\mathbb{P}_1 \times \mathbb{P}_1$ is ruled by two families of «coordinate» lines $\xi \times \mathbb{P}_1$ and $\mathbb{P}_1 \times t$. They are called *a first* and *a second* line families.

Lemma 2.18. *The Segre embedding sends each coordinate line family to the ruling of Q_s by a family of pairwise skew lines. These two line families exhaust all the lines on Q_s. Any two lines from different families are intersecting and each point of Q_s is an intersection point of two lines from different families.*

Proof. The line $\xi \times \mathbb{P}_1$, where $\xi = (\xi_0 : \xi_1) \in \mathbb{P}(U_-^*)$, goes to the set of all degenerated matrices with the fixed column ratio, i. e. to the line given in $\mathbb{P}_3$ by two linear equations $a_{00} : a_{01} = a_{10} : a_{11} = \xi_0 : \xi_1$ Analogously, $s(\mathbb{P}_1 \times t)$, where $t = (t_0 : t_1) \in \mathbb{P}(U_+)$, goes to the line given in $\mathbb{P}_3$ by $a_{00} : a_{10} = a_{01} : a_{11} = t_0 : t_1$. Since the Segre embedding is bijective, each line family consists of pairwise skew lines, any two lines from the different families are intersecting, and for any $x \in Q_s$ there is a pair of the different type lines passing through x. So, $Q_s \cap T_x Q_s$ is the reducible conic, which splits into the last pair of lines. In particular, there are no other lines on Q_s. □

Corollary 2.19. *There exists a unique quadric passing through 3 given pairwise skew lines in $\mathbb{P}_3$. It is ruled by all lines intersecting each of 3 given lines.*

Proof. Let a quadric Q contain 3 given lines. Since Q is non singular, it can be ruled by two line families via Segre. 3 given lines must belong to the same, say, to the first family. Any line from the second family intersect all the lines of the first family (in particular, 3 given lines). Moreover, any line on $\mathbb{P}_3$ which intersect 3 given lines must lay on Q and belong to the second family. □

Exercice 2.20. How many lines intersect 4 given pairwise skew lines in $\mathbb{P}_3$?

Hint. Draw all lines intersecting some 3 of given 4 and look at the interaction of this figure with the remaining 4-th line.

Exercice 2.21*. Do the previous exercise for $\mathbb{A}^3$ instead of $\mathbb{P}_3$ and also over $\mathbb{R}$ instead of $\mathbb{C}$. There will be several answers depending on the line configuration. Find all possible answers and indicate which of them are stable under small perturbations.

Linear subspaces laying on a non singular quadric . Lemma 2.18 can be extended in higher dimensions. Let $Q_n \subset \mathbb{P}_n = \mathbb{P}(V)$ be non singular quadric and $L = \mathbb{P}(W) \subset Q_n$.

Theorem 2.22. $\dim L \leqslant \left[\frac{n-1}{2}\right]$, *where $[\,*\,]$ means the integer part.*

Proof. Let Q_n be given by a quadratic form q with the polarization $\widetilde{q}$. Then

$$L \subset Q_n \iff \widetilde{q}(w_1, w_2) = 0 \;\forall\, w_1, w_2 \in W \iff \widehat{q}(W) \subset \operatorname{Ann}(W) = \{\, \xi \in V^* \mid \xi(w) = 0 \quad \forall\, w \in W \,\},$$

where $\widehat{q} : v \longmapsto \widetilde{q}(v, *)$ is the correlation associated with Q_n. This correlation is injective, because of Q_n is non singular. So, $\dim W \leqslant \dim \operatorname{Ann} W = \dim V - \dim W$. Hence, $\dim L = \dim W - 1 \leqslant (\dim V)/2 - 1 = [(n-1)/2]$. □

Lemma 2.23. *$cork\,(H \cap Q_n) \leqslant 1$ for any codimension 1 hyperplane $H \subset \mathbb{P}_n$.*

Proof. If $H = \mathbb{P}(W)$, then $\dim\ker(q|_W) \leqslant \dim\big(W \cap \widehat{q}^{-1}(\operatorname{Ann} W)\big) \leqslant \dim \widehat{q}^{-1}(\operatorname{Ann} W) = \dim \operatorname{Ann} W = 1$. □

Lemma 2.24. *For any $x \in Q_n$, $Q_n \cap T_x Q_n$ is a simple cone with the vertex x over a non singular quadric Q_{n-2} in a $(n-2)$-dimensional projective subspace in $T_x Q_n \setminus x$.*

Proof. Since $T_x Q = \mathbb{P}(\ker \widetilde{q}(x, *))$ and $\widetilde{q}(x, x) = q(x) = 0$, the restriction of q onto $T_x Q$ has at least 1-dimensional kernel presented by x itself. By the previous lemma this kernel is spaned by x. □

Theorem 2.25. *Let $d_n = \left[\frac{n-1}{2}\right]$ be the upper bound from Theorem 2.22. The d_n-dimensional subspaces $L \subset Q_n$ passing through a given point on Q_n are parameterized bijectively by the all $(d_n - 1)$-dimensional subspaces laying on Q_{n-2}.*

Proof. Fix some $(n-1)$-dimensional projective subspace $H \subset T_x Q \setminus x$ and present $Q_n \cap T_x Q_n$ as a simple cone ruled by lines passing through x and some $Q_{n-2} \subset H$. Since any $L \subset Q_n$ which pass through x is contained inside $Q_n \cap T_x Q_n$, it has to be the linear span of x and some $(d_n - 1)$-dimensional subspace $L' \subset Q_{n-2}$. □

For example, there are only 0-dimensional subspaces on Q_1 and Q_2. Next two quadrics, Q_3 and Q_4, do not contain planes. But any point $x \in Q_3$ lies on two lines passing through x and 2 points of $Q_1 \subset T_x Q_3 \setminus \{x\}$ and any point of Q_4 belongs to 1-dimensional family of lines parameterized by the points of a non singular conic $Q_2 \subset T_x Q_4 \setminus \{x\}$. Further, non singular quadric $Q_5 \subset \mathbb{P}_5$ does not contain 3-dimensional subspaces but for any point $x \in Q_5$ there are two 1-dimensional families of planes passing through x. Each family is parameterized by the corresponding family of lines on $Q_3 \subset T_x Q_5 \setminus \{x\}$, i. e. by $\mathbb{P}_1$ actually.

Plane drawings: line pencils and conics[1].

<u>PROJECTIVE DUALITY.</u> For any $0 \leqslant m \leqslant (n-1)$ there is a canonical bijection between the m-dimensional projective subspaces in $\mathbb{P}_n = \mathbb{P}(V)$ and the $(n-1-m)$-dimensional ones in $\mathbb{P}_n^\times \overset{\text{def}}{=} \mathbb{P}(V^*)$. It sends a subspace $L = \mathbb{P}(U)$ to the subspace $L^\times \overset{\text{def}}{=} \mathbb{P}(\operatorname{Ann}(U))$, where $\operatorname{Ann}(U) \overset{\text{def}}{=} \{\,\xi \in V^* \mid \xi(u) = 0 \quad \forall u \in U\,\}$ is *an anihilator* of U. Since under the natural identification $V^{**} = V$ we have $\operatorname{Ann}\operatorname{Ann} U = \{\,v \in V \mid \xi(v) = 0 \quad \forall \xi \in \operatorname{Ann} U\,\} = U$, then $L^{\times\times} = L$. The correspondence $L \leftrightarrow L^\times$ is called *a projective duality*. It inverts the inclusions[2] and the linear incidence: $L_1, L_2, \dots, L_r$ are contained in some m-dimensional subspace L iff $L_1^\times, L_2^\times, \dots, L_r^\times$ contain some $(n-m-1)$-dimensional subspace $L^\times$ ([3]). The projective duality translates the geometry on $\mathbb{P}_n$ to the one on $\mathbb{P}_n^\times$ and back. For example, in $\mathbb{P}_2$-case we have the following dictionary:

a line $\ell \subset \mathbb{P}_2$	$\longleftrightarrow$	a point $\ell^\times \in \mathbb{P}_2^\times$
the points p of the above line ℓ	$\longleftrightarrow$	the lines $p^\times$ passing through the above point $\ell^\times$
the line passing through two points $p_1, p_2 \in \mathbb{P}_2$	$\longleftrightarrow$	the intersection point for two lines $p_1^\times, p_2^\times \subset \mathbb{P}_2^\times$
the points p of some conic Q	$\longleftrightarrow$	the tangent lines $p^\times$ of some conic $Q^\times$
the tangent lines ℓ of Q	$\longleftrightarrow$	the points $\ell^\times$ of $Q^\times$

Exercice 3.1. Explain the last two items by proving that the tangent spaces of a non singular quadric $Q \subset \mathbb{P}_n$ correspond to the points of some non singular quadric $Q^\times \subset \mathbb{P}_n^\times$. Show also that Q and $Q^\times$ have inverse Gram matrices in dual bases of V and V^*.

HINT. If $\xi = (\xi_0, \xi_1, \dots, \xi_n) \in V^*$ and $x = (x_0, x_1, \dots, x_n) \in V$ in dual coordinate systems, then $\xi(x) = \xi \cdot {}^t x$. Let Q be given by $x \cdot A \cdot {}^t x = 0$. Then $T_x Q = \mathbb{P}(\operatorname{Ann} \xi) \iff \xi = x \cdot A \iff x = \xi \cdot A^{-1}$. Hence,

$$x \in Q \iff x \cdot A \cdot {}^t x = 0 \iff \left(\xi \cdot A^{-1}\right) \cdot A \cdot {}^t\!\left(\xi \cdot A^{-1}\right) = 0 \iff \xi \cdot A^{-1\,t}\xi = 0 \iff \xi \in Q^\times\,,$$

where $Q^\times$ is given by the Gram matrix A^{-1}.

<u>Corollary 3.2.</u> *Any 5 lines without triple intersections in $\mathbb{P}_2$ are tangent to a unique non singular conic.*

Proof. This assertion is projectively dual to the Proposition 2.16. Namely, let $\ell_i \in \mathbb{P}_2$ be the given lines. There exists a unique conic $Q^\times$ passing through the 5 points $\ell_i^\times \in \mathbb{P}_2^\times$. Then $\ell_i = \ell_i^{\times\times}$ are tangent to the dual conic $Q = Q^{\times\times} \subset \mathbb{P}_2$. $\square$

<u>PROJECTIVE LINEAR ISOMORPHISMS $\mathbb{P}_1 \longrightarrow \mathbb{P}_1$ VIA CONICS.</u> Consider a non singular conic Q and a line ℓ and write $Q \xrightarrow{\pi_\ell^p} \ell$ for the bijective map given by projection from a point $p \in Q$ extended into p by sending $p \longmapsto \ell \cap T_p Q$.

Exercice 3.3. Show that this bijection is given by some rational algebraic functions, which express coordinates of the corresponding points through each other.

HINT. To express the coordinates of $t = \pi_\ell^p(x)$ through x, you take special coordinates where ℓ is given by $x_0 = 0$ and $p = (1:0:0)$; then $t = (0 : x_1 : x_2)$. In order to get the inverse expression, only two skills are quite enough: linear equations solving and finding the second root for a quadratic equation with the first root known. The both procedures have a rational output.

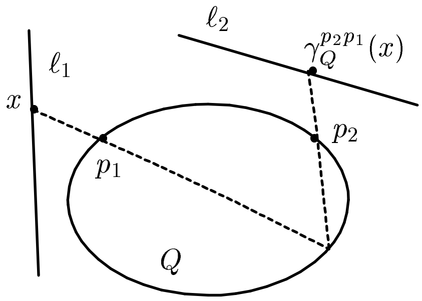

Fig. 3.1. Composing projections.

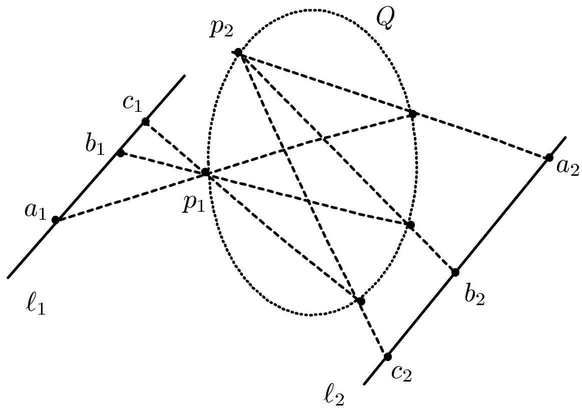

Fig. 3.2. How to find p_1, p_2.

[1] WARNING: in this lecture we always suppose that $\operatorname{char} k \neq 2$
[2] i. e. $L_1 \subset L_2 \iff L_1^\times \supset L_2^\times$
[3] for example, 3 points are collinear iff the dual 3 hyperplanes have common subspace of codimension 2

So, if ℓ_1, ℓ_2 are two lines and p_1, p_2 are two distinct points on some non singular conic Q, then the composition $\gamma_Q^{p_2 p_1} \overset{\text{def}}{=} \pi_{\ell_2}^{p_2} \circ \left(\pi_{\ell_1}^{p_1}\right)^{-1}$ gives the projective linear isomorphism $\ell_1 \xrightarrow{\sim} \ell_2$ (see fig. 3.1). In fact, any projective linear isomorphism $\ell_1 \xrightarrow{\gamma} \ell_2$ can be presented[4] as $\gamma_Q^{p_2 p_1}$ for some Q and $p_1, p_2 \in Q$. Indeed, if γ sends, say $a_1, b_1, c_1 \in \ell_1$ to $a_2, b_2, c_2 \in \ell_2$, we pick *any* p_1, p_2 such that no 3 out of 5 points p_1, p_2, $(a_1\, p_1) \cap (a_2\, p_2)$, $(b_1\, p_1) \cap (b_2\, p_2)$, $(c_1\, p_1) \cap (c_2\, p_2)$ are collinear (see fig. 3.2) and draw Q through these 5 points. Then $\gamma = \gamma_Q^{p_2 p_1}$, because both have the same action on 3 points a_1, b_1, c_1.

<u>DRAWING A CONIC BY THE RULER.</u> Let 5 distinct points p_1, p_2, a, b, c lay on a non singular conic Q. Denote the line $(a\,c)$ by ℓ_1, the line $(b\,c)$ by ℓ_2, and the intersection point $(a\,p_2) \cap (b\,p_1)$ by O (see fig. 3.3). Then the projective linear isomorphism $\ell_1 \xrightarrow{\gamma_Q^{p_2 p_1}} \ell_2$ coincides with the simple projective linear isomorphism $\ell_1 \xrightarrow{\gamma_O} \ell_2$, which takes $x \in \ell_1$ to $\gamma_O(x) = (x\,O) \cap \ell_2$ (indeed, the both send $c \longmapsto c$, $a \longmapsto d$, $e \longmapsto b$, where $d = (a\,p_2) \cap \ell_2$ and $e = (b\,p_1) \cap \ell_1$ — see fig. 3.3).

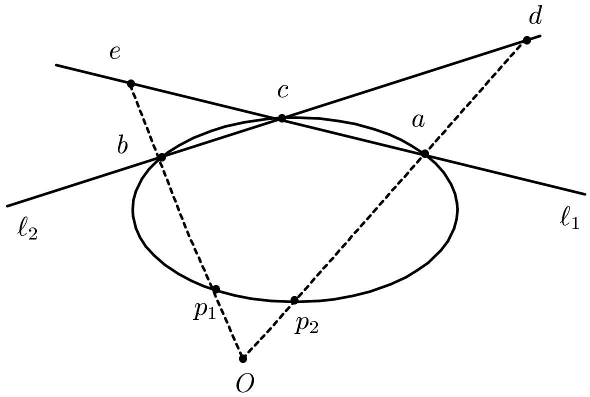

Fig. 3.3. Remarkable coincidence.

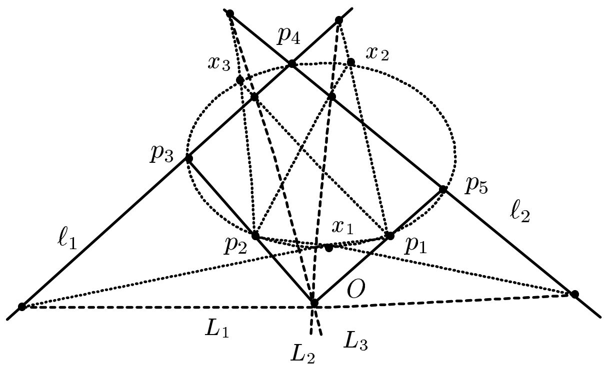

Fig. 3.4. Tracing a conic.

This simple remark allows us to trace, using only the ruler, a dense point set on the conic passing through 5 given points $p_1, p_2, \ldots, p_5$ (see fig. 3.4). Namely, let $\ell_1 = (p_3 p_4)$, $\ell_2 = (p_4 p_5)$, $O = (p_1 p_6) \cap (p_2 p_3)$. Then any line $L \ni O$ gives two intersection points $\ell_1 \cap L$ and $\ell_2 \cap L$. These points are sent to each other by the projective linear isomorphism $\gamma_O = \gamma_Q^{p_2 p_1}$. So, if we draw lines through p_1 and $\ell_1 \cap L$, and trough p_2 and $\ell_2 \cap L$, then the intersection point x of these two lines has to lay on Q. On the fig. 3.4 the points x_1, x_2, x_3 are constructed by this way starting from the lines L_1, L_2, L_3 passing through O.

<u>Proposition 3.4.</u> [5] *A hexagon $p_1, p_2, \ldots, p_6$ is inscribed into a non singular conic iff the points*[6] $(p_1 p_2) \cap (p_4 p_5)$, $(p_2 p_3) \cap (p_5 p_6)$, $(p_3 p_4) \cap (p_6 p_1)$ *are collinear (see fig. 3.5).*

Proof. Draw the conic Q through 5 of p_i except for p_4 and put $\ell_1 = (p_1 p_6)$, $\ell_2 = (p_1 p_2)$, $y = (p_5 p_6) \cap (p_2 p_3)$, $x = (p_3 p_4) \cap \ell_1$, $z = (xy) \cap \ell_2$. Then $z \in \ell_2$ is the image of $x \in \ell_1$ under the projective linear isomorphism $\gamma_y = \gamma^{p_5 p_3} : \ell_1 \xrightarrow{\sim} \ell_2$ like before. In particular, the intersection point $(p_3 p_4) \cap (p_5 z)$ lays on Q. Hence, $p_4 \in Q$ iff $p_4 = (p_3 p_4) \cap (p_5 z)$. $\square$

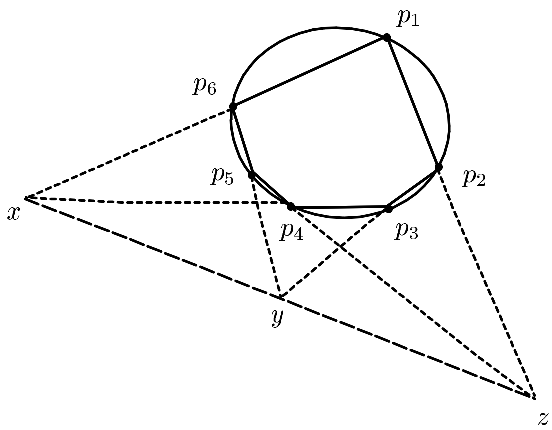

Fig. 3.5. Inscribed hexagon.

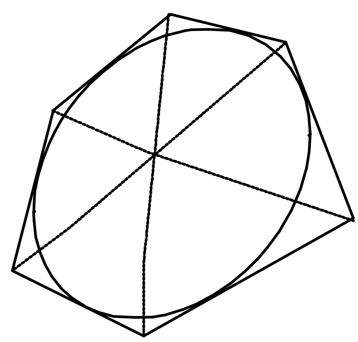

Fig. 3.6. Circumscribed hexagon.

<u>Corollary 3.5.</u> [7] *A hexagon $p_1, p_2, \ldots, p_6$ is circumscribed around a non singular conic iff its main diagonals* $(p_1 p_4)$, $(p_2 p_5)$,

[4] in infinitely many different ways
[5] It is known as *B. Pascal's theorem*
[6] i. e. 3 intersection points of the line pairs passing through the opposite sides of the hexagon
[7] It is known as *Ch. J. Brianchon's theorem*

(p_3p_6) are intersecting at one point (see fig. 3.6).

Proof. This is just the projectively dual version of the Pascal theorem. $\square$

<u>Linear isomorphisms of pencils.</u> A family of geometrical figures is referred as *a pencil*, if it is naturally parameterized by the projective line. For example, all lines passing through a given point $p \in \mathbb{P}_2$ form a pencil, because their equations run trough the line $p^\times \in \mathbb{P}_2^\times$ by projective duality. More generally, there is a pencil of hyperplanes $H \subset \mathbb{P}_n$ passing through a given subspace $L \subset \mathbb{P}_n$ of codimension 2. Such a pencil is denoted by $|h - L|$ (read: «all hyperplanes containing L») or by $L^\times \in \mathbb{P}_n^\times$. Given two such pencils, say $L_1^\times$, $L_2^\times$ and 3 points $a, b, c \in \mathbb{P}_n \setminus (L_1 \cup L_2)$ such that 3 hyperplanes from $L_i^\times$ passing through them are distinct in the both pencils, then these 3 points define a projective linear isomorphism $L_1^\times \xrightarrow{\gamma_{abc}} L_2^\times$ that sends 3 hyperplanes of the first pencil passing through a, b, c to the corresponding ones from the second pencil.

<u>Example 3.6:</u> linear identification of two pencils $p_1^\times$ and $p_2^\times$ on $\mathbb{P}_2$ is given by any 3 points a, b, c such that any 2 of them are not collinear with p_1 or p_2. It sends $(p_1a) \longmapsto (p_2a)$, $(p_1b) \longmapsto (p_2b)$, $(p_1c) \longmapsto (p_2c)$. Let Q be the (unique!) conic passing trough 5 points p_1, p_2, a, b, c. There are two different cases (see fig. 3.7–3.8).

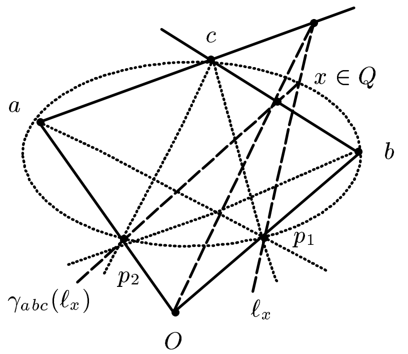

Fig. 3.7. Elliptic isomorphism of pencils.

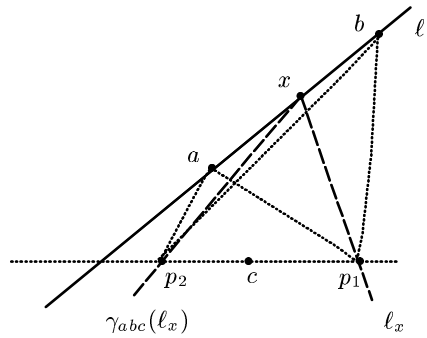

Fig. 3.8. Parabolic isomorphism of pencils.

(A) Elliptic case: Q is non singular, i. e. all 5 points are linearly general. In this case *the incidence graph*[8] of γ_{abc} coincides with Q, because the points of Q give the projective linear isomorphism $p_1^\times \longrightarrow p_2^\times$ that has the same action on a, b, c. Moreover, the above discussions let us draw the line $\gamma_{abc}(\ell_x)$ for a given $(\ell_x \in p_1^\times)$ by the ruler as follows (see fig. 3.7). First mark the point $O = (p_1b) \cap (p_2c)$; then find the intersection point $\ell_x \cap (a\,c)$, join it with O by a line, and mark the point where this line crosses $(b\,c)$; then the line $\gamma_{abc}(\ell_x)$ goes through this marked point.

(B) Parabolic case: Q is reducible, i. e. splits in two lines: (p_1p_2) and, say, $\ell = (a\,b)$. This happens when $c \in (p_1p_2)$ (recall that no 2 points from a, b, c are collinear with any of p_i). In this case the incidence graph for γ_{abc} coincides with the line ℓ (see fig. 3.8).

Dualizing these examples, we get the geometrical classification of projective linear isomorphisms $\ell_1 \xrightarrow{\sim} \ell_2$ between two given lines on $\mathbb{P}_2$.

<u>Corollary 3.7.</u> *There are exactly two types of projective linear isomorphisms $\ell_1 \xrightarrow{\sim} \ell_2$. Elliptic isomorphisms γ_Q correspond bijectively to the non singular conics Q touching both ℓ_1 and ℓ_2. Such γ_Q sends $x \longmapsto y$ iff the line $(x\,y)$ is tangent to Q (see fig. 3.9). Parabolic isomorphisms γ_L are parameterized by the points $L \in \mathbb{P}_2 \setminus (\ell_1 \cup \ell_2)$. Such γ_L sends $x \longmapsto y$ iff the line $(x\,y)$ pass through L (see fig. 3.10).* $\square$

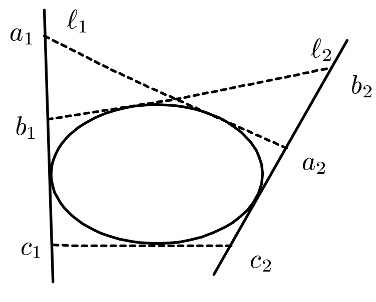

Fig. 3.9. Elliptic isomorphism of lines.

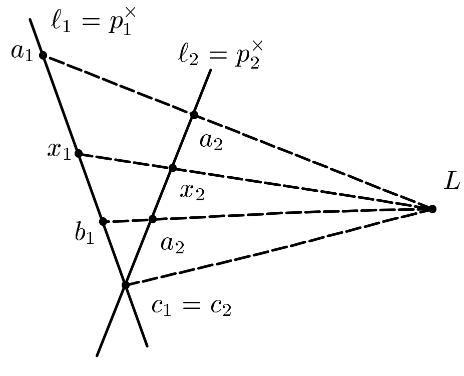

Fig. 3.10. Parabolic isomorphism of lines.

[8] i. e. a curve traced by the intersection points $\ell \cap \gamma_{abc}(\ell)$ while ℓ runs through $p_1^\times$

Towards Poncelet's porism. Given two non singular conics Q, Q', we can try to draw an n-gone simultaneously inscribed in Q' and circumscribed about Q: starting from some point $p_1 \in Q'$ draw a tangent line from p_1 to Q until it meets Q' in p_2, then draw a tangency from p_2 e. t. c. Poncelet's theorem says that if this procedure comes back to $p_n = p_1$ after n steps, then the same holds for any choice of the starting point p_1 maybe except for some finite set. The next two corollaries explain Poncelet porism for triangles (i. e. for $n = 3$)

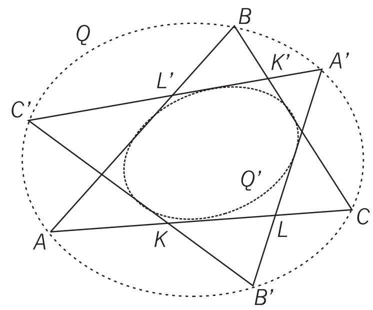

Fig. 3.11. Inscribed-circumscribed triangles.

Corollary 3.8. *Two triangles ABC and $A'B'C'$ are both inscribed into the same conic Q iff they are both circumscribed around the same conic Q'.*

Proof. We check only «$\Rightarrow$» implication, then the opposite implication comes by projective duality. Consider two lines $\ell = (AB)$, $\ell' = (A'B')$ and elliptic projective linear isomorphism $\ell \xrightarrow{\gamma} \ell'$ composed as the projection of ℓ onto Q from B' followed by the projection of Q onto ℓ' from B (see fig. 3.11). Since it takes $A \mapsto L'$, $C \mapsto K'$, $K \mapsto C'$, $L \mapsto A'$, all the sides of the both triangles should touch the conic associated with γ via Corollary 3.7. □

Corollary 3.9. *Given two conics Q, Q' such that there exists a triangle ABC inscribed into Q and circumscribed around Q', then any point $A' \in Q$ is a vertex of a triangle $A'B'C'$ inscribed into Q and circumscribed around Q'.*

Proof. Take any $A' \in Q$ and pick $B', C' \in Q$ such that the lines $(A'B')$, $(A'C')$ are tangent to Q' (see fig. 3.11 again). By the previous corollary, both ABC and $A'B'C'$ are circumscribed around some conic, which must coincide with Q', because there exist a unique conic touching 5 lines (AB), (BC), (CA), $(A'B')$, $(A'C')$. □

Exercice 3.10. Make Corollary 3.2 more precise by finding necessary and sufficient condition on 5 lines in $\mathbb{P}_2$ for existence of a unique non singular conic touching all of them.

Hint. This is projectively dual to Proposition 2.16

Making a projective isomorphism by the ruler. If an isomorphism $\ell_1 \xrightarrow{\gamma} \ell_2$ is given by its action on some 3 points, say: $a_1 \longmapsto a_2$, $b_1 \longmapsto b_2$, $c_1 \longmapsto c_2$, then we can find the image $\gamma(x)$ of any $x \in \ell_1$ by the ruler. In parabolic case this is trivial (see fig. 3.10). In elliptic case the drawing algorithm is projectively dual to the one discussed in Example 3.6 (A). Namely, draw the line $O^\times = (b_1 a_2)$; then pass the line through x and $(a_1 a_2) \cap (c_1 c_2)$ and mark its intersection point with $O^\times$; now $\gamma(x)$ is the intersection of ℓ_2 with the line passing through the last marked point and $(b_1 b_2) \cap (c_1 c_2)$ (compare fig. 3.7 and fig. 3.12).

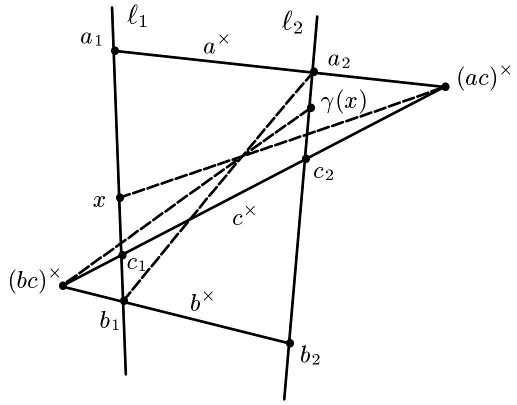

Fig. 3.12. Finding $\gamma_Q(x)$.

Exercice 3.11. Let $Q \subset \mathbb{P}_2$ be non singular conic considered together with some rational parameterization $\mathbb{P}_1 \xrightarrow{\sim} Q$. Show that for any two points $p_1, p_2 \in Q$ and a line $\ell \subset \mathbb{P}_2$ a map $Q \xrightarrow{\gamma} Q$ given by prescription: $x \overset{\gamma}{\longmapsto} y \iff \pi_\ell^{p_1} x = \pi_\ell^{p_2} y$ is induced by some linear automorphism of $\mathbb{P}_1$ (i. e. by some linear fractional reparameterization). Find the images of p_1, p_2 and the fixed points of the above map. Show that any bijection $Q \xrightarrow{\sim} Q$ induced by a linear automorphism of $\mathbb{P}_1$ can be (not uniquely) realized geometrically by two points $p_1, p_2 \in Q$ and a line $\ell \subset \mathbb{P}_2$ in the way described above. Is it possible, using only the ruler, to find (some) p_1, p_2, ℓ for a bijection $Q \xrightarrow{\sim} Q$ given by its action on 3 points $a, b, c, \in Q$?

Hint. Try $p_2 = a$.

Exercice 3.12*. Given a non singular conic Q and three points A, B, C, draw (using only the ruler) a triangle inscribed in Q with sides passing through A, B, C. How many solutions may have this problem?

Hint. Start «naive» drawing from any $p \in Q$ and denote by $\gamma(p)$ your return point after passing trough A, B, C. Is $p \longmapsto \gamma(p)$ a projective isomorphism of kind described in Exercice 3.11?

Exercice 3.13*. Formulate and solve projectively dual problem to the previous one.

Tensor Guide

<u>Multilinear maps.</u> Let $V_1, V_2, \dots, V_n$ and W be vector spaces of dimensions $d_1, d_2, \dots, d_n$ and m over an arbitrary field k with $\operatorname{char} k \neq 2$. A map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W$ is called *multilinear*, if in each argument

$$\varphi(\dots, \lambda v' + \mu v'', \dots) = \lambda\, \varphi(\dots, v', \dots) + \mu\, \varphi(\dots, v'', \dots)$$

when all the other remain to be fixed. The multilinear maps $V_1 \times V_2 \times \cdots \times V_n \longrightarrow W$ form a vector space of dimension $m \cdot \prod d_\nu$. Namely, if we fix a base $\{e_1^{(i)}, e_2^{(i)}, \dots, e_{d_i}^{(i)}\}$ for each V_i and a base $\{e_1, e_2, \dots, e_m\}$ for W, then any multilinear map φ is uniquely defined by its values at all combinations of the base vectors:

$$\varphi(\, e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \dots, e_{\alpha_n}^{(n)}\,) = \sum_\nu a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)} \cdot e^\nu \ \in W$$

As soon as $m \cdot \prod d_\nu$ numbers $a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)} \in k$ are given, the map φ is well defined by the multilinearity. It sends a collection of vectors $(v_1, v_2, \dots, v_n)$, where $v_i = \sum\limits_{\alpha_i=1}^{d_i} x_{\alpha_i}^{(i)} e_{\alpha_i}^{(i)} \in V_i$ for $1 \leqslant i \leqslant n$, to

$$\varphi(v_1, v_2, \dots, v_n) = \sum_{j=1}^{m} \Big(\sum_{\alpha_1, \alpha_2, \dots, \alpha_n} a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)} \cdot x_{\alpha_1}^{(1)} \cdot x_{\alpha_2}^{(2)} \cdot \ \cdots \ \cdot x_{\alpha_n}^{(n)} \Big) \cdot e^\nu \ \in W$$

the numbers $a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)}$ can be considered as elements of some «$(n+1)$-dimensional format matrix of the size $m \times d_1 \times d_2 \times \cdots \times d_n$», if you can imagine such a thing[1].

Exercice 4.1. Check that a collection $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \cdots \times V_n$ doesn't contain zero vector iff there exists a multilinear map φ (to somewhere) such that $\varphi(v_1, v_2, \dots, v_n) \neq 0$.

Exercice 4.2. Check that a multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} U$ followed by *a linear operator* $U \xrightarrow{F} W$ is the multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{F \circ \varphi} W$ as well.

<u>Tensor product of vector spaces.</u> Let $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} U$ be a fixed multilinear map. Then for any vector space W we have a composition operator

$$\begin{pmatrix} \text{the space } \operatorname{Hom}(U, W) \text{ of all} \\ \text{linear operators } U \xrightarrow{F} W \end{pmatrix} \xrightarrow{F \longmapsto F \circ \tau} \begin{pmatrix} \text{the space of all multilinear maps} \\ V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W \end{pmatrix} \tag{5}$$

A multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} U$ is called *universal* if the composition operator (5) is an isomorphism for any vector space W. In other words, the multilinear map τ is universal, if for any W and any multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W$ there exist a unique linear operator $U \xrightarrow{F} W$ such that $\varphi = F \circ \tau$, i. e. the commutative diagram

$$\begin{tikzcd}
 & U \arrow[dd, dashed, "F"] \\
V_1 \times V_2 \times \cdots \times V_n \arrow[ur, "\tau"] \arrow[dr, "\varphi"'] & \\
 & W
\end{tikzcd}$$

can be always closed by a unique linear dotted row.

<u>Claim 4.3</u>. *Let $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau_1} U_1$ и $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau_2} U_2$ be two universal multilinear maps. Then there exists a unique linear isomorphism $U_1 \xrightarrow{\iota} U_2$ such that $\tau_2 = \iota\tau_1$.*

[1] usual $d \times m$ - matrices, which present the *linear* maps $V \longrightarrow W$, have just 2-dimensional format

Proof. Since both U_1, U_2 are universal, there are unique linear operators $U_1 \xrightarrow{F_{21}} U_2$ and $U_2 \xrightarrow{F_{12}} U_1$ mounted in the diagrams

$$\begin{tikzcd}
U_1 \arrow[rr, equal, "\mathrm{Id}_{U_1}"] & & U_1 \arrow[dd, dashed, "F_{21}"] \\
& V_1 \times V_2 \times \cdots \times V_n \arrow[ul, "\tau_1"'] \arrow[ur, "\tau_1"] \arrow[dl, "\tau_2"] \arrow[dr, "\tau_2"'] & \\
U_2 \arrow[uu, dashed, "F_{12}"] \arrow[rr, equal, "\mathrm{Id}_{U_2}"'] & & U_2
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
& & U_2 \arrow[dl, dotted, "F_{12}"'] \arrow[dd, equal, "\mathrm{Id}_{U_2}"] \\
V_1 \times V_2 \times \cdots \times V_n \arrow[urr, "\tau_2"] \arrow[r, "\tau_1"] \arrow[drr, "\tau_2"'] & U_1 \arrow[dr, dotted, "F_{21}"] & \\
& & U_2
\end{tikzcd}$$

So, the composition $F_{21}F_{12} = \mathrm{Id}_{U_2}$, because of the uniqueness property in the universality of U_2. Similarly, $F_{12}F_{21} = \mathrm{Id}_{U_1}$. □

Claim 4.4. *Let $\{e_1^{(i)}, e_2^{(i)}, \dots, e_{d_i}^{(i)}\} \in V_i$ be a base (for $1 \leqslant i \leqslant n$). Denote by $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ a $(\prod d_i)$ - dimensional vector space whose base vectors are the symbols*

$$e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \ldots \otimes e_{\alpha_n}^{(n)}\,, \quad 1 \leqslant \alpha_i \leqslant d_i \tag{6}$$

(all possible formal «tensor products» of base vectors $e_\nu^{(\mu)}$). Then the multilinear map

$$V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} V_1 \otimes V_2 \otimes \cdots \otimes V_n$$

which sends a base vector collection $(e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_n}) \in V_1 \times V_2 \times \cdots \times V_n$ to the corresponding base vector (6) is universal.

Proof. Let $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W$ be a multilinear map and $V_1 \otimes V_2 \otimes \cdots \otimes V_n \xrightarrow{F} W$ be a linear operator. Comparing the values at the base vectors, we see that

$$\varphi = F \circ \tau \iff F(\,e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \ldots \otimes e_{\alpha_n}^{(n)}\,) = \varphi(e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \ldots, e_{\alpha_n}^{(n)})\,.$$

□

<u>THE SEGRE EMBEDDING.</u> The vector space $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is called *a tensor product* of $V_1, V_2, \dots, V_n$. The universal multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is called *a tensor multiplication*. For a collection of vectors $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \cdots \times V_n$ the image $\tau(v_1, v_2, \dots, v_n)$ is denoted by $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ and called *a tensor product* of these vectors. All such products are called *decomposable tensors*. Of course, not all the vectors of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ are decomposable and $\operatorname{im} \tau$ isn't a vector subspace in $V_1 \otimes V_2 \otimes \cdots \otimes V_n$, because τ is *multi*linear but not linear. However, the linear span of decomposable tensors exhausts the whole of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$. Geometrically, the tensor multiplication gives a map

$$\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \cdots \times \mathbb{P}(V_n) \xhookrightarrow{s} \mathbb{P}(V_1 \otimes V_2 \otimes \cdots \otimes V_n)$$

called *a Segre embedding*. If $d_i = \dim V_i = m_i + 1$, then the Segre embedding is a bijection between $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2} \times \cdots \times \mathbb{P}_{m_n}$ and *a Segre subvariety* which consist of all decomposable tensors considered up to proportionality. The Segre subvariety has dimension $(\sum m_i)$, doesn't belong to any hyperplane, is ruled by n families of linear subspaces, and lives in $\mathbb{P}_N$ with $N = -1 + \prod(1 + m_i)$.

Example 4.5: the Segre embedding $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2} \hookrightarrow \mathbb{P}_{m_1+m_2+m_1m_2}$ sends $x = (x_0 : x_1 : \ldots : x_{m_1}) \in \mathbb{P}_{m_1}$ and $y = (y_0 : y_1 : \ldots : y_{m_2}) \in \mathbb{P}_{m_2}$ to the point $s(x, y) \in \mathbb{P}_{m_1+m_2+m_1m_2}$ whose $(1+m_1)(1+m_2)$ homogeneous coordinates are all possible products $x_i y_j$ with $0 \leqslant i \leqslant m_1$ and $0 \leqslant j \leqslant m_2$. To visualize this thing, take $\mathbb{P}_{m_1} = \mathbb{P}(V^*)$, $\mathbb{P}_{m_2} = \mathbb{P}(W)$, and $\mathbb{P}_{m_1+m_2+m_1m_2} = \mathbb{P}(\mathrm{Hom}(V, W))$, where $\mathrm{Hom}(V, W)$ is the space of all linear maps. Then the Segre map sends a pair $(\xi, w) \in V^* \times W$ to the linear map $\xi \otimes w$, which acts by the rule $v \longmapsto \xi(v) \cdot w$.

Exercice 4.6. Check that a map $V^* \times W \longrightarrow \mathrm{Hom}(V, W)$ which sends (ξ, w) to the operator $v \longmapsto \xi(v) \cdot w$ is *the universal bilinear map* (so, there is a canonical isomorphism $V^* \otimes W \simeq \mathrm{Hom}(V, W)$)

Exercice 4.7. Check that for $\xi = (x_0, x_1, \ldots, x_{m_1}) \in V^*$ and $w = (y_0, y_1, \ldots, y_{m_2}) \in W$ operator $\xi \otimes w$ has the matrix $a_{ij} = x_j y_i$.

Since any operator $\xi \otimes w$ has 1-dimensional image, the corresponding matrix has rank 1. On the other side, any rank 1 matrix has proportional columns. Hence, the corresponding operator has 1-dimensional image, say spaned by $w \in W$, and takes $v \longmapsto \xi(v)\, w$, where the coefficient $\xi(v) \in k$ depends on v linearly. So, the image of the Segre embedding consists of all rank 1 operators up to proportionality. In particular, it may be presented as the intersection of quadrics $\det \begin{pmatrix} a_{ij} & a_{ik} \\ a_{\ell j} & a_{\ell k} \end{pmatrix} = a_{ij} a_{\ell k} - a_{ik} a_{\ell j} = 0$ (i. e. by vanishing all 2×2 - minors for the matrix $(a_{\mu\nu})$).

TENSOR ALGEBRA OF A VECTOR SPACE. If $V_1 = V_2 = \cdots = V_n = V$, then $V^{\otimes n} \stackrel{\text{def}}{=} \underbrace{V \otimes V \otimes \cdots \otimes V}_{n}$ is called *an n-th tensor power* of V. All tensor powers are combined in the infinite dimensional non commutative graded algebra $\mathsf{T}^\bullet V = \bigoplus_{n\geqslant 0} V^{\otimes n}$, where $V^{\otimes 0} \stackrel{\text{def}}{=} k$.

Exercice 4.8. Using the universality, show that there are canonical isomorphisms

$$\left(V^{\otimes n_1} \otimes V^{\otimes n_2}\right) \otimes V^{\otimes n_3} \simeq V^{\otimes n_1} \otimes \left(V^{\otimes n_2} \otimes V^{\otimes n_3}\right) \simeq V^{\otimes (n_1+n_2+n_3)}$$

which make the vector's tensoring to be well defined associative multiplication on $\mathsf{T}^\bullet V$.

Algebraically, $\mathsf{T}^\bullet V$ is what is called «a free associative k-algebra generated by V» ([2]). Practically, this means that as soon as a basis $\{e_1, e_2, \dots, e_d\} \subset V$ is fixed, then $\mathsf{T}^\bullet V$ turns into the space of the formal finite linear combinations of words which consist of the letters e_i separated by $\otimes$. The words are multiplied by writing after one other consequently. This multiplication is extended on the linear combinations of the words by the usual distributivity rules.

DUALITY. The spaces $V^{\otimes n} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n}$ and $V^{*\otimes n} = \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{n}$ are canonically dual to each other. The pairing between $v = v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n}$ and $\xi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in V^{*\otimes n}$ is given by *a full contraction*

$$\langle\, v \,,\, \xi \,\rangle \stackrel{\text{def}}{=} \prod_{i=1}^{n} \xi_i(v_i)\ . \tag{7}$$

Let $\{e_1, e_2, \dots, e_n\} \subset V$ and $\{\xi_1, \xi_2, \dots, \xi_n\} \subset V^*$ be dual bases, i. e. $\xi_i(e_j) = 0$ for $i \neq j$ and $\xi_i(e_i) = 1$. Then the base words $\{\, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \,\}$ and $\{\, \xi_{j_1} \otimes \xi_{j_2} \otimes \cdots \otimes \xi_{j_s} \,\}$ form the dual bases for $\mathsf{T}^\bullet V$ and $\mathsf{T}^\bullet V^*$ with respect to the full contraction. So, $V^{\otimes n *} \simeq V^{*\otimes n}$. On the other side, the space $(V^{\otimes n})^*$ is naturally identified with the space of all multilinear forms $\underbrace{V \times V \times \cdots \times V}_{n} \longrightarrow k$, because $V^{\otimes n}$ is universal. So, there exists a canonical isomorphism between $V^{*\otimes n} = \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{n}$ and the space of multilinear forms in n arguments from V. It sends a tensor $\xi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in V^{*\otimes n}$ to the form $(v_1, v_2, \dots, v_n) \longmapsto \prod_{i=1}^{n} \xi_i(v_i)$.

PARTIAL CONTRACTIONS. Let $\{1, 2, \dots, p\} \xleftarrow{I} \{1, 2, \dots, m\} \xrightarrow{J} \{1, 2, \dots, q\}$ be two injective (not necessary monotonous) maps. We write i_ν and j_ν for $I(\nu)$ and $J(\nu)$ respectively and consider $I = (i_1, i_2, \dots, i_m)$ and $J = (j_1, j_2, \dots, j_m)$ as two ordered (but not necessary monotonous) index collections of the same cardinality. A linear operator

$$\underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{p} \otimes \underbrace{V \otimes V \otimes \cdots \otimes V}_{q} \xrightarrow{c^I_J} \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{p-m} \otimes \underbrace{V \otimes V \otimes \cdots \otimes V}_{q-m}$$

which sends $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_p \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_q$ to $\prod_{\nu=1}^{m} \xi_{i_\nu}(v_{j_\nu}) \cdot \bigotimes_{i \notin \operatorname{im}(I)} \xi_i \otimes \bigotimes_{j \notin \operatorname{im}(J)} v_j$ is called *a partial contraction* in the indexes I and J.

Example 4.9: the contraction between a vector and a multilinear form. Consider a multilinear form $\varphi(v_1, v_2, \dots, v_n)$ as the tensor from $V^{*\otimes n}$ and contact it in the first index with a vector $v \in V$. The result belongs to $V^{*\otimes(n-1)}$ and gives a multilinear form in $(n-1)$ arguments. This form is denoted by $i_v\varphi$ and called an *inner product* of v and φ.

Exercice 4.10. Check that $i_v\varphi(w_1, w_2, \dots, w_{n-1}) = \varphi(v,\, w_1, w_2, \dots, w_{n-1}\,)$, i. e. the inner multiplication by v is just the fixation of v in the first argument.

LINEAR SPAN OF A TENSOR. Let $U, W \subset V$ be any two subspaces. Writing down the standard monomial bases, we see immediately that $(U \cap W)^{\otimes n} = U^{\otimes n} \cap W^{\otimes n}$ in $V^{\otimes n}$. So, for any $t \in V^{\otimes n}$ there is a *minimal* subspace *span*$(t) \subset V$ whose n-th tensor power contains t. It is called *a linear span* of t and coincides with the intersection of all $W \subset V$ such that $t \in W^{\otimes n}$. To describe *span*(t) more constructively, for any injective (not necessary monotonous) map

$$J = (j_1, j_2, \dots, j_{n-1}) \,:\, \{1,\, 2,\, \dots\,,\, (n-1)\} \hookrightarrow \{1,\, 2,\, \dots\,,\, n\}$$

[2] If you like it, make the following formal exercise: deduce from the universality that for any associative k-algebra A and vector space map $V \xrightarrow{f} A$ there exists a unique algebra homomorphism $\mathsf{T}^\bullet V \xrightarrow{\alpha} A$ such that $\alpha|_V = f$

consider a linear map $V^{*\otimes(n-1)} \xrightarrow{c_t^J} V$ defined by complete contraction with t: it sends a decomposable tensor $\varphi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_{n-1}$ to a vector obtained by coupling ν-th factor ξ_ν, of φ, with j_ν-th factor of t for all $1 \leqslant \nu \leqslant (n-1)$, i.e.

$$c_t^J(\varphi) = c^{(1,2,\dots,(n-1))}_{(j_1,j_2,\dots,j_{n-1})}(\varphi \otimes t)\ .$$

Claim 4.11. *$span(t) \subset V$ is spanned by the images $c_t^J\left(V^{*\otimes(n-1)}\right) \subset V$ taken for all possible J.*

Proof. Let $span(t) = W \subset V$. Then $t \in W^{\otimes n}$ and $\operatorname{im}\left(c_t^J\right) \subset W\ \forall J$. It remains to prove that W is annihilated by any linear form $\xi \in V^*$ which annihilate all the subspaces $\operatorname{im}\left(c_t^J\right)$. Suppose the contrary: let $\xi \in V^*$ have non zero restriction on W but annihilate all $c_t^J\left(V^{*\otimes(n-1)}\right)$. Then there exist a base $\{w_1, w_2, \dots, w_k\}$ for W and a base $\{\xi_1, \xi_2, \dots, \xi_d\}$ for V^* such that: $\xi_1 = \xi$, the restrictions of $\xi_1, \xi_2, \dots, \xi_k$ onto W form the base of W^* dual to $\{w_\nu\}$, and $\xi_{k+1}, \dots, \xi_d$ annihilate W. Now, for any J and $\xi_{i1}, \xi_{i2}, \dots, \xi_{in-1}$ we have

$$0 = \left\langle\, \xi\,,\, c_t^J\left(\xi_{i1} \otimes \xi_{i2} \otimes \cdots \otimes \xi_{in-1}\right)\right\rangle = \left\langle\, \xi_{i_{j_1^{-1}}} \otimes \cdots \otimes \xi_{i_{j_{s-1}^{-1}}} \otimes \xi \otimes \xi_{i_{j_{s+1}^{-1}}} \otimes \cdots \otimes \xi_{i_{j_n^{-1}}}\,,\, t\right\rangle \tag{8}$$

where $s = \{1, 2, \dots, n\} \setminus \operatorname{im}(J)$ and $J^{-1} = (j_1^{-1}, j_2^{-1}, \dots, j_n^{-1})$ is the inverse to J map from $\operatorname{im}(J) \subset \{1, 2, \dots, n\}$ to $\{1, 2, \dots, (n-1)\}$. Note that each basic monomial of $V^{*(n-1)}$ containing as a factor $\xi_1 = \xi$ can appear as the first operand in the right side of (8). But if we expand t trough the basic monomials $w_{i1} \otimes w_{i2} \otimes \cdots \otimes w_{in}$, then the coefficients of this expansion can be computed as full contractions of t with the corresponding elements $\xi_{i1} \otimes \xi_{i2} \otimes \cdots \otimes \xi_{in}$ from the dual base for $W^{*\otimes n}$. By (8), such a contraction equals zero as soon one of ξ_{i_α} equals $\xi_1 = \xi$, which is dual to w_1. So, $span(t)$ is contained in the linear span of $w_2, \dots, w_k$ but this contradicts our assumption. ☐

Symmetry properties. A multilinear map $\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\varphi} W$ is called *symmetric* if it doesn't change its value under any permutations of the arguments. If the value of φ is stable under the even permutations and changes the sign under the odd ones, then φ is called *skew symmetric*. Since the composition operator (5) preserves the symmetry properties, for the (skew)symmetric φ the composition operator (5) turns into

$$\begin{pmatrix}\text{the space } \operatorname{Hom}(U,W) \text{ of all}\\ \text{linear operators } U \xrightarrow{F} W\end{pmatrix} \xrightarrow{F \longmapsto F\circ\varphi} \begin{pmatrix}\text{the space of all (skew)symmetric}\\ \text{multilinear maps } \underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\psi} W\end{pmatrix} \tag{9}$$

A (skew)symmetric multilinear map $\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\varphi} U$ is called *universal* if (9) is an isomorphism for any W. In the symmetric case the universal target space is denoted by S^nV and called n-th *symmetric power* of V. In the skew symmetric case it is called n-th *exterior power* of V and denoted by Λ^nV.

Exercice 4.12. Show that, if exist, S^nV and Λ^nV are unique up to unique isomorphism commuting with the universal maps.

Symmetric algebra $S^\bullet V$ of V is a factor algebra of the free associative algebra $\mathsf{T}^\bullet V$ by a commutation relations $vw = wv$. More precisely, denote by $\mathcal{I}_{\text{sym}} \subset \mathsf{T}^\bullet V$ a linear span of all tensors

$$\cdots \otimes v \otimes w \otimes \cdots \quad - \quad \cdots \otimes w \otimes v \otimes \cdots\ ,$$

where the both terms are decomposable, have the same degree, and differ only in order of v, w. Clearly, $\mathcal{I}_{\text{sym}}$ is a double-sided ideal in $\mathsf{T}^\bullet V$ generated by a linear span of all the differences $v \otimes w - w \otimes v \in V \otimes V$. The factor algebra $S^\bullet V \stackrel{\text{def}}{=} \mathsf{T}^\bullet V/\mathcal{I}_{\text{sym}}$ is called *a symmetric algebra* of the vector space V. By the construction, it is commutative[3]. Since $\mathcal{I}_{\text{sym}} = \bigoplus_{n\geqslant 0}(\mathcal{I}_{\text{sym}} \cap V^{\otimes n})$ is the direct sum of its homogeneous components, the symmetric algebra is graded: $S^\bullet V = \bigoplus_{n\geqslant 0} S^nV$, where $S^nV \stackrel{\text{def}}{=} V^{\otimes n}/(\mathcal{I}_{\text{sym}} \cap V^{\otimes n})$.

Claim 4.13. *The tensor multiplication followed by the factorization map:*

$$\underbrace{V \times V \times \cdots \times V}_{n} \overset{\tau}{\dashrightarrow} V^{\otimes n} \xrightarrow{\pi} S^n(V) \tag{10}$$

gives the universal symmetric multilinear map.

[3] Again, if you like it, prove that for any commutative k-algebra A and a vector space map $V \xrightarrow{f} A$ there exists a unique homomorphism of commutative algebras $S^\bullet V \xrightarrow{\alpha} A$ such that $\alpha|_V = f$

Proof. *Any* multilinear map $V\times V\times\cdots\times V \xrightarrow{\varphi} W$ is uniquely decomposed as $\varphi = F\circ\tau$, where $V^{\otimes n}\xrightarrow{F} W$ is linear. F is factored through π iff $F(\cdots\otimes v\otimes w\otimes\cdots) = F(\cdots\otimes w\otimes v\otimes\cdots)$, i. e. iff $\varphi(\dots,v,w,\dots) = \varphi(\dots,w,v,\dots)$ □

The graded components S^nV are called *symmetric powers* of V and the map (10) is called *a symmetric multiplication*. If a base $\{e_1,e_2,\dots,e_d\}\subset V$ is fixed, then S^nV is naturally identified with the space of all homogeneous polynomials of degree n in e_i. Namely, consider the polynomial ring $k[e_1,e_2,\dots,e_d]$ (whose «variables» are the base vectors e_i) and identify V with the space of all linear homogeneous polynomials in e_i.

Exercice 4.14. Check that the multiplication map

$$\underbrace{V\times V\times\cdots\times V}_{n}\xrightarrow{(\ell_1,\ell_2,\dots,\ell_n)\longmapsto\prod_{\nu=1}^{n}\ell_\nu}\begin{pmatrix}\text{the homogeneous polynomials}\\ \text{of degree } n \text{ in } e_i\end{pmatrix}$$

is universal and show that $\dim S^nV = \binom{d+n-1}{n}$.

<u>Exterior algebra $\Lambda^\bullet V$ of V</u> is a factor algebra of the free associative algebra $\mathsf{T}^\bullet V$ by a skew commutation relations $vw = -wv$. More precisely, consider a double-sided ideal $\mathcal{I}_{\text{skew}}\subset\mathsf{T}^\bullet V$ generated by all sums $v\otimes w + w\otimes v\in V\otimes V$ and put $\Lambda^\bullet V \overset{\text{def}}{=} \mathsf{T}^\bullet V/\mathcal{I}_{\text{skew}}$. Exactly as in the symmetric case, the ideal $\mathcal{I}_{\text{skew}}$ is *homogeneous*: $\mathcal{I}_{\text{skew}} = \bigoplus_{n\geqslant 0}(\mathcal{I}_{\text{skew}}\cap V^{\otimes n})$, where $(\mathcal{I}_{\text{skew}}\cap V^{\otimes n})$ is the linear span of all sums

$$\cdots\otimes v\otimes w\otimes\cdots \quad + \quad \cdots\otimes w\otimes v\otimes\cdots$$

(the both items have degree n and differ only in the order of v,w). So, the factor algebra $\Lambda^\bullet V$ is graded by the subspaces $\Lambda^nV \overset{\text{def}}{=} V^{\otimes n}/(\mathcal{I}_{\text{skew}}\cap V^{\otimes n})$.

Exercice 4.15. Prove that the tensor multiplication followed by the factorization

$$\underbrace{V\times V\times\cdots\times V}_{n}\xrightarrow{\tau} V^{\otimes n}\xrightarrow{\pi}\Lambda^n(V) \tag{11}$$

gives the universal skew symmetric multilinear map.

The map (11) is called *an exterior* or *skew* multiplication. The skew product of vectors $(v_1,v_2,\dots,v_n)$ is denoted by $v_1\wedge v_2\wedge\cdots\wedge v_n$. By the construction, it changes the sign under the transposition of any two consequent terms. So, under any permutation of terms the skew product is multiplied by the sign of the permutation.

Exercice 4.16. For any $U,W\subset V$ check that $S^nU\cap S^nW = S^n(U\cap U)$ in S^nV and $\Lambda^nU\cap\Lambda^nW = \Lambda^n(U\cap U)$ in Λ^nV.

<u>Grassmannian polynomials.</u> Let $\{e_1,e_2,\dots,e_d\}\subset V$ be a basis. Then the exterior algebra $\Lambda^\bullet V$ is identified with *a grassmannian polynomial ring* $k\langle e_1,e_2,\dots,e_d\rangle$ whose «variables» are the base vectors e_i which *skew commute*, that is, $e_i\wedge e_j = -e_j\wedge e_i$ for all i,j. More precisely, it is linearly spanned by the grassmannian monomials $e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_n}$. It follows from skew commutativity that $e_i\wedge e_i = 0$ for all i, that is, a grassmannian monomial vanishes as soon as it becomes of degree more then 1 in some e_i. So, any grassmannian monomial has a unique representation $e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_n}$ with $1\leqslant i_1<i_2<\cdots<i_n\leqslant d$.

<u>Claim 4.17</u>. *The monomials $e_I \overset{\text{def}}{=} e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_n}$, where $I = (i_1,i_2,\dots,i_n)$ runs through the increasing n-element subsets in $\{1,2,\dots,d\}$, form a base for Λ^nV. In particular, $\Lambda^nV = 0$ for $n>\dim V$, $\dim\Lambda^nV = \binom{d}{n}$, and $\dim k\langle e_1,e_2,\dots,e_d\rangle = 2^d$.*

Proof. Consider $\binom{d}{n}$-dimensional vector space U whose base consists of the symbols ξ_I, where $I = (i_1,i_2,\dots,i_n)$ runs through the increasing n-element subsets in $\{1,2,\dots,d\}$. Define a skew symmetric multilinear map

$$V_1\times V_2\times\cdots\times V_n\xrightarrow{\alpha} U\ :\ (e_{j_1},e_{j_2},\dots,e_{j_n})\longmapsto\operatorname{sgn}(\sigma)\cdot\xi_I\,,$$

where $I = (j_{\sigma(1)},j_{\sigma(2)},\dots,j_{\sigma(n)},)$ is an *increasing* collection obtained from $(j_1,j_2,\dots,j_n)$ by a (unique) permutation σ. This map is universal. Indeed, for any skew symmetric multilinear map $\underbrace{V\times V\times\cdots\times V}_{n}\xrightarrow{\varphi} W$ there exists at most one linear operator $U\xrightarrow{F} W$ such that $\varphi = F\circ\alpha$, because it has to act on the base as $F(\xi_I) = \varphi(e_{i_1},e_{i_2},\dots,e_{i_n}))$ for all increasing $I = (i_1,i_2,\dots,i_n)$. On the other side, such F really decomposes φ, because $F(\alpha(e_{j_1},e_{j_2},\dots,e_{j_n})) = \varphi(e_{j_1},e_{j_2},\dots,e_{j_n}))$ for all *not increasing* base collections $(e_{j_1},e_{j_2},\dots,e_{j_n})\subset\underbrace{V\times V\times\cdots\times V}_{n}$ as well. By the universality, there exists a canonical isomorphism between U and Λ^nV which sends ξ_I to $e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_n} = e_I$. □

Exercice 4.18. Check that $f(e)\wedge g(e) = (-1)^{\deg(f)\cdot\deg(g)}\,g(e)\wedge f(e)$ for all homogeneous $f(e),g(e)\in k\langle e_1,e_2,\dots,e_d\rangle$. In particular, each *even degree* homogeneous polynomial commutes with any grassmannian polynomial.

Exercice 4.19. Describe the center of $k\langle e_1, e_2, \dots, e_d\rangle$, i. e. all grassmannian polynomials which commute with everything.

Example 4.20: linear base change in grassmannian polynomial. Under the linear substitution $e_i = \sum\limits_{j=1}^{d} a_{ij}\,\xi_j$ the base monomials e_I are changed by the new base monomials ξ_I as follows:

$$e_I = e_{i_1}\wedge e_{i_2}\wedge\,\cdots\,\wedge e_{i_n} = \Bigl(\sum_{j_1} a_{i_1 j_1}\,\xi_j\Bigr)\wedge\Bigl(\sum_{j_2} a_{i_2 j_2}\,\xi_j\Bigr)\wedge\,\cdots\,\wedge\Bigl(\sum_{j_n} a_{i_n j_n}\,\xi_j\Bigr) =$$

$$= \sum_{1\leqslant j_1<j_2<\cdots<j_n\leqslant n}\ \sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\,a_{i_1 j_{\sigma(1)}}a_{i_2 j_{\sigma(2)}}\cdots a_{i_n j_{\sigma(n)}}\,\xi_{j_1}\wedge\xi_{j_2}\wedge\,\cdots\,\wedge\xi_{j_n} = \sum_J a_{IJ}\,\xi_J\ ,$$

where a_{IJ} is $(n\times n)$-minor of (a_{ij}) placed at $(i_1, i_2, \dots, i_n)$ rows and $(j_1, j_2, \dots, j_n)$ columns, and J runs through all increasing index collections of *the length* $\#J = n$.

Exercice 4.21. Let $|I| = \sum\limits_{\nu} i_\nu$ denote *a weight* of the increasing index collection $I = (i_1, i_2, \dots, i_n)$ of length $\#I = n$. Check that

$$e_I\wedge e_{\widehat I} = (-1)^{|I|+\frac{1}{2}\#I(1+\#I)}\cdot e_1\wedge e_2\wedge\,\cdots\,\wedge e_d \tag{12}$$

for any two *complementary* index collections I and $\widehat I \overset{\text{def}}{=} \{1, 2, \dots, n\}\setminus I$.

Example 4.22: Laplace relations via grassmannian polynomials. Let us take two complementary index collections I and $\widehat I \overset{\text{def}}{=} \{1, 2, \dots, n\}\setminus I$ and do a base change $e_i = \sum\limits_{j=1}^{d} a_{ij}\,\xi_j$ in the identity (12). Its left side $e_I\wedge e_{\widehat I}$ turns to

$$\Bigl(\sum_{\substack{K:\\ \#K=\#I}} a_{IK}\,\xi_K\Bigr)\wedge\Bigl(\sum_{\substack{L:\\ \#L=(d-\#I)}} a_{L\widehat I}\xi_L\Bigr) = (-1)^{\frac{1}{2}\#I(1+\#I)}\sum_{\substack{K:\\ \#K=\#I}}(-1)^{|K|}a_{IK}a_{\widehat I\widehat K}\,\xi_1\wedge\xi_2\wedge\,\cdots\,\wedge\xi_d\ ,$$

where $\widehat K = \{1, 2, \dots, d\}\setminus K$. The right side of (12) gives $(-1)^{\frac{1}{2}\#I(1+\#I)}(-1)^{|I|}\det(a_{ij})\xi_1\wedge\xi_2\wedge\,\cdots\,\wedge\xi_d$. So, for any any collection I of rows in any square matrix (a_{ij}) the following relation holds:

$$\sum_{\substack{K:\\ \#K=\#I}}(-1)^{|K|+|I|}a_{IK}\widehat a_{IK} = \det(a_{ij})\ , \tag{13}$$

where $\widehat a_{IK}\overset{\text{def}}{=} a_{\widehat I\widehat K}$ denotes the $(d-n)\times(d-n)$ - minor which is complementary[4] to a_{IK} and the summation runs over all $(n\times n)$ - minors a_{IK} contained in the rows $(i_1, i_2, \dots, i_n)$.

If we take $I\cap J\neq\varnothing$, then starting from $e_I\wedge e_J = 0$ instead of (12) we get by the same calculation the relation

$$\sum_{\substack{K:\\ \#K=\#I}}(-1)^{|K|+|I|}a_{IK}\widehat a_{JK} = 0\ , \tag{14}$$

The identities (13) and (14) are known as *Laplace relations*. Let us fix, say lexicografical, order on the set of indices I and arrange all $(n\times n)$-minors a_{IJ} as $\binom{d}{n}\times\binom{d}{n}$ - matrix $A^{(n)}\overset{\text{def}}{=}(a_{IJ})$. If we denote by $\widehat A^{(n)}$ a matrix whose (IJ)-entry equals $\bigl((-1)^{|I|+|J|}\widehat a_{JI}\bigr)$, then all the Laplace relations are expressed by the single matrix equality $A^{(n)}\cdot\widehat A^{(n)} = \det(a_{ij})\cdot E$.

Example 4.23: Reduction of grassmannian quadratic forms. Any homogeneous grassmannian polynomial of degree 2 can be written as

$$\xi_1\wedge\xi_2 + \xi_3\wedge\xi_4 + \,\cdots\, + \xi_{r-1}\wedge\xi_r \tag{15}$$

in some base (over any field k). Namely, we can suppose[5] that our grassmannian quadratic form is

$$q(e) = e_1\wedge(\alpha_2 e_2 + \,\cdots\, + \alpha_n e_n) + (\text{terms without } e_1)$$

where $\alpha_2\neq 0$ and $\xi_2\overset{\text{def}}{=}\alpha_2 e_2 + \cdots + \alpha_n e_n$ does not contain e_1, that is it can be included in the new base $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ with $\zeta_i = e_i$ for $i\neq 2$. After the substitution $e_2 = \alpha_2^{-1}(\xi_2 - \alpha_3\zeta_3 - \,\cdots\, - \alpha_n\zeta_n)$, $e_i = \zeta_i$ for $i\neq 2$, we can write q as $q(\zeta) = \zeta_1\wedge\zeta_2 + \zeta_2\wedge(\beta_3\zeta_3 + \,\cdots\, + \beta_n\zeta_n) + (\text{terms without } \zeta_1 \text{ and } \zeta_2)$. So, in the next new base: $\{\xi_1, \xi_2, \dots, \xi_n\}$ with $\xi_1 = \zeta_1 - \beta_3\zeta_3 - \,\cdots\, - \beta_n\zeta_n$, $\xi_i = \zeta_i$ for $i\neq 1$ our q turns to

$$q(\zeta) = \xi_1\wedge\xi_2 + (\text{terms without } \xi_1 \text{ and } \xi_2)$$

and this procedure can be repeated inductively for the remaining terms.

Exercice 4.24. Let $A = (a_{ij})$ be a skew symmetric matrix (i. e. $a_{ij} = -a_{ji}$) and $q(e) = \sum\limits_{ij} a_{ij}\,e_i\wedge e_j$ be a grassmannian quadratic form. Show that in the representation (15) the number r doesn't depend on the base choice and equals $\operatorname{rk} A$. (In particular, $\operatorname{rk} A$ is always even.)

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[4] i. e. sitting in the complementary rows and columns

[5] may be after appropriate renumbering of the base vectors

Polarizations and contracions for (skew) polynomials[1]

<u>(Skew) symmetric tensors.</u> A symmetric group $\mathfrak{S}_n$ acts on $V^{\otimes n}$ permuting factors in the decomposable tensors: $\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_n) \stackrel{\text{def}}{=} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \quad \forall\, \sigma \in \mathfrak{S}_n$. Subspaces

$$\mathsf{Skew}^n V = \{\, t \in V^{\otimes n} \mid \sigma(t) = \operatorname{sgn}(\sigma) \cdot t \quad \forall\, \sigma \in \mathfrak{S}_n \,\}$$
$$\mathsf{Sym}^n V = \{\, t \in V^{\otimes n} \mid \sigma(t) = t \quad \forall\, \sigma \in \mathfrak{S}_n \,\}$$

are called the spaces of *skew symmetric* and *symmetric* tensors.

<u>Claim 5.1</u>. *If* char $(k) = 0$*, then, restricting the canonical factorization maps:* $V^{\otimes n} \xrightarrow{\pi_{skew}} \Lambda^n V$, $V^{\otimes n} \xrightarrow{\pi_{sym}} S^n V$ *, onto the spaces of (skew) symmetric tensors, we get canonical isomorphisms:*

$$\mathsf{Skew}^n V \xrightarrow{\pi_{skew}} \Lambda^n V \qquad \textit{and} \qquad \mathsf{Sym}^n V \xrightarrow{\pi_{sym}} S^n V\,. \tag{16}$$

Proof. In the skew symmetric case, a base of $\mathsf{Skew}^n V$ is formed by the tensors

$$e_{\langle i_1, i_2, \dots, i_n \rangle} \stackrel{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \cdot e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}} \otimes \cdots \otimes e_{i_{\sigma(n)}}$$

(sum of all the tensor monomials sent to the basic Grassmannian monomial $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$ by π_{skew}). So, $\pi_{\text{skew}}\left(e_{\langle i_1, i_2, \dots, i_n \rangle}\right) = n!\, e_I$. In the symmetric case, let us write $e_{[m_1, m_2, \dots, m_d]}$ for the sum of all tensor monomials containing m_1 factors e_1, m_2 factors e_2, … , m_d factors e_d, where $\sum\limits_{\nu} m_\nu = n$. These monomials form one $\mathfrak{S}_n$-orbit, which consists of $\frac{n!}{m_1!\, m_2! \cdots m_d!}$ elements and collects all the decomposable tensors sent to $e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$ by π_{sym}. As above, the tensors $e_{[m_1, m_2, \dots, m_d]}$ form a base for $\mathsf{Sym}^n V$ and $\pi_{\text{sym}}\left(e_{[i_1, i_2, \dots, i_n]}\right) = \frac{n!}{m_1!\, m_2! \cdots m_d!} e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$. $\square$

Exercice 5.2. Verify that the above sums $e_{\langle i_1, i_2, \dots, i_n \rangle}$ and $e_{[m_1, m_2, \dots, m_d]}$ really give the bases for $\mathsf{Sym}^n V$ (over any field of an arbitrary characteristic). Also note that if char $(k) > 0$ divides n, then all these basic (skew) symmetric tensors are annihilated by factorization through (skew) symmetric relations.

Exercice 5.3. Verify that if char $(k) = 0$, then $V^{\otimes n} = \mathcal{I}^{(n)}_{\text{skew}} \oplus \mathsf{Skew}^n V = \mathsf{Sym}^n V \oplus \mathcal{I}^{(n)}_{\text{sym}}$, where the projection $V^{\otimes n} \twoheadrightarrow \mathsf{Sym}^n V$ along $\mathcal{I}^{(n)}_{\text{sym}}$ is given by *the symmetrization map*

$$\operatorname{sym}_n \,:\, \tau \longmapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma(\tau)$$

and the projection $V^{\otimes n} \twoheadrightarrow \mathsf{Skew}^n V$ along $\mathcal{I}^{(n)}_{\text{skew}}$ given by *the alternation map*

$$\operatorname{alt}_n \,:\, \tau \longmapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \cdot \sigma(\tau)\,.$$

<u>Polarization of (skew) polynomials.</u> The inverse maps to the isomorphisms (16) take

$$\begin{aligned} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} &\longmapsto \frac{1}{n!} \cdot e_{\langle i_1, i_2, \dots, i_n \rangle} \\ e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d} &\longmapsto \frac{m_1!\, m_2! \cdots m_d!}{n!} \cdot e_{[m_1, m_2, \dots, m_d]}\,. \end{aligned} \tag{17}$$

The both maps are called *complete polarizations* of (skew) polynomials and are denoted by $f \longmapsto \operatorname{pl}(f)$.

<u>Example 5.4:</u> (skew) polynomials and (skew) symmetric multilinear forms. Full polarization $\operatorname{pl}(f)$ of a (skew) homogeneous degree n polynomial f of one argument on V can be considered as a multilinear form of n arguments on V. It sends $(v_1, v_2, \dots, v_n)$ to the full contraction $\widetilde{f}(v_1, v_2, \dots, v_n) \stackrel{\text{def}}{=} \langle\, v_1 \otimes v_2 \otimes \cdots \otimes v_n \,,\, \operatorname{pl}(f) \,\rangle$ and has the same symmetry properties as f, because for all $\sigma \in \mathfrak{S}_n$, $t \in V^{\otimes n}$, $\xi \in V^{*\otimes n}$ we have $\langle\, \sigma(t) \,,\, \sigma(\xi) \,\rangle = \langle\, t \,,\, \xi \,\rangle$, which implies $\langle\, \sigma(t) \,,\, \xi \,\rangle = \langle\, t \,,\, \sigma^{-1}(\xi) \,\rangle$.

Exercice 5.5. Check that for a symmetric quadratic form $q(x) \in S^2 V^*$ we have

$$\widetilde{q}(x, y) = \frac{q(x+y) - q(x-y)}{4} = \frac{q(x+y) - q(x) - q(y)}{2} = \frac{1}{2} \sum_{\nu=1}^{\dim V} y_\nu \frac{\partial q}{\partial x_\nu}\,.$$

[1] WARNING!!! In this lecture we suppose that the ground field k is algebraically closed and has zero characteristic.

Since any multilinear form φ may be presented via full contraction

$$\varphi(v_1, v_2, \dots, v_n) = \langle\, v_1 \otimes v_2 \otimes \cdots \otimes v_n \,,\, \xi \,\rangle$$

with some $\xi \in V^{*\otimes n}$, the complete polarization identifies S^nV^* and Λ^nV^* with the spaces of all symmetric and skew symmetric multilinear forms $\underbrace{V \times V \times \cdots \times V}_{n} \longrightarrow k$, in n arguments on V.

Exercice 5.6. In symmetric case, show that a homogeneous degree n polynomial $f(x)$ (in one argument $x \in V$) coincides with the restriction of the corresponding symmetric multilinear form $\widetilde{f}(x_1, x_2, \dots, x_n)$ (in n arguments $x_\nu \in V$) onto the diagonal: $f(x) = \widetilde{f}(x, x, \dots, x)$.

Example 5.7: duality on polynomials Using the complete polarization and the full contraction between $V^{\otimes n}$ and $V^{*\otimes n}$ we obtain (over a field of zero characteristic) a natural non degenerate pairing between $\Lambda^n(V)$ and $\Lambda^n(V^*)$ as well as between $S^n(V)$ and $S^n(V^*)$. Namely, for two (skew) polynomials f, in $e_i \in V$, and ξ, in $x_i \in V^*$, we put $\langle\, f \,,\, \xi \,\rangle \stackrel{\text{def}}{=} \langle\, \mathrm{pl}(f) \,,\, \mathrm{pl}(\xi) \,\rangle$.

Exercice 5.8. Let $\{e_1, e_2, \dots, e_n\} \subset V$ and $\{x_1, x_2, \dots, x_n\} \subset V^*$ be dual bases. Check that:

$$\langle\, e_I \,,\, x_J \,\rangle = \begin{cases} 1/n! & \text{, for } I = J \\ 0 & \text{, for } I \neq J \end{cases} \tag{18}$$

$$\langle e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d} \,,\, x_1^{\ell_1} x_2^{\ell_2} \cdots x_d^{\ell_d} \rangle = \begin{cases} \frac{m_1!\, m_2! \cdots m_d!}{n!} & \text{, if } m_\nu = \ell_\nu \ \forall \nu \\ 0 & \text{, otherwise} \end{cases} \tag{19}$$

Partial derivatives (symmetric case) For any vector $v \in V$ and any polynomial $f \in S^nV^*$ the contraction $c_1^i(\mathrm{pl}(f) \otimes v) \in V^{*n-1}$ does not depend on the choice of index i in $\mathrm{pl}(f)$ to be contracted, because $\mathrm{pl}(f) \in \mathsf{Sym}^n V^*$ is symmetric. Its projection to $S^{n-1}V^*$ is called *a polar* of v w.r.t. f and is denoted by $\mathrm{pl}_v f$.

Exercice 5.9. Show that $\deg(f) \cdot \mathrm{pl}_v f = \partial_v f$, where ∂_v is the derivative in v-direction, which takes f to $\sum\limits_{i=1}^{\dim V} v_i \frac{\partial f}{\partial x_i}$ (here $v = \sum v_i e_i$, $\{e_i\}$ is a base for V and $\{x_i\}$ is a dual base for V^*).

Exercice 5.10. Show that $n! \cdot \widetilde{f}(v_1, v_2, \dots, v_n) = \partial_{v_1} \partial_{v_2} \cdots \partial_{v_n} f$ for any $f \in S^nV^*$, $v_1, v_2, \dots, v_n \in V$.

In particular, $\partial_v f$ does not depend on a base choice, is bilinear in v, f and satisfies the following symmetry property:

$$m_2!\; \partial_{v_1}^{m_1} f(v_2) = (m_1 + m_2)!\; \widetilde{f}(\underbrace{v_1, v_1, \dots, v_1}_{m_1}, \underbrace{v_2, v_2, \dots, v_2}_{m_2}) = m_1!\; \partial_{v_2}^{m_2} f(v_1)\;, \tag{20}$$

where $(m_1 + m_2) = n = \deg f$. In particular, the both, left and right sides, are bihomogeneous of bidegree (m_1, m_2) in (v_1, v_2).

Exercice 5.11. Show that *multiple polars*: $\mathrm{pl}_{\,v_1, v_2, \dots, v_m} f \stackrel{\text{def}}{=} \mathrm{pl}_{v_1} \mathrm{pl}_{v_2} \cdots \mathrm{pl}_{v_1} f$ are symmetric and multilinear in v_i and linear in f, that is come from a linear map $S^mV \otimes S^nV^* \longrightarrow S^{n-m}V^*$, which sends $v_1 v_2 \cdots v_m \otimes f$ to $\mathrm{pl}_{\,v_1, v_2, \dots, v_m} f(w) = \widetilde{f}(v_1, v_2, \dots, v_m, w, w, \dots, w)$.

Example 5.12: Taylor formula. For $f \in S^nV^*$ the value $f(v + w)$ can be computed as the full contraction of f with $(v + w)^m = \sum\limits_m \binom{n}{m} v^m w^{n-m} \in S^nV$. This can be arranged as the *Taylor formula*:

$$f(v + w) = \sum_{\nu=0}^{n} \frac{1}{\nu!}\, \partial_w^\nu f(v)\;. \tag{21}$$

Example 5.13: *span*(f) for $f \in S^nV^*$, that is a minimal subspace $W \subset V^*$ such that $f \in S^nW$, coincides with *span*$(\mathrm{pl}(f))$ described in Claim 4.11 as an image of the contraction map $S^{n-1}V \otimes S^nV^* \longrightarrow V^*$. So, in terms of partial derivatives, *span*(f) is generated by the linear forms

$$\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_{n-1}}} f(x) \tag{22}$$

obtained from f by all possible $(n-1)$-fold differentiations.

Veronese variety in $\mathbb{P}_N = \mathbb{P}(S^nV^*)$, where $N = \binom{n}{d} - 1$ and $\dim V = d + 1$, consists of pure n-th powers of linear forms, that is, of $f(x) = \xi^n(x)$, where $\xi \in \mathbb{P}(V^*)$. It has a rational parameterization given by the Veronese map

$$\mathbb{P}(V^*) \xhookrightarrow{\ \xi \longmapsto \xi^n\ } \mathbb{P}(S^nV^*)\;.$$

Example 5.13 presents the Veronese variety by a system of quadratic equations. Namely, $f(x) = \xi^n(x)$ for some $\xi \in V^*$ iff *span*(f) is 1-dimensional subspace generated by ξ. So, f is pure n-th power iff any two linear forms (22) are proportional to each other. If we arrange the coefficients of two linear forms as the rows of some $2 \times (d+1)$-matrix, then their proportionality means that all 2×2-minors of this matrix vanish.

Example 5.14: Veronese's curve. Let $\dim U = 2$, $\mathbb{P}_1 = \mathbb{P}(U^*)$, $\mathbb{P}_n = \mathbb{P}(S^n U^*)$, $\{t_0, t_1\}$ be a base of U^*, and $\binom{n}{i} t_0^i t_1^{n-i}$ for $0 \leqslant i \leqslant n$ be the corresponding base of $S^n V^*$. Then the Veronese embedding $\mathbb{P}_1 \hookrightarrow \mathbb{P}_n$ sends a linear form $(\alpha_0 t_0 + \alpha_1 t_1)$ to

$$(\alpha_0 t_0 + \alpha_1 t_1)^n = \sum_i \alpha_0^i \alpha_1^{n-i} \cdot \binom{n}{i} t_0^i t_1^{n-i}$$

Its image is a rational curve $C_n \subset \mathbb{P}_n$ called *Veronese curve* or *rational normal curve* of degree n. If we use the coefficients $(a_0 : a_1 : \ \dots \ : a_n)$ and $(\alpha_0 : \alpha_1)$ as homogeneous coordinates for a polynomial $\sum\limits_i a_i \cdot \binom{n}{i} t_0^i t_1^{n-i} \in S^n U^*$ and for a linear form $\alpha_0 t_0 + \alpha_1 t_1 \in \mathbb{P}(U^*)$, then C_n will be presented parametrically as $a_i = \alpha_0^i \alpha_1^{n-i}$.

On the other hand, for binary form $f(t) = \sum\limits_i a_i \cdot \binom{n}{i} t_0^i t_1^{n-i}$ the linear forms (22) are exhausted by

$$\frac{\partial^i}{\partial t_0^i} \frac{\partial^{n-1-i}}{\partial t_1^{i-1-n}} f(t) = a_i t_0 + a_{i+1} t_1 \quad , \quad \text{where} \quad 0 \leqslant i \leqslant (n-1) \,.$$

So, $f \in C_n$ iff $\det \begin{pmatrix} a_i & a_{i+1} \\ a_{j-1} & a_j \end{pmatrix} = a_i a_j - a_{i+1} a_{j-1} = 0$ for all $0 \leqslant i < j \leqslant n$, i. e. iff

$$\operatorname{rk} \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_n \end{pmatrix} = 1 \,.$$

For $n = 2$ we get the Veronese quadric $a_0 a_2 = a_1^2$ in $\mathbb{P}_2$. For $n = 3$ we have a rational cubic curve, which is called *a twisted cubic*, given as an intersection of 3 quadrics $a_0 a_2 = a_1^2$, $a_1 a_3 = a_2^2$, $a_0 a_3 = a_1 a_2$.

Exercice 5.15. Draw the picture and discover that the first two quadrics are simple cones with vertices at $(0:0:0:1)$ and at $(1:0:0:0)$; the third quadric is Segre's one and is ruled by two line families; two cones have a common line element $a_1 = a_2 = 0$, which joins the vertices but does not lie on the Segre quadric; the line element $a_0 = a_1 = 0$ of the first cone and the one $a_2 = a_3 = 0$ of the second do lie on the Segre quadric in the same ruling family. So, any two of 3 quadrics are intersected along the twisted cubic and one more line, that is, the Veronese curve can not be given by use of 2 equations only!

Partial derivatives (skew-symmetric case). The contraction $c_1^i(\mathrm{pl}(f) \otimes v) \in V^{*n-1}$ between a vector $v \in V$ and a skew polynomial $f \in \Lambda^n V^*$ slightly depends on the choice of index i: it changes the sign when i is incremented or decremented by 1, because $\mathrm{pl}(f) \in \mathsf{Skew}^n V^*$ is skew symmetric. We choose $i = 1$ and write $\mathrm{pl}_v f$ for the projection of $c_1^1(\mathrm{pl}(f) \otimes v)$ into $\Lambda^{n-1} V^*$. Now consecutive polarizations anti commute: $\mathrm{pl}_v \mathrm{pl}_w f = -\mathrm{pl}_w \mathrm{pl}_v f$, i. e. the multilinear map

$$\underbrace{V \times V \times \cdots \times V}_{m} \times \Lambda^n V^* \xrightarrow{(v_1, v_2, \dots, v_m \,,\, f) \mapsto \mathrm{pl}_{v_1} \mathrm{pl}_{v_2} \dots \mathrm{pl}_{v_m} f} \Lambda^{n-m} V^*$$

comes from the linear map $\Lambda^m V \otimes \Lambda^n V^* \longrightarrow \Lambda^{n-m} V^*$.

Exercice 5.16. Show that $\deg f \cdot \mathrm{pl}_v f = \partial_v f$, where $\partial_v = \sum v_i \frac{\partial}{\partial x_i}$, as in Exercice 5.9, and verify the following properties of skew partial derivatives:
a) $\partial_v \partial_w = -\partial_w \partial_v$ (in particular, $\partial_v^2 = 0 \quad \forall v \in V$);
b) $\widetilde{f}(v_1, v_2, \dots, v_n) = \frac{1}{n!} \partial_{v_1} \partial_{v_2} \cdots \partial_{v_n} f$ for any $f \in \Lambda^n V^*$, $v_1, v_2, \dots, v_n \in V$.
c) $\partial_v (f_1 \wedge f_2) = (\partial_v f_1) \wedge f_2 - (-1)^{\deg f_1} f_1 \wedge (\partial_v f_2)$, in particular,

$$\partial_{e_{i_\nu}} x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} = (-1)^{\nu-1} x_{i_1} \wedge \cdots \wedge x_{i_{\nu-1}} \wedge x_{i_{\nu+1}} \wedge \cdots \wedge x_{i_n} \,.$$

Example 5.17: *span*(f) for grassmannian polynomial $f \in \Lambda^n V$ is linearly generated by partial derivatives

$$\partial_J f = \partial_{x_{j_{n-1}}} \partial_{x_{j_{n-2}}} \dots \partial_{x_{j_1}} f = n! \left\langle x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_{n-1}} \,,\, \mathrm{pl}(f) \right\rangle$$

where $J = (j_1, j_2, \dots, j_{n-1}) \subset \{1, 2, \ \dots \ , d\}$ runs through all $\binom{d}{n-1}$ increasing ordered collections of $(n-1)$ indexes and $\{x_i\} \in V^*$ form a dual base to some base $\{e_i\} \in V$. If $f = \sum\limits_I \alpha_I \, e_I$ w. r. t. the last base, then

$$\partial_J f = n! \cdot \sum_{i \notin J} (-1)^{n - p(i,J)} \, \alpha_{J \sqcup \{i\}} \, e_i \tag{23}$$

where $p(i, J)$ is the number of place where i stays in the increasing permutation I of $J \sqcup \{i\}$ (because x_I is the only monomial in $\Lambda^n V^*$ whose complete contraction with e_I is non zero, see (18)).

Example 5.18: Plücker relations. A skew polynomial $f \in \Lambda^n V$ is called *completely decomposable*, if $f = v_1 \wedge v_2 \wedge \cdots \wedge v_n$ is a product of n linear factors, or equivalently, if $\dim span(f) = n$ is minimal possible.

Exercice 5.19. Show that f is completely decomposable iff $f \wedge v = 0\ \forall v \in span(f)$.

So, like in Example 5.13 — Example 5.14, the set of completely decomposable polynomials $f \in \Lambda^n V$ is described by a system of quadratic equations $(\partial_J f) \wedge f = 0$. By (23), a basic monomial $e_K \in \Lambda^{n+1} V$ appears in

$$(\partial_J f) \wedge f = n! \cdot \left(\sum_{i \notin J} (-1)^{p(i,J)}\, \alpha_{J \sqcup \{i\}}\, e_i \right) \wedge \left(\sum_I a_I e_I \right)$$

as $n! \cdot \sum\limits_{\alpha=1}^{n+1} (-1)^{n-p(k_\alpha,J)} \cdot \alpha_{J \sqcup \{k_\alpha\}} \alpha_{K \setminus \{k_\alpha\}} \cdot e_{k_\alpha} \wedge e_{K \setminus \{k_\alpha\}}$, i.e. with the coefficient, which up to a constant factor equals

$$P_{JK}(f) \stackrel{\text{def}}{=} \sum_{i \in K \setminus (K \cap J)} (-1)^{p(i,J,K)}\, \alpha_{J \sqcup i}\, \alpha_{K \setminus i}\ , \tag{24}$$

where $p(i,J,K)$ is the sum of the place numbers where i stays in the increasing version of $J \sqcup \{i\}$ and in K.

Exercice 5.20. Show that this coefficient vanishes, if $J \subset K$

System of quadratic equations $P_{JK}(f) = 0$, which defines the variety of completely decomposable Grassmannian polynomials, is known as the *Plücker relations*. Note that these quadratic equations are not «independent», even not pairwise different (see Example 6.10 below).

Working example: Grassmannians

<u>PLÜCKER QUADRIC IN $\mathbb{P}_5$.</u> Let V be a 4-dimensional vector space and $\mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$. Then the *Plücker quadric*

$$Q_P \stackrel{\text{def}}{=} \{\,\omega \in \Lambda^2 V \mid \omega \wedge \omega = 0\,\}$$

is a non singular quadric in $\mathbb{P}_5$. Fixing a base $\{e_0, e_1, e_2, e_3\}$ for V and induced base $e_{ij} = e_i \wedge e_j$ for $\Lambda^2 V$ and writing x_{ij} for the homogeneous coordinate along e_{ij}, we have $\Big(\sum\limits_{i<j} x_{ij} \cdot e_i \wedge e_j\Big) \wedge \Big(\sum\limits_{i<j} x_{ij} \cdot e_i \wedge e_j\Big) = 2\,(x_{01}x_{23} - x_{02}x_{13} + x_{03}x_{12}) \cdot e_0 \wedge e_1 \wedge e_2 \wedge e_3$, i.e. Q_P has the equation

$$x_{02}x_{13} = x_{01}x_{23} + x_{03}x_{12}\ .$$

In coordinateless terms, Q_P is given by quadratic form $q(\omega) = \widetilde{q}(\omega, \omega)$ whose polarization $\widetilde{q}(\omega_1, \omega_2)$ is a bilinear form on $\Lambda^2 V$ defined up to a scalar factor by the prescription

$$\omega_1 \wedge \omega_2 = \widetilde{q}(\omega_1, \omega_2) \cdot \Omega\ ,$$

where $\Omega \in \Lambda^4 V \simeq k$ is any fixed non-zero vector. This form is symmetric, because even degree Grassmannian polynomials commute: $\omega_1 \wedge \omega_2 = \omega_2 \wedge \omega_1$.

<u>PLÜCKER EMBEDDING.</u> By the definition, *a grassmannian* $\mathrm{Gr}(2,4)$ is a set of all lines $\ell \subset \mathbb{P}_3 = \mathbb{P}(V)$, or equivalently, a set of all 2-dimensional vector subspaces $U \subset V$. *A Plücker map*

$$\mathrm{Gr}(2,4) \xhookrightarrow{\mathfrak{u}} \mathbb{P}(\Lambda^2 V)$$

sends a 2-dimensional subspace $U \subset V$ to the 1-dimensional subspace $\Lambda^2 U \subset \Lambda^2 V$. If U is spaned by a pair of vectors u_1, u_2 (i.e. $\ell = \mathbb{P}(U)$ pass trough $u_1, u_2 \in \mathbb{P}(V)$), then $\mathfrak{u}(\ell) = \mathfrak{u}(U) = u_1 \wedge u_2$ up to proportionality.

<u>Lemma 6.1</u>. *Two lines $\ell_1, \ell_2 \subset \mathbb{P}_3$ are intersecting iff* $\widetilde{q}(\mathfrak{u}(\ell_1), \mathfrak{u}(\ell_2)) = \mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell_2) = 0$.

Proof. Let $\ell_1 = \mathbb{P}(U_1)$, $\ell_2 = \mathbb{P}(U_2)$. If $U_1 \cap U_2 = 0$, then $V = U_1 \oplus U_2$ and there exist a base $\{e_i\} \subset V$ such that U_1 is spaned by e_0, e_1 and U_2 is spaned by e_2, e_3. So, $\mathfrak{u}(U_1) = e_0 \wedge e_1$, $\mathfrak{u}(U_2) = e_2 \wedge e_3$ and $\mathfrak{u}(U_1) \wedge \mathfrak{u}(U_2) = e_0 \wedge e_1 \wedge e_2 \wedge e_3 \neq 0$. If $U_1 \cap U_2 \neq 0$, then taking $u_0 \subset U_1 \cap U_2$ we can write $\mathfrak{u}(U_1) = u_0 \wedge u_1$, $\mathfrak{u}(U_2) = u_0 \wedge u_2$ for some u_1, u_2. So, $\mathfrak{u}(U_1) \wedge \mathfrak{u}(U_2) = u_0 \wedge u_1 \wedge u_0 \wedge u_2 = 0$. $\square$

<u>Lemma 6.2</u>. *If* $\dim V = 4$, *then $\omega \in \Lambda^2 V$ is decomposable*[1] *iff* $\omega \wedge \omega = 0$.

Proof. If ω is decomposable, say $\omega = u_1 \wedge u_2$, then $\omega \wedge \omega = u_1 \wedge u_2 \wedge u_1 \wedge u_2 = 0$, because of skew symmetry. To get the inverse, take a base $\{\xi_i\}$ such that ω turns into either $\omega = \xi_0 \wedge \xi_1 + \xi_2 \wedge \xi_3$ or $\omega = \xi_0 \wedge \xi_1$. If $\omega \wedge \omega = 0$, then the first case is impossible, because of $(\xi_0 \wedge \xi_1 + \xi_2 \wedge \xi_3) \wedge (\xi_0 \wedge \xi_1 + \xi_2 \wedge \xi_3) = 2\,\xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 \neq 0$. $\square$

<u>Corollary 6.3</u>. *The Plücker map is a bijection between the grassmannian* $\mathrm{Gr}(2,4)$ *and the Plücker quadric* $Q_P \subset \mathbb{P}_5$.

Proof. $\mathfrak{u}$ is injective, because for any two lines $\ell_1 \neq \ell_2$ on $\mathbb{P}_3$ there exists a third line ℓ which intersect ℓ_1 and doesn't intersect ℓ_2. Then $\mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell) = 0$ and $\mathfrak{u}(\ell_2) \wedge \mathfrak{u}(\ell) \neq 0$, so, $\mathfrak{u}(\ell_1) \neq \mathfrak{u}(\ell_2)$. Surjectivity follows from Lemma 6.2. $\square$

<u>Corollary 6.4</u>. *For any point $p = \mathfrak{u}(\ell) \in Q_P$ the intersection $Q_P \cap T_p Q_P$ consists of all $\mathfrak{u}(\ell')$ such that $\ell \cap \ell' \neq \varnothing$.*

Proof. $T_p Q_P$ is a zero set of the linear form $\widetilde{q}(\mathfrak{u}(\ell), *)$. But by Lemma 6.1, $\widetilde{q}(\mathfrak{u}(\ell), \mathfrak{u}(\ell')) = 0 \iff \ell \cap \ell' \neq \varnothing$. $\square$

<u>LINE NETS AND LINE PENCILS IN $\mathbb{P}_3$.</u> A set of lines on $\mathbb{P}_3$ is called *a net* if it is represented by a plane $\pi \subset Q_P \subset \mathbb{P}_5$. If $\pi \subset Q_P$ is spaned by 3 non collinear points $p_i = \mathfrak{u}(\ell_i)$, $i = 1, 2, 3$, then

$$\pi = Q_P \cap T_{p_1} Q_P \cap T_{p_2} Q_P \cap T_{p_3} Q_P\ .$$

So, by Lemma 6.1 and Corollary 6.4 the corresponding line net consist of all lines which intersect 3 given pairwise intersecting lines. Hence, there are exactly two geometrically different line nets on $\mathbb{P}_3$:

α-net is a set of all lines passing through a given point $O \in \mathbb{P}_3$; the corresponding plain $\pi_\alpha(O) \subset Q_P$ is called *α-plane*. It is spanned by the Plücker images of any 3 non coplanar lines passing through O.

[1] recall that homogeneous polynomial is called decomposable if it is represented as a product of linear forms

β-net is a set of all lines laying on a given plane $\Pi \in \mathbb{P}_3$; the corresponding plain $\pi_\beta(\Pi) \subset Q_P$ is called *β-plane*. It is spanned by the Plücker images of any 3 lines which lay on Π and don't have a common intersection.

Note that any two planes of the same type have exactly 1-point intersection, namely:

$$\pi_\beta(\Pi_1) \cap \pi_\beta(\Pi_2) = \mathfrak{u}(\,\Pi_1 \cap \Pi_2\,)$$
$$\pi_\alpha(O_1) \cap \pi_\alpha(O_2) = \mathfrak{u}(\,(O_1 O_2)\,)$$

Two planes of different types, say $\pi_\beta(\Pi)$ and $\pi_\alpha(O)$, don't intersect each other, if $O \notin \Pi$. If $O \in \Pi$, then $\pi_\beta(\Pi) \cap \pi_\alpha(O)$ is a pencil of lines $\ell \subset \mathbb{P}_3$ such that $O \in \ell \subset \Pi$.

Exercice 6.5. Show that there are no other line pencils in $\mathbb{P}_3$, i. e. each line on $Q_P \subset \mathbb{P}_5$ has the form $\pi_\beta(\Pi) \cap \pi_\alpha(O)$ for some O and Π.

HINT. Consider the cone $C = Q_P \cap T_p Q_P$. It has a vertice at p and consists of all lines which pass through p and lay on Q_p. Fix a 3-dimensional hyperplane $H \subset T_p Q_P$ which doesn't contain p. Then $G = C \cap H$ is non singular quadric on H. So, any line passing through p has a form $(pp') = \pi_\alpha \cap \pi_\beta$, where $p' \in G$ and the planes π_α, π_β are spaned by p and two lines passing through p' on G (see 6.1).

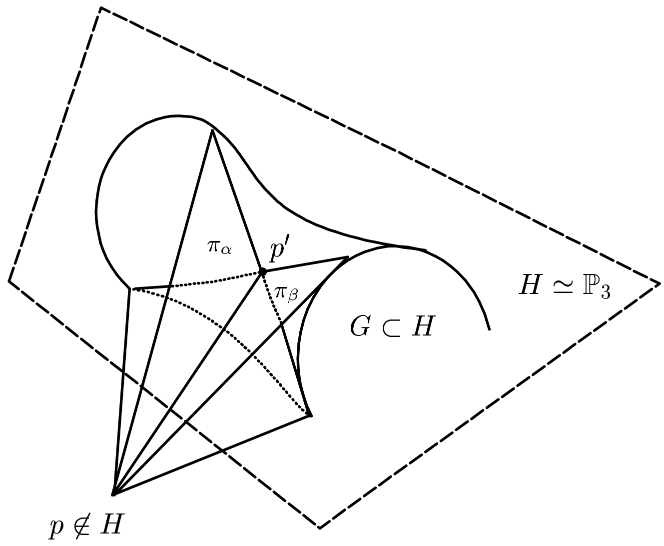


Fig. 6.1. The cone $C = Q_P \cap T_p Q_P$.

AFFINE CELL DECOMPOSITION OF $\mathrm{Gr}(2,4)$. Let $H \subset T_p Q_P$ be a 3-dimensional projective hyperplane such that $p \notin H$, as in above exercise, $C = Q_P \cap T_p Q_P$, and $G = H \cap Q_P$. Then C is the simple cone with vertex p over G (see fig. 6.1) and we have the following diagram of inclusions

$$\begin{tikzcd} & & \pi_\alpha \arrow[dr, hook] & & \\ p \arrow[r, hook] & \pi_\alpha \cap \pi_\beta \arrow[ur, hook] \arrow[dr, hook] & & C \arrow[r, hook] & Q_P \\ & & \pi_\beta \arrow[ur, hook] & & \end{tikzcd} \quad \Rightarrow \quad \mathrm{Gr}(2,4) \;=\; \mathbb{A}^0 \sqcup \mathbb{A}^1 \sqcup \begin{pmatrix} \mathbb{A}^2 \\ \sqcup \\ \mathbb{A}^2 \end{pmatrix} \sqcup \mathbb{A}^3 \sqcup \mathbb{A}^4$$

The right side decomposition is produced via replacing each stratum in the left side by the complement to all the smallest strata it contain and identifying the resulting disjoint cells with affine spaces as follows: $(\pi_\alpha \cap \pi_\beta) \setminus p \simeq \mathbb{A}^1$ (because this is a projective line without a point), $\pi_\alpha \setminus (\pi_\alpha \cap \pi_\beta) \simeq \pi_\beta \setminus (\pi_\alpha \cap \pi_\beta) \simeq \mathbb{A}^2$ (because the both are projective planes without a line), $C \setminus (\pi_\alpha \cup \pi_\beta) \simeq \mathbb{A}^1 \times \Big(G \setminus (G \cap T_{p'} G)\Big)$ (because C is a cone over G), and finally, $G \setminus (G \cap T_{p'} G) \simeq \mathbb{A}^2$ and $Q \setminus C \simeq \mathbb{A}^4$, because of the lemma below.

Lemma 6.6. *Let $Q \subset \mathbb{P}_n$ be a quadric, $p \in Q$ be a non singular point, and $H \not\ni p$ be a codimension 1 hyperplane. Then the projection from p onto H induces a bijection between $Q \setminus (Q \cap T_p Q)$ and $\mathbb{A}^{n-1} = H \setminus (H \cap T_p Q)$.*

Proof. Any non tangent line passing through p have to intersect Q precisely ones more. All such lines are 1–1 parameterized by the points of $\mathbb{A}^{n-1} = H \setminus (H \cap T_p Q)$. $\square$

Exercice 6.7*. If you have some experience in topology, show that over $\mathbb{C}$ all odd integer homologies of $\mathrm{Gr}(2,4)$ vanish and the even ones are $H_0 = H_2 = H_6 = H_8 = \mathbb{Z}$, $H_4 = \mathbb{Z} \oplus \mathbb{Z}$. Also, try to compute the homologies for the real grassmannian, where the boundary maps are non trivial.

GENERAL GRASSMANNIAN $\mathrm{Gr}(m,d)$ is defined as the set of all m-dimensional vector subspaces in a given d-dimensional vector space V. If the nature of V is important, we write $\mathrm{Gr}(m,V)$ instead of $\mathrm{Gr}(m,d)$. In the projective language, $\mathrm{Gr}(m,d)$ is the set of all $(m-1)$-dimensional projective subspaces in $\mathbb{P}_{d-1}$. If $m=1$, then $\mathrm{Gr}(m,d) = \mathbb{P}_{d-1}$. There is a canonical bijection $\mathrm{Gr}(m,V) \simeq \mathrm{Gr}(d-m,V^*)$ induced by duality. It sends $U \subset V$ to $\mathrm{Ann}\,U \subset V^*$ and wise versa.

Exercice 6.8. Let $\dim V = 4$. Fix an isomorphism $V \xrightarrow{\widehat{q}} V^*$, say presented by a non singular quadric $Q \subset \mathbb{P}(V)$, and consider an automorphism of $\mathrm{Gr}(2,V)$ given by $U \longmapsto \mathrm{Ann}\,\widehat{q}(U)$. Show that it maps the α-planes on $\mathrm{Gr}(2,4)$ to the β-planes and wise versa.

PLÜCKER EMBEDDING $\mathrm{Gr}(m,V) \xhookrightarrow{\mathfrak{u}} \mathbb{P}(\Lambda^m V)$ sends m-dimensional subspace $U \subset V$ to the 1-dimensional subspace $\Lambda^m U \subset \Lambda^m V$. If U is based by the vectors $\{u_1, u_2, \dots, u_m\} \subset U$, then $\mathfrak{u}(U) = u_1 \wedge u_2 \wedge \cdots \wedge u_m$ up to proportionality, because taking an other base, say $v_i = \sum a_{ij} u_j$, we get

$$\mathfrak{u}(U) = v_1 \wedge v_2 \wedge \cdots \wedge v_m = \det(a_{ij}) \cdot u_1 \wedge u_2 \wedge \cdots \wedge u_m\,.$$

Lemma 6.9. *The Plücker embedding is really injective.*

Proof. If $U_1 \neq U_2$, then there exist a base in V such that some vectors $w_1, w_2, \dots, w_r$ of this base give a base for $U_1 \cap U_2$, some other $u_1, u_2, \dots, u_{m-r}$ together with $\{w_\nu\}$ give a base for U_1, some other $v_1, v_2, \dots, v_{m-r}$ together with $\{w_\nu\}$ give a base for U_2, and the rest $e_1, e_2, \dots, e_{d+r-2m}$ are complementary to $U_1 + U_2$. Let $\omega \in \Lambda^{d-m}V$ be the skew product of all v_ν and e_ν. The skew multiplication by ω $\Lambda^m V \xrightarrow{\xi \mapsto \xi \wedge \omega} \Lambda^d V \simeq k$ turns into a linear form on $\Lambda^m V$ as soon as a base vector for $\Lambda^d V$ is fixed. This linear form does vanish at $\mathfrak{u}(U_2)$ and doesn't at $\mathfrak{u}(U_1)$. □

Example 6.10: 2×2-minors of 2×4-matrices. Let $\dim V = 4$ and a base $\{e_1, e_2, e_3, e_4\} \subset V$ be fixed. Then a subspace $U \subset V$ based by u_1, u_2 can be presented as 2×4-matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$ whose rows are the coordinates of u_1, u_2. This matrix is defined by U up to the left multiplication $A \mapsto C \cdot A$ by any $C \in GL_2(k)$ (this corresponds to a base change in U). The Plücker embedding sends A to

$$u_1 \wedge u_2 = \sum_{i<j} \det \begin{pmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{pmatrix} e_i \wedge e_j \,.$$

So, the homogeneous coordinates of $\mathfrak{u}(U) \in \mathbb{P}_5$ in the base $\{e_{ij} = e_i \wedge e_j\}$ are six 2×2-minors of A. In particular, the left multiplications by $C \in GL_2$ doesn't effect on the ratios between 2×2-minors of A. An other claim: six numbers $x_1, x_2, \dots, x_6$ give a collection of 2×2-minors for some 2×4 matrix iff they satisfy (maybe, after some renumbering) the Plücker equation $x_1x_2 = x_3x_4 + x_5x_6$.

Exercice 6.11. Is there 2×4 - matrix with minors a) $\{2, 3, 4, 5, 6, 7\}$ b) $\{3, 4, 5, 6, 7, 8\}$?

Matrix notations and Plücker coordinates on $\mathrm{Gr}(m,d)$. A point $U \in \mathrm{Gr}(m,d)$ can be presented by $(m \times d)$-matrix A_U whose rows are the coordinates of some base vectors $\{u_1, u_2, \dots, u_m\} \subset U$ with respect to a fixed base $\{e_1, e_2, \dots, e_d\} \subset V$. Such a matrix is not unique and is defined by U only up to the left multiplication by any $C \in GL_m$ (this corresponds to a base change in U). So, the grassmannian $\mathrm{Gr}(m,d)$ can be considered as a factor space of $Mat_{m\times d}(k)$ by the left action of $GL_m(k)$. Under the Plücker embedding , the homogeneous coordinates of $\mathfrak{u}(U) \in \Lambda^m V$ in the standard base $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$ are equal to the maximal minors of A. They are stable under the left GL_m-action and are called *Plücker coordinates* of U.

Affine covering and affine coordinates on $\mathrm{Gr}(m,d)$. Consider the standard affine card $\mathscr{U}_I \subset \mathbb{P}(\Lambda^m V)$ given by $x_I = 1$, where x_I is the coordinate along $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$. The inverse image $\mathfrak{U}_I \stackrel{\text{def}}{=} \mathfrak{u}^{-1}(\mathscr{U}_I) \subset \mathrm{Gr}(m,d)$ consists of all U such that A_U has non zero maximal minor in the columns $(i_1, i_2, \dots, i_m)$. Any such U has *a unique* matrix representation $A_U^{(I)} = \left(a_{ij}^{(I)}\right)$ with *the identity $m \times m$-submatrix* staying in these columns. It is given by $A_U^{(I)} = A_{U,I}^{-1} \cdot A_U$, where A_U is an arbitrary matrix representation for U and $A_{U,I} \subset A_U$ is $m \times m$-submatrix formed by the columns $(i_1, i_2, \dots, i_m)$. So, the points of $\mathfrak{U}_I \subset \mathrm{Gr}(m,d)$ are 1–1 parametrized by $m\,(d-m)$ matrix elements $\left(a_{\mu\nu}^{(I)}\right)$ staying outside the columns $(i_1, i_2, \dots, i_m)$ in $A_U^{(I)}$. In other words, we have *an affine chart* $\mathbb{A}^{m(d-m)} \xrightarrow{\sim} \mathfrak{U}_I \subset \mathrm{Gr}(m,d)$ which covers an open dense subset of the grassmannian. The charts $\mathfrak{U}_I$ are called *standard* and cover the whole of $\mathrm{Gr}(m,d)$ when I runs through the length m increasing subsets in $\{1, 2, \dots, d\}$.

Exercice 6.12. Write down the explicit *transition functions* between the standard affine charts $\mathfrak{U}_{12}$ and $\mathfrak{U}_{23}$ on $\mathrm{Gr}(2,4)$.

Exercice 6.13*. If you had deal with smooth topology, check that real and complex grassmannians are the smooth (moreover, analytic) manifolds.

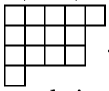
Cell decomposition. The Gauss method shows that for any $U \subset V$ there exists *a unique* base $\{u_1, u_2, \dots, u_m\}$, of U, such that the corresponding matrix $A_U = (a_{\mu\nu})$ is *a strong step matrix*, that is, $(a_{\mu\nu})$ contains the identity $m \times m$-submatrix, say in columns $(j_1, j_2, \dots, j_m)$, such that each row vanishes at the left of the unity coming from this identity submatrix (i. e. for all $i = 1, 2, \dots, m$ we have $a_{ij} = 0 \quad \forall\, j < j_i$).

Exercice 6.14. Prove that different strong step matrices give different subspaces in V.

So, there exist a bijection between $\mathrm{Gr}(m,d)$ and the set of all strong step matrices. The last one splits into disjoint union of the affine spaces. Namely, all strong step matrices that contain the identity submatrix in the fixed columns $I = (i_1, i_2, \dots, i_m)$ have exactly

$$mn - m^2 - (i_1 - 1) - (i_2 - 2) - \cdots - (i_m - m) = \dim \mathrm{Gr}(m,d) - \sum_{\nu=1}^{m} (i_\nu - \nu)$$

free entries to put there any numbers from k. Hence, topologically, $\mathrm{Gr}(m,d)$ is a disjoint union of $\binom{d}{m}$ affine cells $\mathfrak{A}_I$ enumerated by length m increasing subsets $I \subset \{1, 2, \dots, d\}$. The I-th cell is homeomorphic to the affine space and has codimension $\sum\limits_{\nu=1}^{m} (i_\nu - \nu)$ in $\mathrm{Gr}(m,d)$.

Young diagram notations. Traditionally, the ν-th difference $(i_\nu - \nu)$ in a length m increasing subset $I \subset \{1, 2, \dots, d\}$ is denoted by $\lambda_{m+1-\nu}$ in order to have *a partition* $(d-m) \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_m \geqslant 0$ instead of the increasing collection $1 \leqslant i_1 < i_2 < \dots < i_m \leqslant d$. By the definition, a partition λ is a not increasing collection of non negative integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$. A *length* $\ell(\lambda)$ is a number of the last non zero element in λ. A *weight* of λ is $|\lambda| \overset{\text{def}}{=} \sum\limits_\nu \lambda_\nu$. A *Young diagram* of λ is a `flushleft`'ed collection of cell rows whose lengths are $\lambda_1, \lambda_2, \dots, \lambda_m$. For example, the partition $\lambda = (5, 4, 4, 1)$ has length $\ell(\lambda) = 4$ weight $|\lambda| = 14$ and Young diagram  . In other words, the partition is just the Young diagram, its weight is the number of cells, and its length is the number of rows.

Exercice 6.15. Check that there is a bijection between the length m increasing subsets $I \subset \{1, 2, \dots, d\}$ and the Young diagrams contained in the rectangle of size $m \times (d-m)$.

In terms of Young diagrams, the grassmannian $\mathrm{Gr}(m, d)$ is decomposed into the disjoint union of affine cells enumerated by the Young diagrams contained in the $m \times (d-m)$ - rectangle. λ-th cell has codimension $|\lambda|$ and is isomorphic to $\mathbb{A}^{m(d-m)-|\lambda|}$. In particular there is a unique 1-point cell, which has codimension $m\,(d-m)$ and corresponds to the rectangle itself, and a unique open dense cell of codimension zero, which corresponds to the empty diagram and coincides with the standard affine chart $\mathfrak{U}_{\{1,2,\dots,m\}}$. A topological closure of λ-th affine cell is called *a Schubert cycle* and denoted by σ_λ.

Exercice 6.16. Check that 6 Schubert cycles on the Plücker quadric $\mathrm{Gr}(2, 4) \simeq Q_P \subset \mathbb{P}_5$ are: $\sigma_{00} = Q_P$; $\sigma_{22} = p = (0 : 0 : 0 : 0 : 0 : 1) \in \mathbb{P}_5$; $\sigma_{10} = Q_P \cap T_p Q_P$; $\sigma_{11} = \pi_\alpha(O)$, where $O = (0 : 0 : 0 : 1) \in \mathbb{P}_3$; $\sigma_{20} = \pi_\beta(\Pi)$, where $\Pi \subset \mathbb{P}_3$ is given by $x_0 = 0$; $\sigma_{21} = \pi_\alpha(O) \cap \pi_\beta(\Pi)$.

Remark for who studied the topology. Clearly, the Schubert cycles generate the homologies of $\mathrm{Gr}(m, d)$. Moreover, for the complex grassmannian they form the base of $H_*(\mathrm{Gr}(m, \mathbb{C}^d), \mathbb{Z})$ over $\mathbb{Z}$. It is a nice (but not simle) combinatorial problem, to express the (topological) intersections of the Schubert cycles in terms of the Schubert cycles. The corresponding technique is known as *a Schubert calculus* and is described in Griffits-Harris, Fulton-Harris and Macdonald. Roughly speaking, *the homology ring* $H_*(\mathrm{Gr}(m, \mathbb{C}^d), \mathbb{Z})$ is isomorphic to the truncated ring of symmetric polynomials via sending the Schubert cycles σ_λ to the Schur polynomials s_λ.

Exercice 6.17*. Show that $\sigma_{10}\sigma_{21} = \sigma_{20}^2 = \sigma_{11}^2 = \sigma_{22}$, $\sigma_{20}\sigma_{11} = 0$, and $\sigma_{10}^2 = \sigma_{20} + \sigma_{11}$ in the integer homology ring of complex $\mathrm{Gr}(2, 4)$.

Hint. To calculate σ_{10}^2 realize σ_{10} as $\sigma_{1,0}(\ell) = Q_P \cap T_{\mathfrak{u}(\ell)} Q_P = \{\ell'' \subset \mathbb{P}_3 \mid \ell \cap \ell'' \neq \varnothing\}$. Then, taking two *intersecting* lines ℓ and ℓ' in $\mathbb{P}_3$ we get $\sigma_{10}(\ell) \cap \sigma_{10}(\ell') = \pi_\alpha(O) \cup \pi_\beta(\Pi)$, where $O = \ell \cap \ell'$ and Π is spaned by ℓ and ℓ'.

Plücker equations The image of general Plücker embedding $\mathrm{Gr}(m, V) \xhookrightarrow{\mathfrak{u}} \mathbb{P}(\Lambda^m V)$, that is a variety of decomposable skew-symmetric quadratic polynomials, can be explicitly described by Plücker quadratic relations considered in Example 5.18 and generalizing those we written in Example 6.10, Lemma 6.2 for $\dim V = 4$. In this case, potentially, we can write 4 general Plücker relations from Example 5.18: they correspond to all possible distributions of $4 = 3 + 1$ indexes $\{1, 2, 3, 4\}$ between K and J.

Exercice 6.18. Check that they all produce the same quadratic equation $A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = 0$ on 2×2 - minors A_{ij} of 2×4 - matrix A.

Working example: Veronese curves[1]

<u>Linear span of Veronese curve.</u> Recall (see Example 5.14) that the Veronese curve $C_n \subset \mathbb{P}_n$ is the image of Veronese's map

$$\mathbb{P}_1 = \mathbb{P}(U^*) \xhookrightarrow{v_n} \mathbb{P}_n = \mathbb{P}(S^n U^*) \ ,$$

which takes a linear form $\alpha = \alpha_0 t_0 + \alpha_1 t_1 \in U^*$ to its n-th power:

$$v_n(\alpha) = \alpha^n = \sum \binom{n}{i} \alpha_0^i \alpha_1^{n-i}\, t_0^i t_1^{n-i} \ .$$

As in Example 5.14, we write polynomials $f \in S^n U^*$ in the form $f = \sum \binom{n}{i} a_i\, t_0^i t_1^{n-i}$ and use a_i as homogeneous coordinates on $\mathbb{P}_n = \mathbb{P}(S^n U^*)$.

Given an arbitrary hyperplane $\pi = \{a \in \mathbb{P}_n \mid \sum A_i a_i = 0\}$, the intersection $C_n \cap \pi$ consists of all $a = a(\alpha)$ that satisfy the equation $\sum\limits_i A_i \alpha_0^i \alpha_1^{n-i} = 0$ whose left side is non zero polynomial of degree n. So, a hyperplane section of Veronese's curve always consists[2] of n points counted with appropriate multiplicities (typically, of n *distinct* points). In particular, the linear span of any $(n+1)$ Veronese's curve points gives the whole of $\mathbb{P}_n$.

<u>Corollary 7.1.</u> [3] *To prove that some linear in f assertion holds for all polynomials f, it is enough to verify it only for all powers of all linear forms.* □

Exercice 7.2. Use Arongold principle to give another proof for Taylor formula (21).

<u>Example 7.3: Projecting twisted cubic.</u> Let us describe all plane projections of the twisted cubic $C_3 \subset \mathbb{P}_3 = \mathbb{P}(S^3 U^*)$. Up to a projective isomorphism, the projection does not depend on the choice of a target plane as soon as the center is fixed, because the projection of one target plane onto another gives an linear isomorphism between the projection images. Let $p = p(t) = \ell_1(t)\ell_2(t)\ell_3(t) \in \mathbb{P}_3 = \mathbb{P}(S^3 V^*)$ be a projection center. After some parameter change we can suppose that either $\ell_1 = \ell_2 = \ell_3 = t_0^2$, (this means that $p \in C$), or $\ell_1 = \ell_2 = t_0$, $\ell_3 = t_1$ ([4]), or $\ell_1 = (t_0 + t_1)$, $\ell_2 = (t_0 + \omega\, t_1)$, $\ell_3 = (t_0 + \omega^2 t_1)$, where $\omega = \sqrt[3]{1} \neq 1$ (that is, $p(t) = t_0^3 + t_1^3$ has 3 distinct roots).

In the first case $p = (1:0:0:0)$; take a target plane to be $a_0 = 0$ with the coordinates $(x_0 : x_1 : x_2) = (a_1 : a_2 : a_3)$. Then the projection is given by parametric equations $(x_0 : x_1 : x_2) = (\alpha_0^2 : \alpha_0\alpha_1 : \alpha_1^2)$ and coincides with the plane Veronese conic $x_0 x_2 = x_1^2$.

In the second case $p = (0:1:0:0)$; take a target plane to be $a_1 = 0$ with the coordinates $(x_0 : x_1 : x_2) = (a_0 : a_2 : a_3)$. Then the projection is given by parametric equations $(x_0 : x_1 : x_2) = (\alpha_0^3 : \alpha_0\alpha_1^2 : \alpha_1^3)$ and in the affine chart $\{x_0 = 1\}$ it turns into $x = \alpha^2$, $y = \alpha^3$, where $x = x_1/x_0$, $y = x_2/x_0$, and $\alpha = \alpha_1/\alpha_0$. So, we get a curve $y^2 = x^3$ or, in the homogeneous coordinates, $x_2^3 = x_3^2 x_0$. This curve is called *a cuspidal cubic*, because of the singularity form at the origin.

In the third case $p = (1:0:0:1)$; take a target plane $\pi = \{a_2 = 0\}$ with the coordinates $(x_0 : x_1 : x_2) = ((a_0 - a_1) : a_1 : a_2)$ (the first three coordinates w.r.t. the base $\{t_0^3 , 3\,t_0^2 t_1 , 3\,t_0 t_1^2 , t_0^3 + t_1^3\}$). So, the projection from $p = t_0^3 + t_1^3$ gives the parameterized curve $(x_0 : x_1 : x_2) = \big((\alpha_0^3 - \alpha_1^3) : \alpha_0^2\alpha_1 : \alpha_0\alpha_1^2\big)$. In affine chart $x_0 = 1$ we get, like above, $x = \alpha/(1-\alpha^3)$, $y = \alpha^2/(1-\alpha^3)$ or $xy = x^3 - y^3$. This curve has a self intersection point at the origin and is called *a nodal* cubic.

<u>Geometric description of rational curves.</u> A plane curve $C \subset \mathbb{P}_2$ is called *rational* if there are 3 coprime homogeneous polynomials $p_0(t)$, $p_1(t)$, $p_2(t)$ of the same degree in $t = (t_0 : t_1)$ such that a map $\mathbb{P}_1 \xrightarrow{\alpha \mapsto (p_0(\alpha):p_1(\alpha):p_2(\alpha))} \mathbb{P}_2$ gives (maybe, after removing some finite sets of points from $\mathbb{P}_1$ and C) a bijection between $\mathbb{P}_1$ and C .

Exercice 7.4. Intersecting C with lines, show that $\deg C = \deg p_i$.

When $\deg p_i = d$, a map $(t_0^d : t_0^{d-1} t_1 : \ldots : t_0 t_1^{d-1} : t_1^d) \longmapsto (p_0(t) : p_1(t) : p_2(t))$ defines a projection of the Veronese curve $C_d \subset \mathbb{P}_d$ into some plane $\mathbb{P}_2 \subset \mathbb{P}_d$. So, we have

<u>Claim 7.5.</u> *Each rational plane curve of degree d is an appropriate projection of the Veronese curve $C_d \subset \mathbb{P}_d$.* □

<u>Corollary 7.6.</u> *A smooth plane cubic curve is not rational.*

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[1] WARNING!!! In this lecture we suppose that the ground field k is algebraically closed and has zero characteristic.

[2] recall that we suppose the ground field to be algebraically closed

[3] it is known as *Arongold principle*

[4] geometrically, this means that p lies on a tangent line to C_3

Proof. Rational cubic curve is a plane projection of the twisted cubic $C_3 \subset \mathbb{P}_3$. But such a projection is either a conic or a singular cubic. $\square$

Simplices inscribed into Veronese curve. For $1 \leqslant i \leqslant n$ let $p_i = \xi_i^n$ be any n distinct points on the Veronese curve $C_n \subset \mathbb{P}_n = \mathbb{P}(S^nU^*)$. For each i consider the pencil of hyperplanes passing through $(n-2)$-dimensional face $(p_1, \dots p_{i-1}, p_{i+1}, \dots, p_n)$ opposite to p_i in the $(n-1)$ dimensional simplex $(p_1, p_2, \dots, p_n)$. These n pencils are parameterized uniformly by the points of $\mathbb{P}_1^\times = \mathbb{P}(U)$ as follows. For any $\xi \in \mathbb{P}(U^*)$ denote by $\widehat{\xi} \in \mathbb{P}(U)$ the annihilator[5] $\operatorname{Ann}(\xi)$ and for each i take the product $\tau_i = \widehat{\xi}_1, \cdots, \widehat{\xi}_{i-1}\widehat{\xi}_{i+1}, \cdots, \widehat{\xi}_1 \in S^{n-1}U$. Define a plane $\pi_i(u) \subset \mathbb{P}(S^nU^*)$, which corresponds to $u \in \mathbb{P}(U)$ in i-th pencil, as the annihilator of $u\tau_i \in S^nU$. This means that

$$\pi_i(u) = \left\{ f(t) \in S^nU^* \mid \widetilde{f}(\widehat{\xi}_1, \dots, \widehat{\xi}_{i-1}, u, \widehat{\xi}_{i+1}, \dots, \widehat{\xi}_n) = 0 \right\}$$

where $\widetilde{f}$ is the full polarization of f considered as a symmetric multilinear form on U. In particular, for $f(t) = \zeta^n(t) \in C_n$ we have

$$(\zeta^n)^\sim(\widehat{\xi}_1, \dots, \widehat{\xi}_{i-1}, u, \widehat{\xi}_{i+1}, \dots, \widehat{\xi}_n) = \zeta(u) \prod_{\nu \neq i} \zeta\left(\widehat{\xi}_\nu\right)$$

So, for any $u \in \mathbb{P}(U)$ the plane $\pi_i(u)$ pass through all $p_\nu = \xi_\nu^n$ with $\nu \neq i$ and through the point $p = \xi^n \in C_n$ whose ξ annihilate u (i. e. such that $u = \widehat{\xi}$). In other words,

$$C_n = \bigcup_u \pi_1(u) \cap \pi_2(u) \cap \cdots \cap \pi_n(u) \tag{25}$$

Since a linear isomorphism between projective lines is uniquely defined by the images of any 3 distinct points, we get the following corollary.

Claim 7.7. *The Veronese curve is uniquely recovered from any its $(n+3)$ distinct points $a, b, c, p_1, p_2, \dots, p_n$ as follows. Consider n hyperplane pencils through the $(n-2)$-dimensional faces of the inscribed simplex $(p_1, p_2, \dots, p_n)$ and parameterize them uniformly by $u \in \mathbb{P}_1$ in such a way that the hyperplanes passing through a, b, c appear in each pencil when $u = 0, 1, \infty$. Then C_n coincides with the incidence graph (25), where u runs through the parameter line $\mathbb{P}_1$.* $\square$

Projective linear group action. The group $PGL_2(k)$ acts on $\mathbb{P}(S^nU^*)$ via linear variable changes $(t_0, t_1) \longmapsto (at_0 + bt_1, ct_0 + dt_1)$. Projective automorphisms $\mathbb{P}(S^nU^*) \xrightarrow{\sim} \mathbb{P}(S^nU^*)$ which are induced by these substitutions send the Veronese curve to itself. We call them *reparameterizations* of the Veronese curve.

Claim 7.8. *Let $p_1, p_2, \dots, p_n, a, b, c \in \mathbb{P}_n = \mathbb{P}(S^nU^*)$ be any $n+3$ points with no $(n+1)$ on the same hyperplane. Then there exists a projective linear isomorphism $\mathbb{P}_n \xrightarrow{\sim} \mathbb{P}_n$ that sends these points onto Veronese curve C_n; this isomorphism is unique up to a reparameterization of the Veronese curve.*

Proof. For each $i = 1, 2, \dots, n$ identify $\mathbb{P}_1 = \mathbb{P}(U)$ with a pencil of hyperplanes through $p_1, \dots p_{i-1}, p_{i+1}, \dots, p_n$ by sending $u = e_0, e_1, (e_1 - e_0)$ to the hyperplanes that contain a, b, c and denote by $\pi_i(u)$ the u-th hyperplane in the i-pencil. Let the hyperplane $(p_1, p_2, \dots, p_n)$ appear in i-th pencil when $u = u_i$. Then $u_1, u_2, \dots, u_n \in \mathbb{P}(U)$ are mutually distinct by the following reason. Consider 2-dimensional plane $\Pi = (a, b, c)$ and denote by q_1 and q_2 its intersection with i-th and j-th $(n-2)$-dimensional faces of $(p_1, p_2, \dots, p_n)$. Sitting in this plane, we will see the picture shown on 7.1. Our i-th and j-th pencils of hyperplanes are represented inside Π by the pencils of lines passing through q_1, q_2 and the hyperplane $(p_1, p_2, \dots, p_n)$ is represented by the line (q_1q_2) (compare this construction with drawings from the **lecture 2**).

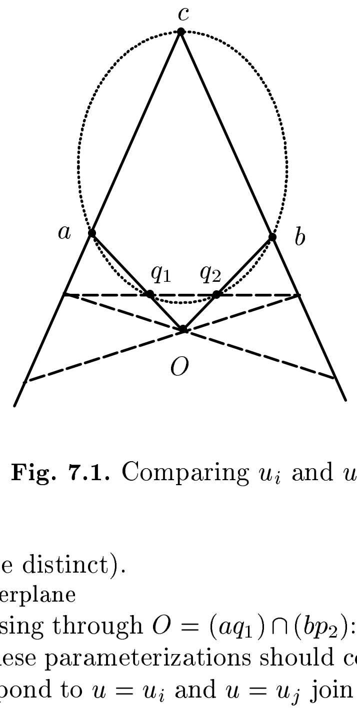

Fig. 7.1. Comparing u_i and u_j.

Exercice 7.9. Show that no 3 of q_1, q_2, a, b, c are collinear (in particular, q_1, q_2 are distinct).

Hint. In the contrary case some $(n+1)$ of $p_1, p_2, \dots, p_n, a, b, c$ would lie on a hyperplane

There are two ways to identify the parameter line $\mathbb{P}(U)$ with the pencil of lines passing through $O = (aq_1) \cap (bp_2)$: we can send u either to the line through $\pi_i(u) \cap (bc)$ or to the line through $\pi_j(u) \cap (ac)$. These parameterizations should coincide, because they attach the same u to the lines through a, b, c. The lines which correspond to $u = u_i$ and $u = u_j$ join O with $\pi_i(u_i) \cap (bc) = (q_1q_2) \cap (bc)$ and with $\pi_j(u_j) \cap (ac) = (q_1q_2) \cap (ac)$. Since O, q_1 and q_2 are not collinear, these 2 lines are distinct, i. e. $u_i \neq u_j$.

Now, denote by Γ the incidence graph (25) build from our current pencils of hyperplanes. Let $u_i = \operatorname{Ann}(\xi_i)$ for $\xi_1, \xi_2, \dots, \xi_n \in U^*$. By the Lemma 1.11, there exists a unique projective linear automorphism $\mathbb{P}_n \xrightarrow{\sim} \mathbb{P}_n$ which sends

[5] if $\xi = \alpha_0 t_0 + \alpha_1 t_1$, then $\widehat{\xi} = \alpha_1 e_0 - \alpha_0 e_1$, where $\{e_o, e_1\} \subset U$ is the base of U dual to $\{t_0, t_1\} \subset U^*$

$p_i \longmapsto \xi_i^n$ for $1 \leqslant i \leqslant n$, $a \longmapsto t_1^n$, and $b \longmapsto t_0^n$. It identifies Γ with the Veronese curve, because it sends the hyperplane pencil through $p_1, \dots p_{i-1}, p_{i+1}, \dots, p_n$ to the one through $\xi_1^n, \dots \xi_{i-1}^n, \xi_{i+1}^n, \xi_n^n$ in such a way that $\pi_i(u)$ goes to the hyperplane through ξ^n as soon as $\xi = \operatorname{Ann} u$. Indeed, this takes place for $u = e_1$, $u = e_0$ and $u = u_i$ when a, b and p_i go to t_0^n, t_1^n and ξ_i^n. Hence, this holds for each u and for $u = e_1 - e_0$ we have $c \longmapsto (t_0 + t_1)^n$. This proves the existence.

Uniqueness follows from the above construction as well. Namely, after appropriate parameter change we can suppose that the isomorphism in question sends a, b, c to t_0^n, t_1^n and $(t_0 + t_1)^n$. So, it induces the uniform parameterization of hyperplane pencils trough p_ν and this parameterization coincides with the above one. So, the images of $p_1, p_2, \dots, p_n$ are uniquely recovered as $\xi_i = \operatorname{Ann}(u_i)$. $\square$

Commutative algebra draught

<u>Noetherian rings.</u> We write $(f_1, f_2, \dots, f_m)$ for the ideal $\{g_1 f_1 + g_2 f_2 + \cdots + g_m f_m \mid g_\nu \in A\}$ spanned (as A-module) by $\{f_1, f_2, \dots, f_m\} \subset A$. A commutative ring A is called *Noetherian*, if it satisfies the next lemma:

<u>Lemma 8.1</u>. *The following properties of a commutative ring A are mutually equivalent*

1) *any set of elements f_ν contains some finite subset that generates the same ideal as the initial set;*
2) *any ideal admits a finite set of generators;*
3) *for any infinite chain of embedded ideals $I_1 \subset I_2 \subset I_3 \subset \cdots$ there exists $n \in \mathbb{N}$ such that $I_\nu = I_n \quad \forall\, \nu \geqslant n$.*

Proof. Clearly, (1) $\Rightarrow$ (2). To deduce (3) from (2), take a finite set of generators for the ideal $I = \bigcup I_\nu$; since they belong to some I_n, we have $I_\nu = I_n = I \;\forall\, \nu \geqslant n$. Finally, (1) follows from (3) applied to the chain $I_n = (f_1, f_2, \dots, f_n)$, where f_i are chosen from $\{f_\nu\}$ in order to have $f_\nu \notin (f_1, f_2, \dots, f_{\nu-1})$. $\square$

<u>Theorem 8.2</u>. (Hilbert's base theorem). *If A is Noetherian, then $A[x]$ is Noetherian too.*

Proof. Let $I \subset A[x]$ be an ideal. We write $L_d \subset A$ for a set of leading coefficients of all degree d polynomials in I. Clearly, each L_d and $L_\infty \overset{\text{def}}{=} \bigcup_d L_d$ are ideals in A. Let L_∞ be generated by $a_1, a_2, \dots, a_s \in A$ coming from $f_1^{(\infty)}, f_2^{(\infty)}, \dots, f_{s_\infty}^{(\infty)} \in I$ and let $\max_\nu(\deg f_\nu) = m$. Similarly, write $f_1^{(k)}, f_2^{(k)}, \dots, f_{s_k}^{(k)}$ for the polynomials whose leading coefficients span the ideal L_k for $0 \leqslant k \leqslant m-1$. To finish the proof, make: $\square$

Exercice 8.3. Show that I is spanned by $s_0 + \cdots + s_{m-1} + s_\infty$ polynomials $f_\nu^{(\mu)}$ just constructed.

<u>Corollary 8.4</u>. *A polynomial ring $A[x_1, x_2, \dots, x_n]$ over a Noetherian ring A is Noetherian as well.* $\square$

<u>Corollary 8.5</u>. *Any finitely generated k-algebra is Noetherian for any field k.*

Proof. A polynomial algebra $k[x_1, x_2, \dots, x_n]$ is Noetherian by the previous corollary. Any its factor algebra A is Noetherian as well: full preimage of any ideal $I \subset A$ under the factorizing morphism $k[x_1, x_2, \dots, x_n] \twoheadrightarrow A$ is an ideal in $k[x_1, x_2, \dots, x_n]$, i. e. admits a finite set of generators, whose classes span I over A, certainly. $\square$

<u>Integrality.</u> Let $A \subset B$ be two commutative rings. An element $b \in B$ is called *integer* over A, if it satisfies the conditions from Lemma 8.6 below. If all $b \in B$ are integer over A, then B is called an *integer extension* of A or an *integer A-algebra*.

<u>Lemma 8.6</u>. *The following properties of an element $b \in B$ are pairwise equivalent:*

1) $b^m = a_1 b^{m-1} + \cdots + a_{m-1} b + a_0$ *for some $m \in \mathbb{N}$ and some $a_1, a_2, \dots, a_m \in A$;*
2) *A-module spanned by all nonnegative powers $\{b^i\}_{i \geqslant 0}$ admits a finite set of generators;*
3) *there exist a finitely generated faithful*[1] *A-submodule $M \subset B$ such that $bM \subset M$.*

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are trivial. To deduce (1) from (3), let $\{e_1, e_2, \dots, e_m\}$ generate M over A and let the multiplication $M \xrightarrow{m \mapsto bm} M$ be presented by a matrix Y, i. e. write $(be_1, be_2, \dots, be_m) = (e_1, e_2, \dots, e_m) \cdot Y$.

Note that if A-linear map $M \xrightarrow{\varkappa} M$ takes $(e_1, e_2, \dots, e_m) \longmapsto (e_1, e_2, \dots, e_m) \cdot X$, where X is a square matrix with entries in A, then there is an inclusion $(\det X) \cdot M \subset \operatorname{im} \varkappa$, which follows from the Laplace identity $\det X \cdot \mathrm{Id} = \widehat{X} \cdot X$ applied to $(e_1, e_2, \dots, e_m)$.

In our case the zero operator $M \xrightarrow{\varkappa} 0$ can be presented by the matrix $X = b \cdot \mathrm{Id} - Y$. So, the multiplication by $\det(b \cdot \mathrm{Id} - Y)$ annihilates M. Since M is faithful, $\det(b \cdot \mathrm{Id} - Y) = 0$. This is a polynomial equation on b with the coefficients in A and the leading term b^n as required in (1). $\square$

<u>Example 8.7</u>: Integer algebraic numbers. Let $K \supset \mathbb{Q}$ be a finite dimensional[2] field extension; then elements $z \in K$ are called ***algebraic numbers***. Such a number z is integer over $\mathbb{Z}$ iff there are some $\vartheta_1, \vartheta_2, \dots, \vartheta_m \in K$ such that the multiplication by z sends their $\mathbb{Q}$-linear span to itself and is presented there by a matrix whose entries belong to $\mathbb{Z}$.

<u>Example 8.8</u>: Invariants of a finite group action. Let a finite group $\mathfrak{G}$ act on a k-algebra B via k-algebra automorphisms $B \xrightarrow{g} B$, $g \in \mathfrak{G}$, and let $A = B^{\mathfrak{G}} = \{a \in B \mid ga = a \;\; \forall\, g \in \mathfrak{G}\,\}$ be the subalgebra of $\mathfrak{G}$-invariants. Then B is an integer extension of A. Indeed, if $b_1, b_2, \dots, b_s \in B$ form a $\mathfrak{G}$-orbit of any given $b = b_1 \in B$, then the polynomial $\beta(t) = \prod (t - b_i)$ is monic[3], lies in $A[t]$, and annihilates b.

[1] A-module M is called *faithful*, if $aM = 0$ implies $a = 0$ for $a \in A$

[2] as a vector space over $\mathbb{Q}$

[3] a polynomial is called *monic* or *unitary*, if its leading coefficient equals 1

Integer closures. A set of all $b \in B$ that are integer over a subring $A \subset B$ is called *an integer closure* of A in B. If this closure coincides with A, then A is called *integrally closed* in B.

Lemma 8.9. *The integer closure of A is a subring in B (in particular, ab is integer for any $a \in A$ as soon as b is integer). If $C \supset B$ is an other commutative ring and $c \in C$ is integer over an integer closure of A in B, then c is integer over A as well (in particular, any integer B-algebra is an integer A-algebra as soon as B is an integer A-algebra).*

Proof. If $p^m = x_{m-1}\, p^{m-1} + \cdots + x_1\, p + x_0$, $q^n = y_{n-1}\, q^{n-1} + \cdots + y_1\, q + y_0$ for $p, q \in B$, $x_\nu, y_\mu \in A$, then A-module spanned by $p^i q^j$ with $0 \leqslant i \leqslant (m-1)$, $0 \leqslant j \leqslant (n-1)$ is faithful (it contains 1) and goes to itself under the multiplication by both $p+q$ and pq. Similarly, if $c^r = z_{r-1}\, c^{r-1} + \cdots + z_1\, c + z_0$ and all z_ν are integer over A, then a multiplication by c preserves a faithful A-module spanned by a sufficient number of products $c^i z_1^{j_1} z_2^{j_2} \cdots z_r^{j_r}$. $\square$

Corollary 8.10. [4] *For any two commutative rings $A \subset B$ let $f(x), g(x) \in B[x]$ be two monic polynomials. Then all coefficients of $h(x) = f(x)g(x)$ are integer over A iff all coefficients of both $f(x)$, $g(x)$ are integer over A.*

Proof. There exists[5] a ring $C \supset B$ such that $f(x) = \prod(t - \alpha_\nu)$ and $g(x) = \prod(t - \beta_\mu)$ in $C[x]$ for some $\alpha_\nu, \beta_\mu \in C$. By Lemma 8.9, all coefficients of $h(x) = \prod(t - \alpha_\nu) \prod(t - \beta_\mu)$ are integer over $A \iff$ all α_ν, β_μ are integer over $A \iff$ all coefficients of $f(x)$ and $g(x)$ are integer over A. $\square$

Lemma 8.11. *Let $B \supset A$ be integer over A. If B is a field, then A is a field. Vise versa, if A is a field and there are no zero divisors in B, then B is a field.*

Proof. If B is a field integer over A, then any non zero $a \in A$ has an inverse $a^{-1} \in B$, which satisfy an equation $a^{-m} = \alpha_1\, a^{1-m} + \cdots + \alpha_{m-1}\, a^{-1} + \alpha_0$ with $\alpha_\nu \in A$. We multiply the both sides by a^{m-1} and get $a^{-1} = \alpha_1 + \cdots + \alpha_{m-1}\, a^{m-2} + \alpha_0\, a^{m-1} \in A$. Conversely, if A is a field and B is an integer A-algebra, then all non negative integer powers b^i of any $b \in B$ form a finite dimensional vector space V over A. If $b \neq 0$ and there are no zero divisors in B, then $x \longmapsto bx$ is an injective linear operator on V, that is, an isomorphism. A preimage of $1 \in V$ is b^{-1}. $\square$

Example 8.12: algebraic elements and minimal polynomials. If $A = k$ is a field and $B \supset k$ is a k-algebra, then $b \in B$ is integer over k iff b satisfy a polynomial equation $f(b) = 0$ for some $f \in k[x]$. Traditionally, such b is called *algebraic* over k rather than *integer*. We write $k[b]$ for a k-linear span of nonnegative integer powers $\{b^n\}_{n \geqslant 0}$ and $k(b)$ for a k-linear span of all integer powers $\{b^n\}_{n \in \mathbb{Z}}$ (if such exist in B). Clearly, $k[b] \subset B$ is the minimal k-subalgebra containing 1 and b; in other terms, $k[b] = \mathrm{im}\,(\mathrm{ev}_b) = k[x]/\ker(\mathrm{ev}_b)$, where $\mathrm{ev}_b : k[x] \xrightarrow{f(x) \mapsto f(b)} B$ is *an evaluation homomorphism*. If b is algebraic, then $\ker(\mathrm{ev}_b) = (f) \overset{\text{def}}{=} f \cdot k[x]$ for some $0 \neq f \in k[x]$, because $k[x]$ is a principal ideal domain. This f is fixed uniquely as a monic polynomial of lowest degree such that $f(b) = 0$; it is called *the minimal polynomial* of b over k. Note that $k[b]$ is a $\deg(f)$-dimensional vector space over k with a basis $1, b, b^2, \dots, b^{\deg(f)-1}$. If B does not have zero divisors, then $k[b]$ is a field by Lemma 8.11; in particular, the minimal polynomial of b is irreducible in this case. If b is not algebraic, then $\ker(\mathrm{ev}_b) = 0$ and $k[b] \simeq k[x]$ is a polynomial ring. In particular, it is an infinite dimensional vector space over k and it is not a field. This phenomenon will be generalized in Lemma 8.17 below.

Lemma 8.13. *Let $k = Q(A)$ be the fraction field for a commutative ring A without zero divisors, B be any k-algebra, and $b \in B$ be algebraic over k with minimal polynomial $f \in k[x]$. If b is integer over A, then all coefficients of f are integer over A.*

Proof. Since b is integer, $g(q) = 0$ for some monic $g \in A[x]$. Then $g = fh$ in $k[x]$ for some monic $h \in k[x]$ and all the coefficients of g, h are integer over A by Corollary 8.10. $\square$

Normal rings. A commutative ring A without zero divisors is called *normal*, if there are no $b \in Q(A) \setminus A$ integer over A. Certainly, any field is normal.

Exercice 8.14. Show that the ring $\mathbb{Z}$ is normal.

Hint. A polynomial $a_0 t^m + a_1 t^{m-1} + \cdots + a_{m-1} t + a_m \in \mathbb{Z}[t]$ annihilates a fraction $p/q \in \mathbb{Q}$ with coprime $p, q \in \mathbb{Z}$ only if $q | a_0$ and $p | a_m$

Corollary 8.15. *Let A be a normal ring and $f \in A[x]$ be factorized in $Q(A)[x]$ as $f = g\,h$. If both g, h are monic, then necessarily $g, h \in A[x]$.*

Proof. Indeed, all the coefficients of g, h are integer over A by Corollary 8.10. $\square$

Corollary 8.16. *Let A be normal ring with the fraction field $k = Q(A)$ and B be any k-algebra. Then $b \in B$ is integer over A iff it is algebraic over k and its minimal polynomial (over k) lies in $A[x]$.*

Proof. This follows immediately from Lemma 8.13. $\square$

[4] it is known as Gauss - Kronecker - Dedekind lemma

[5] For any commutative ring A and any monic non constant $f(x) \in A[x]$ there exists a commutative ring $C \supset A$ such that $f(x) = \prod(x - c_\nu)$ in $C[x]$ for some $c_\nu \in C$. It is constructed inductively as follows. Consider a factor ring $B = A[x]/(f)$ (which contains A as the congruence classes of constants) and put $b \overset{\text{def}}{=} x \pmod f \in B$. Then $f(b) = 0$ in $B[x]$. Hence the residue after dividing $f(x)$ by $(x - b)$ in $B[x]$ vanishes and we get the factorization $f(x) = (x - b)h(x)$ with $h(x) \in B[x]$. Now repeat the procedure for h, B instead of f, A e. t. c.

Finitely generated commutative k-algebras. Let k be any field. A commutative k-algebra B is called *finitely generated*, if there is a k-algebra epimorphism $k[x_1, x_2, \dots, x_m] \xrightarrow{\pi} B$. Images $b_i = \pi(x_i) \in B$ are called *algebra generators* for B over k.

Lemma 8.17. *A finitely generated k-algebra B is a field only if each $b \in B$ is algebraic over k.*

Proof. Let B be generated by $\{b_1, b_2, \dots, b_m\}$. We use induction over m. The base case: $m = 1$, $B = k[b]$, was considered in Example 8.12. Let $m > 1$. If b_m is algebraic over k, then $k[b_m]$ is a field and B is algebraic over $k[b_m]$ by the inductive assumption. Hence, by Lemma 8.9, B is algebraic over k as well.

So, we only have to show that b_m *must* be algebraic over k. Suppose the contrary: let b_m be not algebraic, that is $k(b_m)$ be isomorphic to the field $k(x)$, of rational functions in 1 variable, via sending $b_m \longmapsto x$. Then, by the inductive assumption, B is algebraic over $k(b_m)$ and each of $b_1, b_2, \dots, b_{m-1}$ satisfies a polynomial equation with coefficients in $k(b_m)$. Multiplying these equations by appropriate polynomials in b_m, we can put their coefficients into $k[b_m]$ and make all their leading coefficients to be equal to the same polynomial, which we denote by $p(b_m) \in k[b_m]$.

Now, B is integer over the subalgebra $F \subset B$ generated over k by b_m and $q = 1/p(b_m)$. By Lemma 8.11, F is a field. So, there exists a polynomial $g \in k[x_1, x_2]$ such that $g(b_m, q)$ is inverse to $1 + q$ in F. Let us write the rational function $g(x\,,\,1/p(x)) \in k(x)$ as $h(x)/p^k(x)$, where $h \in k[x]$ is coprime to $p \in k[x]$. Multiplying the both sides of

$$\left(1 + \frac{1}{p(b_m)}\right) \frac{h(b_m)}{p^k(b_m)} = 1$$

by $p^{k+1}(b_m)$, we get for b_m a polynomial equation $h(b_m)\,(p(b_m)+1) = p^{k+1}(b_m)$, which is nontrivial, because $h(x)(1+p(x))$ is not divisible by $p(x)$. Hence, b_m should be algebraic over k. $\square$

Hilbert's Nullstellensatz. Let us write $V(I) = \{a \in \mathbb{A}_n \mid f(a) = 0 \quad \forall f \in I\} \subset \mathbb{A}^n$ for affine algebraic variety defined by a system of polynomial equations[6] $I \subset k[x_1, x_2, \dots, x_n]$. Vise versa, for any subset $V \subset \mathbb{A}^n$ we write $I(V) = \{f \in k[x_1, x_2, \dots, x_n] \mid f|_V \equiv 0\}$ for a set of all polynomials vanishing along V. Clearly, $I(V)$ is always an ideal and $I(V(I)) \supset I$ for any ideal I. In general, then last inclusion is proper. For example, if $I = (x^2) \in \mathbb{C}[x]$, then $V(I) = \{0\} \subset \mathbb{A}^1(\mathbb{C})$ and $I(V(I)) = (x)$.

Theorem 8.18. (week Nullstellensatz). *If k is algebraically closed, then $V(I) = \varnothing \iff 1 \in I$ for an ideal I.*

Proof. We only have to find a point $p \in \mathbb{A}_n$ such that $f(p) = 0$ for any polynomial f from a given proper ideal $I \subset k[x_1, x_2, \dots, x_n]$. Moreover, it is enough to suppose that I is *maximal*, that is, any $f \notin I$ is invertible modulo I. Indeed, otherwise an ideal J generated by f and I would be proper and strictly larger than I, i. e. we could replace I by J; a finite chain of such replacements leads to some maximal ideal. As soon as I is maximal, the factor algebra $K = k[x_1, x_2, \dots, x_n]/I$ is a field. Hence, any element of K is algebraic over $k \subset K$ by Lemma 8.17. Since k is algebraically closed, this means that any polynomial is (mod I)-congruent to some constant. Let $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ be the constants presenting the basic linear forms $x_1, x_2, \dots, x_n$ (mod I). Then any polynomial $f \in k[x_1, x_2, \dots, x_n]$ is (mod I) congruent to $f(\vartheta_1, \vartheta_2, \dots, \vartheta_n) \in k$. In particular, $f(\vartheta_1, \vartheta_2, \dots, \vartheta_n) = 0$ for any $f \in I$ as required. $\square$

Corollary 8.19. (strong Nullstellensatz). *Under conditions of Theorem 8.18, $f \in I(V(I)) \iff \exists\, k \in \mathbb{N} : f^k \in I$.*

Proof. Only the implication '$\Rightarrow$' for *nonempty* $V(I) \subset \mathbb{A}^n$ is nontrivial. To prove it, we identify $\mathbb{A}^n$ with the hyperplane $t = 0$ in $\mathbb{A}^{n+1}$ coordinated by $(t,\ x_1, x_2, \dots, x_n)$. If $f \in k[x_1, x_2, \dots, x_n] \subset k[t,\ x_1, x_2, \dots, x_n]$ vanishes along $V(I)$, then an ideal $J \subset k[t,\ x_1, x_2, \dots, x_n]$ spanned by I and a polynomial $g(t, x) = 1 - t\,f(x)$ has empty zero set $V(J) \subset \mathbb{A}^{n+1}$, because of $g(x, t) \equiv 1$ on $V(I)$. So, $1 \in J$, i. e. $\exists\ q_0, q_1, \dots, q_s \subset k[t,\ x_1, x_2, \dots, x_n],\ f_1, f_2, \dots, f_s \subset I$: $q_0(x,t)(1 - tf(x)) + q_1(t,x)f_1(x) + \dots + q_s(x,t)f_s(x) = 1$. Then a homomorphism $k[t, x_1, x_2, \dots, x_n] \longrightarrow k(x_1, x_2, \dots, x_n)$ sending $t \longmapsto 1/f(x)$, $x_\nu \longmapsto x_\nu$ takes this equality to $q_1(1/f(x)\,,x)\,f_1(x) + \dots + q_s(1/f(x)\,,x)\,f_s(x) = 1$. Since I is proper, some of $q_\nu(1/f(x), x)$ should have nontrivial denominators. Hence, multiplication by appropriate power f^k leads to an expression we are looking for: $\widetilde{q}_1(x)\,f_1(x) + \dots + \widetilde{q}_s(x)\,f_s(x) = f^k(x)$, where $\widetilde{q}_\nu \in k[x_1, x_2, \dots, x_n]$. $\square$

Factorization. Let A be a commutative ring without zero divisors. An element $q \in A$ is called *irreducible*, if it is not invertible and $q = rs \Longrightarrow$ one of r, s is invertible. An element $p \in A$ is called *prime* if it generates a prime ideal in A, i. e. $A/(p)$ is not zero and does not have zero divisors. In other words, p is prime, if it is not invertible and $p|rs \Longrightarrow p$ divides at least one of r, s.

Exercice 8.20. Check that each prime element is irreducible.

A ring A is called *factorial*, if any $a \in A$ is a finite product of irreducible elements $a = q_1\,q_2\,\cdots\,q_m$ and these irreducible factors are defined by a uniquely up to multiplication by invertible elements, i. e. any two irreducible factorizations $q_1\,q_2\,\cdots\,q_m = a = q_1'\,q_2'\,\cdots\,q_n'$ have $m = n$ and (after appropriate renumbering) $q_i = s_i\,q_i'$ for all i and some invertible[7] $s_i \in A$.

Claim 8.21. *Any factorial ring A is normal.*

[6] certainly, we can always extend I to an ideal spanned by I — this does not effect on $V(I)$

[7] if $a = bs$ for an invertible s, then a and b are called *associated* (with each other) elements of ring

Proof. Let $\alpha/\beta \in Q(A)$ satisfy a polynomial equation $t^n + a_1 t^{n-1} + \cdots + a_{n-1}t + a_n = 0$, where $\alpha_i \in A$. Then α^n is divisible by β. Since A is factorial, each irreducible divisor of β divides α, that is, $\alpha/\beta \in A$. $\square$

Claim 8.22. *A Noetherian ring without zero divisors is factorial iff all its irreducible elements are prime.*

Proof. In a Noetherian ring, any element f is a finite product of irreducible elements: in the contrary case f can written as $f = f_1 g_1$, where f_1 is reducible and can be written as $f = f_2 g_2$ an so on *infinitely* many times producing an infinite chain of strictly increasing ideals $(f) \subset (f_1) \subset (f_2) \subset (f_2) \subset \ \ldots$. Further, if there are no zero divisors and all irreducible elements are prime, then two irreducible factorizations $\prod q_i = \prod q'_j$ have the same number of factors and satisfy $q_i = s_i q'_i$ for some invertible s_i (after appropriate renumbering). Indeed, since prime q'_1 divides $\prod q_i$ it divides some q_i, say q_1. So, $q_1 = s_1 q'_1$, where s_1 is invertible, because q_1 is irreducible. Now we have $q'_1 \left(s_1 \prod\limits_{i\geqslant 2} q_i - \prod\limits_{j\geqslant 2} q'_j \right) = 0$, which implies $s_1 \prod\limits_{i\geqslant 2} q_i = \prod\limits_{j\geqslant 2} q'_j$, and we can replace q_2 by $s_1 q_2$ and use induction over the number of factors.

It remains to note that in factorial ring all irreducible elements are prime: if $ab = pq$, where q is irreducible, then irreducible factorization of either a or b should contain an element sq with invertible s. $\square$

Greatest common divisor. Let A be a factorial ring and $a_1, a_2 \in A$ have the prime factorizations:

$$a_1 = q_1 \cdots q_s q'_{s+1} \cdots q'_m \ , \quad a_2 = q_1 \cdots q_s q'_{s+1} \cdots q'_n \ ,$$

where no q'_i, q'_j are associated (the case $s = 0$, without any q's, is also possible). The product $q_1 \cdots q_s$ (or 1, if $s = 0$) is called *the greatest common divisor* of a_1, a_2 and denoted by $GCD(a_1, a_2)$. Note that $GCD(a_1, a_2)$ is defined up to invertible factor. Inductively, $GCD(a_1, a_2, \ldots, a_n) = GCD\,(GCD(a_1, a_2, \ldots, a_{n-1}), a_n)$. Given a polynomial $f = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \in A[x]$, then $GCD(a_1, a_2, \ldots, a_n)$ is called *a content* of f and is denoted by $cont(f)$.

Claim 8.23. *$cont(fg) = cont(f)cont(g)$ for any $f, g \in A[x]$.*

Proof. It is enough to check that $cont(fg) = 1$, if $cont(f) = cont(g) = 1$. If all the coefficients of fg are divisible by some prime $p \in A$, then $fg \pmod p = 0$ in the ring $(A/pA)[x]$, which has no zero divisors, because p is prime. So, either $f \pmod p = 0$ or $g \pmod p = 0$. $\square$

Claim 8.24. *If A is factorial, then $A[x]$ is factorial as well.*

Proof. Let $k = Q(A)$ be the quotient field. By Claim 8.22, it is enough to show that any irreducible $f \in A[x]$ remains to be irreducible inside the factorial ring $k[x]$. Let $f = gh$ in $k[x]$. We can write $g(x) = a^{-1}g'(x)$, $h(x) = b^{-1}h'(x)$ for some $a, b \in A$ and $g', h' \in A[x]$ such that $cont(g') = cont(h') = 1$. Now $ab\,f = g'h'$, where $cont(g'h') = 1$ by Claim 8.23 and $cont(f) = 1$, because f is irreducible in $A[x]$. Hence. ab is invertible and $h'' = (ab)^{-1}h' \in A[x]$. This leads to the decomposition $f = h''g'$ inside $A[x]$. $\square$

Corollary 8.25. *If A is factorial, then $A[x_1, x_2, \ldots, x_n]$ is factorial (in particular, normal).* $\square$

Exercice 8.26. Let k be an algebraically closed field of any characteristic, $X \subset \mathbb{A}_n(k)$ be an algebraic hypersurface given by a polynomial equation $f(x_1, x_2, \ldots, x_n) = 0$, where $f \in k[x_1, x_2, \ldots, x_n]$, and let $g(x) \in k[x_1, x_2, \ldots, x_n]$ vanish at any point of X. Show that g is divisible by any irreducible factor of f

Hint. Since $k[x_1, x_2, \ldots, x_n]$ is factorial, the result follows from Hilbert's Nullstellensatz

Resultant systems. We fix a collection of m degrees $d_1, d_2, \ldots, d_m$ and write $\mathscr{S}_d = \mathbb{P}(S^d V^*)$ for the space of hypersurfaces of degree d in $\mathbb{P}_n = \mathbb{P}(V)$. Let $\mathscr{R} \subset \mathscr{S}_{d_1} \times \mathscr{S}_{d_2} \times \cdots \times \mathscr{S}_{d_m}$ be a set of all hypersurface collections $S_1, S_2, \ldots, S_m \subset \mathbb{P}_n$ such that $\bigcap S_\nu \neq \varnothing$. Then $\mathscr{R}$ is an *algebraic variety*, i. e. can be described by a finite system of multi-homogeneous polynomial equations on the coefficients of forms $(f_1, f_2, \ldots, f_m) \in S^{d_1}V^* \times \cdots \times S^{d_n}V^*$ defining the hypersurfaces $S_1, S_2, \ldots, S_m$. This equation system depends only on n, $d_1, d_2, \ldots, d_m$ and is called *a resultant system*. Indeed, consider an ideal $I \subset k[\,x_0, x_1, \ldots, x_n\,]$ generated by f_ν. Then $\bigcap S_\nu \subset \mathbb{P}(V)$ is empty $\iff$ $V(I) \subset \mathbb{A}(V)$ either is empty or coincides with the origin $O \in \mathbb{A}(V)$. In the both cases each x_i vanishes along $V(I)$, i. e. by Hilbert's Nullstellensatz $x_i^m \in I$ for some m, that is $S^d V^* \subset I\ \forall\, d \gg 0$. Since $V(x_0^m,\, x_1^m,\, \ldots\,,\, x_n^m) = \{O\}$ this condition is also sufficient. So, $\bigcap S_\nu = \varnothing$ iff k-linear map:

$$\mu_d \ : \ S^{d-d_0}V^* \oplus S^{d-d_1}V^* \oplus \cdots \oplus S^{d-d_n}V^* \xrightarrow{(g_0, g_1, \ldots, g_n) \mapsto \sum g_\nu f_\nu} S^d \qquad (26)$$

is non surjective $\forall\, d \gg 0$. In terms of the standard monomial bases, μ_d is presented by a matrix whose entries are linear forms in the coefficients of f_ν. Since for $d \gg 0$ the dimension of the left side in (26) becames greater then the right one[8], $\mathscr{R}$ coincides with the zero set of all $d \times d$ - minors of all μ_d with d large enough. By Hilbert's base theorem, this infinite equation system is equivalent to some finite subsystem. Say, we will see in

[8]the leading terms of their expansions as polynomials in d are $md^n/n!$ and $d^n/n!$ respectively

Example 8.28 that for $n = 1$, $m = 2$ the smallest d producing a non trivial restriction is $d = d_0 + d_1 - 1$, where μ_d becomes a square matrix; in this case $\mathscr{R}$ is a hypersurface given by just one equation $\det \mu_{d_0+d_1-1} = 0$.

Example 8.27: projection $\mathbb{P}_m \times \mathbb{A}^n \xrightarrow{\pi} \mathbb{A}^n$ sends algebraic varieties to algebraic varieties, i.e. as soon as $X = \{ (q,p) \in \mathbb{P}_m \times \mathbb{A}^n \mid f_\nu(q,p) = 0 \}$ is given by polynomial equations[9] $f_\nu(t,x) = 0$ its projection onto $\mathbb{A}^n$ is also a zero set of some system of polynomials. Indeed, consider f_ν as homogeneous polynomials in t with the coefficients in $k[x_1, x_2, \dots, x_n]$. Then the image $\pi(X) \subset \mathbb{A}^n$ consists of all p such that the homogeneous in t polynomials $f_\nu(t,p)$ have a common zero on $\mathbb{P}_m$. As we have seen before, this means that their coefficients, which are some polynomials in p, satisfy the system of resultant equations.

Example 8.28: resultant of two binary forms. If k is algebraically closed, then each polynomial $f(t) = a_0\, t^m + a_1\, t^{m-1} + \dots + a_{m-1}\, t + a_m$ can be factorized as $f(t) = a_0 \prod (t - \vartheta_\nu) = a_m \prod (1 - \vartheta_\nu^{-1} t)$, where $\vartheta_1, \vartheta_2, \dots, \vartheta_m$ are all its roots. In homogeneous world, each degree d homogeneous polynomial

$$A(t_0,t_1) = a_0\, t_1^d + a_1\, t_0\, t_1^{d-1} + a_2\, t_0^2\, t_1^{d-2} + \ \cdots\ + a_{d-1}\, t_0^{d-1}\, t_1 + a_d\, t_0^d$$

has similar decomposition $A(t_0,t_1) = \prod\limits_{i=0}^{d} (\alpha_i'' t_0 - \alpha_i' t_1) = \prod\limits_{i=0}^{d} \det \begin{pmatrix} t_0 & t_1 \\ \alpha_i' & \alpha_i'' \end{pmatrix}$, which means that A vanishes at d points $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{P}_1$ with homogeneous coordinates $\alpha_i = (\alpha_i' : \alpha_i')$. In particular, each coefficient a_i, of $A(t)$, is expressed as bihomogeneous degree $(i, d-i)$ polynomial in (α', α''):

$$a_i = (-1)^{d-i} \sigma_i(\alpha', \alpha'')\,, \quad \text{where} \quad \sigma_i(\alpha', \alpha'') = \sum_{\#I=i} \Big(\prod_{i \in I} \alpha_i' \cdot \prod_{j \notin I} \alpha_j'' \Big)$$

(here I runs through all increasing length i subsets in $\{1, 2, \dots, d\}$ and σ_i is a *bi*homogeneous version of the i-th elementary symmetric function).

Now, let us fix two degrees $m, n \in \mathbb{N}$ and consider a polynomial ring $k[\alpha', \alpha'', \beta', \beta'']$ in four collections of variables $\alpha' = (\alpha_1', \alpha_2', \dots, \alpha_n')$, $\alpha'' = (\alpha_1'', \alpha_2'', \dots, \alpha_n'')$, $\beta' = (\beta_1', \beta_2', \dots, \beta_m')$, $\beta'' = (\beta_1'', \beta_2'', \dots, \beta_m'')$ Then the product

$$R_{AB} \overset{\text{def}}{=} \prod_{i,j} (\alpha_i' \beta_j'' - \alpha_i'' \beta_j') = \prod_{j=1}^{n} A(\beta_j) = (-1)^{mn} \prod_{i=1}^{m} B(\alpha_i)\,,$$

vanishes iff two homogeneous binary forms $A(t_0,t_1) = \sum\limits_{i=0}^{n} a_i\, t_0^i\, t_1^{n-i}$ and $B(t_0,t_1) = \sum\limits_{j=0}^{m} b_j\, t_0^j\, t_1^{m-j}$ (whose coefficients $a_i = (-1)^{n-i} \sigma_i(\alpha', \alpha'')$, $b_j = (-1)^{m-j} \sigma_j(\beta', \beta'')$ have a common root. Clearly, $R_{A,B}$ is bihomogeneous of bidegree (mn, mn) in (α, β) and may be expressed in terms of the coefficients of A, B. This expression is called *a resultant* of the polynomials $A(t_0,t_1)$, $B(t_0,t_1)$ and generates the ideal of all resultant relations for two binary forms. More precisely, $R_{A.B}$ up to a scalar factor coincides with *the Silvester determinant*

$$\det \underbrace{\begin{pmatrix} a_0 & a_1 & \dots & a_n & & & \\ & a_0 & a_1 & \dots & a_n & & \\ & & \ddots & \ddots & & \ddots & \\ & & & a_0 & a_1 & \dots & a_n \\ b_0 & b_1 & \dots & b_m & & & \\ & b_0 & b_1 & \dots & b_m & & \\ & & \ddots & \ddots & & \ddots & \\ & & & b_0 & b_1 & \dots & b_m \end{pmatrix}}_{m+n} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix} b \\ b \\ b \\ b \end{matrix}}\right\} n \end{matrix}$$

Indeed, consider a vector space U with a basis $\{t_0, t_1\}$ and a linear map $S^{m-1}U \oplus S^{n-1}U \xrightarrow{M_{A,B}} S^{m+n-1}U$ which sends a pair of polynomials $(h_1(t), h_2(t))$ to $A(t)h_1(t) + B(t)h_2(t)$ as in 26. The Silvester matrix is transpose to the matrix of $M_{A,B}$ in the standard monomial bases. If a point (α, β) lies on the quadric $\alpha_i' \beta_j'' - \alpha_i'' \beta_j' = 0$, then $(\alpha_i'' t_0 - \alpha_i' t_1) = (\beta_i'' t_0 - \beta_i' t_1)$ up to a scalar factor. This linear form divides $A(t)$, $B(t)$ and any polynomial $A(t)h_1(t) + B(t)h_2(t)$. Hence, $\operatorname{im} M_{A,B} \neq S^{m+n-1}U$. So, the Silvester determinant vanishes along each quadric $\alpha_i' \beta_j'' - \alpha_i'' \beta_j' = 0$ and, by Hilbert's Nullstellensatz, it is divisible by $R_{A,B}$. The quotient has to be a constant, because the both polynomials are bihomogeneous of the same bidegree (mn, mn) in (α, β). Moreover, since the Silvester determinant vanishes as soon $A(t)$ and $B(t)$ are not coprime, R_{AB} spans the ideal of all resultant relations — so, it is principal in the case of two binary forms.

Example 8.29: Elimination technique. Let C_1, C_2 be two plane curves of degrees m and n given by equations $F(x) = 0$ and $G(x) = 0$ in $x = (x_0 : x_1 : x_2)$. Consider F, G as (non homogeneous) polynomials in x_0 with the coefficients in

[9] homogeneous in $t = (t_1, t_2, \dots, t_m) \in \mathbb{P}_m$

$k(x_1, x_2)$ and take their resultant[10] $R_{F,G}(x_1, x_2) \in k[x_1, x_2]$. If it is identically zero, then F and G have a common divisor in $k(x_1, x_2)[x_0]$.

Exercice 8.30. Deduce from the Gauss – Kronecker – Dedekind lemma that this common divisor can be taken with the coefficients in $k[x_1, x_2]$.

So, if $R \equiv 0$, then C_1 and C_2 have a common component. If $R \not\equiv 0$, then (x_1, x_2)-coordinates of any intersection point $p \in C_1 \cap C_2$ have to satisfy the resultant equation $R_{F,G}(x_1, x_2) = 0$ which is homogeneous of degree mn. So, the curves have either a common component or at most mn intersection points, which may be found by solving a homogeneous polynomial equation in x_1, x_2 only. These procedure is called *an elimination of a variable*.

[10] resultant of non-homogeneous polynomials $F(x_0)$ and $G(x_0)$ is defined as the resultant of $t_0^n F(t_1/t_0)$ and $t_0^m G(t_1/t_0)$

Projective hypersurfaces[1]

<u>SPACE OF HYPERSURFACES.</u> Projective space $\mathbb{P}(S^dV^*)$ consists of all non zero d-th degree homogeneous polynomials considered up to a scalar factor. It is called *a space of degree d hypersurfaces* in $\mathbb{P}_n = \mathbb{P}(V)$. Geometrically, each polynomial whose prime factorization is $f(x) = \prod_{i=1}^{s} p_i(x)^{m_i}$ defines a zero set

$$Z_f \stackrel{\text{def}}{=} \{x \in \mathbb{P}_n \mid f(x) = 0\} = \bigcup_{i=1}^{s} m_i \cdot Z_{p_i} \,,$$

which is an union of the *irreducible components* $Z_{p_i} = \{x \in \mathbb{P}(V) \mid p_i(x) = 0\}$ counted with integer multiplicities m_i. We will also write $Z_f = m_1 Z_{p_1} + m_2 Z_{p_2} + \cdots + m_s Z_{p_s}$. By Exercice 8.26, each irreducible component Z_{p_i} does not admit any further decomposition into a sum of proper subsurfaces.

Exercice 9.1. Find $\dim \mathbb{P}(S^dV^*)$.

Traditionally, 1-, 2-, and 3-dimensional projective subspaces in the space of hypersurfaces are called, respectively, *pencils*, *nets* and *webs* of hypersurfaces.

<u>Example 9.2: pencil of plane curves</u> $\ell = (C_1C_2) \subset \mathbb{P}(S^dV^*)$ on $\mathbb{P}_2 = \mathbb{P}(V)$ is defined by any two distinct elements $C_1, C_2 \in \ell$. A curve $\lambda C_1 + \mu C_2 \in \ell$ (whose homogeneous coordinates w.r.t. the base $\{C_1, C_2\}$, of ℓ, are $(\lambda : \mu)$) is given in $\mathbb{P}_2$ by the equation $\lambda f_1(x) + \mu f_2(x) = 0$, where $f_1(x) = 0$ and $f_2(x) = 0$ are the equations of the base curves C_1, C_2. In particular, each curve from the pencil $(C_1C_2$ pass through all the intersection points $C_1 \cap C_2$. Another remarkable property: for any $p \in \mathbb{P}_2$ any pencil of plane curves contains a curve passing through p. Indeed, curves passing through a given point form a codimension 1 hyperplane in the space of curves and this hyperplane intersects each line of curves.

This leads to another fruitful proof of the Pascal theorem. Let a hexagon $p_1p_2p_3p_4p_5p_6$ be inscribed in a non singular conic C and write $x = p_3p_4 \cap p_6p_1$, $y = p_2p_3 \cap p_5p_6$, $z = p_1p_2 \cap p_4p_5$ for the intersection points of its opposite sides. Fix some 7-th point $p_7 \in C$, which differs from $p_1, p_2, \ldots, p_6$, and consider a pencil of cubic curves (Q_1Q_2) spanned by 2 completely splitted cubics: $Q_1 = (p_1p_2) \cup (p_3p_4) \cup (p_5p_6)$ and $Q_2 = (p_2p_3) \cup (p_4p_5) \cup (p_6p_1)$ formed by 'opposite triples' of sides. All cubics of this pencil pass through 9 intersection points $Q_1 \cap Q_2 = \{p_1, p_2, \ldots, p_6, x, y, z\}$ and at least one of them, say Q, pass through p_7 as well. Since the conic C either intersects the cubic Q in $\leqslant 6$ points or is contained in Q, we have perforce $Q = C + \text{some line}$, where the line has to pass through x, y, z.

<u>INTERACTION WITH LINES.</u> Let $S \subset \mathbb{P}_n$ be a surface given by a homogeneous equation $F(x) = 0$ of degree d and $\ell = (pq) \in \mathbb{P}_n$ be a line spanned by $p, q \in V$. Write $(\lambda : \mu)$ for internal homogeneous coordinates of a point $\lambda p + \mu q \in \ell$. In these coordinates, $\ell \cap S$ is given by the equation $f(\lambda, \mu) = 0$ obtained from $F(x) = 0$ by the substitution $x = \lambda p + \mu q$. By the Newton - Taylor formula,

$$f(\lambda, \mu) = F(\lambda p + \mu q) = \sum_{i=0}^{d} \lambda^i \mu^{n-i} \binom{d}{i} \widetilde{F}(p^i, q^{n-i}) \,, \tag{27}$$

where the coefficient $\widetilde{F}(p^i, q^{n-i}) \stackrel{\text{def}}{=} \text{pl}(F)(\underbrace{p, p, \ldots, p}_{i}, \underbrace{q, q, \ldots, q}_{d-i}) = \dfrac{(d-i)!}{d!} \dfrac{\partial^i F}{\partial p^i}(q) = \dfrac{i!}{d!} \dfrac{\partial^{d-i} F}{\partial q^{d-i}}(p)$ is bihomogeneous of degree $(i, n-i)$ in (p, q). If $f(\lambda, \mu) \equiv 0$, i.e. $\widetilde{F}(p^i, q^{n-\mu}) = 0$ for all i, then $\ell \subset S$. If $f(\lambda, \mu) \not\equiv 0$, then $f(\lambda, \mu) = \prod_i (\alpha_i'' \mu - \alpha_i' \lambda)^{s_i}$ is a product of linear forms[2]. Each linear form corresponds to an intersection point $\alpha = (\alpha' : \alpha'') = \alpha' p + \alpha'' q \in \ell \cap S$. The maximal power s_i such that $f(\lambda, \mu)$ is divisible by $(\alpha_i'' \mu - \alpha_i' \lambda)^{s_i}$ is called *a local intersection index* between S and ℓ at α. It is denoted by $(S, \ell)_\alpha$. So, $\deg S = \sum_{\alpha \in S \cap \ell} (S, \ell)_\alpha$ as soon as $\ell \not\subset S$. i.e. a line either lies on S or intersect S in $\deg S$ points counted with multiplicities.

<u>TANGENT LINES AND TANGENT SPACE.</u> A line ℓ is called *a tangent line* to S, if there is a point $p \in S \cap \ell$ with $(S, \ell)_p \geqslant 2$. We say that ℓ does *touch* S at p or that p is a *tangency point*.

<u>Claim 9.3</u>. *For any $p \in S$ and any $q \in \mathbb{P}_n$ the line (pq) touch S at p iff $\widetilde{F}(p^{n-1}, q) = 0$.*

Proof. If $p \in S$, that is $\widetilde{F}(p^n) = F(p) = 0$, the affine version of (27) near p takes the form:

$$F(p + tq) = t \binom{d}{1} \widetilde{F}(p^{n-1}, q) + t^2 \binom{d}{2} \widetilde{F}(p^{n-2}, q^2) + \cdots$$

[1] WARNING!!! In this lecture we suppose that the ground field k is algebraically closed and has zero characteristic.

[2] namely, $f(\lambda, \mu) = \mu^d f(t, 1)$, where $t = \lambda/\mu$ and $f(t, 1) \in k[t]$; now, $f(t, 1) = \prod (t - \alpha_i)^{m_i}$

and $(S,(pq))_p$ is the maximal power of t factored out of $F(p+tq)$. It is $\geqslant 2$ iff $\widetilde{F}(p^{n-1},q)=0$. $\square$

Corollary 9.4. *The union of all tangent lines through $p\in S$ is a projective space $T_pS \overset{\text{def}}{=} \left\{ y\in\mathbb{P}_n \;\middle|\; \sum_{i=0}^{n} y_i \frac{\partial F}{\partial x_i}(p)=0 \right\}$. It is either a hyperplane or the whole of $\mathbb{P}_n$. The last happens iff $\frac{\partial F}{\partial x_i}(p)=0\ \forall i$.* $\square$

The space T_pS is called *a tangent space* to S at p. If $T_pS=\mathbb{P}_n$, then S is called *singular* at p and p is called *a singular point* of S. Otherwise p is called *a smooth point* of S. S is called *smooth*, if all its points are smooth.

Corollary 9.5. *Let q be either a smooth point on S or any point outside S. Then the apparent contour[3] of S visible from q is slashed by the hypersurface $S_q^{(d-1)} \overset{\text{def}}{=} \left\{ y\in\mathbb{P}_n \;\middle|\; \sum_{i=0}^{n} q_i \frac{\partial F}{\partial x_i}(y)=0 \right\}$; of degree $(d-1)$ (in particular, $\sum_{i=0}^{n} q_i \frac{\partial F}{\partial x_i}(y)\not\equiv 0$ as a polynomial in y).*

Proof. Indeed, (qy) touch S at y, if $0=\widetilde{F}(y^{n-1},q)=\mathrm{pl}_qF(y)=\frac{1}{d}\sum_{i=0}^{n} q_i\frac{\partial F}{\partial x_i}(y)$. If this polynomial vanishes identically in y, then taking $y=q$ we get $F(q)=0$, i.e. $q\in S$. At the same time $F(q,q,\dots,q,y)=\mathrm{pl}_q^{n-1}F(y)=\mathrm{pl}_q^{n-2}\mathrm{pl}_qF(y)\equiv 0$, because of $\widetilde{F}(y^{n-1},q)\equiv 0$. So, q is singular point of S. $\square$

Point multiplicities. A number $\mathrm{mult}_S(p) \overset{\text{def}}{=} \min\limits_{\ell\ni p}(\ell,S)_p$ is called *a multiplicity* of p on S. A point $p\in S$ is singular iff any line through p intersects S with index $\geqslant 2$ at p. So, $p\in S$ is smooth iff p has the multiplicity 1. A point p has multiplicity $\geqslant m$ iff all possible $(m-1)$-type partial derivatives of F vanish at p.

Polar hypersurfaces. A hypersurface $S_q^{(r)} \overset{\text{def}}{=} \left\{ y\in\mathbb{P}_n \mid \widetilde{F}(q^{n-r},y^r)=0 \right\}$ is called *a r-th degree polar* of S with respect to p. If $F(q^{n-r},y^r)$ vanishes identically in y, we say that the polar is *trivial*, i.e. coincides with the whole of $\mathbb{P}_n$. Intuitively, for a smooth point $q\in S$, the polar $S_q^{(r)}$ is a degree r surface which gives the most closed approximation for S near q in a sense that the both have at q the same tangent hyperplanes (i.e. their linear polars at q coincide), the same 'tangent quadrics' (i.e. their quadratic polars at q coincide), and so on up to coincidence of $(r-1)$-th degree polars. If $q\in S$ is singular of multiplicity $m\geqslant 2$, then all the polars of degree $\leqslant (m-1)$ w.r.t. p are trivial and the m-th degree polar is non trivial but singular at p.

Example 9.6: space of singular conics. Let V be 3D vector space, $\mathbb{P}_5=\mathbb{P}(S^2V^*)$ be the space of conics on $\mathbb{P}_2=\mathbb{P}(V)$, and $S\subset\mathbb{P}_5$ be a locus of the singular conics. Let us fix some coordinates and present quadratic forms $q(x)\in S^2V^*$ as $q(x)=x\cdot A\cdot {}^tx$ with symmetric 3×3-matrices A. Since q is singular iff $\det A=0$, we see that S is an irreducible cubic hypersurface in $\mathbb{P}_5$. We would like to find its singular points and describe non singular tangent hyperplanes. By Laplace's relations, $\det A=\sum_\nu (-1)^{i+\nu}a_{i\nu}A_{i\nu}$, where $A_{i\nu}$ is 2×2-minor situated outside i-th row and j-th column. So, $\frac{\partial \det A}{\partial a_{ij}}=(-1)^{i+j}A_{ij}$ and a point $q\in S$ is singular iff $\mathrm{rk}\,A=1$.

Exercice 9.7. Show that any $m\times n$ matrix a_{ij} of rank 1 has $a_{ij}=\lambda_i\mu_j$ for some $\lambda_1,\lambda_2,\dots,\lambda_m$, $\mu_1,\mu_2,\dots,\mu_n\in k$.

Hint. A linear operator $k^n \xrightarrow{x\mapsto Ax} k^m$ has $\mathrm{rk}\,A=1$ iff $\dim\mathrm{im}\,A=1$; if $w=(\lambda_1,\lambda_2,\dots,\lambda_m)$ generates $\mathrm{im}\,A$, then $A(v)=\alpha(v)\,w$, where $k^n\xrightarrow{\alpha}k$ is linear form, say $\alpha=(\mu_1,\mu_2,\dots,\mu_n)\,\dots$

In our case $A=(a_{ij})$ is symmetric and we should have $\mu_i=\lambda_i$, i.e. $a_{ij}=\lambda_i\lambda_j$ for some $\lambda_0,\lambda_1,\lambda_2\in k$. So, $A\in S$ is singular iff $q(x)=\left(\sum\lambda_ix_i\right)^2$ is a double line. Thus, the set of singular points of S coincides with 2-dimensional Veronese's surface $\mathcal{V}\subset S$, which parameterizes double lines in $\mathbb{P}_2$.

Now, let $q(x)=x\cdot A\cdot {}^tx$ be a smooth point of S, i.e. a pair of distinct lines $\ell_1\cup\ell_2\subset\mathbb{P}_2$. Then the correlation map $V\xrightarrow{x\mapsto x\cdot A}V^*$ has 1-dimensional kernel spanned by $v=\ell_1\cap\ell_2$, that is $\mathrm{rk}\,(A)=2$ and the adjoint matrix $\widehat{A}=\left((-1)^{i+j}A_{ij}\right)$ is non zero. By the Laplace relations: $A\cdot\widehat{A}=\widehat{A}\cdot A=\det(A)\cdot\mathrm{Id}=0$, each row and each column of $\widehat{A}$ lies in the kernel of A, i.e. is proportional to v. Thus, $\mathrm{rk}\,\widehat{A}=1$ and $(-1)^{i+j}A_{ij}=\mu_i\mu_j$, where $(\mu_0:\mu_1:\mu_2)$ are homogeneous coordinates of v. So, $B=(b_{ij})\in T_qS_n \iff \sum_{ij} b_{ij}\cdot(-1)^{i+j}A_{ij}=0 \iff \sum_{ij} b_{ij}\,\mu_i\mu_j=0 \iff v\cdot B\cdot {}^tv=0$. In other words, the tangent space T_qS at $q=\ell_1\cup\ell_2\subset\mathbb{P}_2$ consists of all conics passing through the point $\ell_1\cap\ell_2\in\mathbb{P}_2$.

Exercice 9.8. Extend this result to general case $\dim V=n+1$, i.e. show that a point q on the hypersurface $S\subset\mathbb{P}(S^2V^*)$, of singular quadrics on $\mathbb{P}_n=\mathbb{P}(V)$, is non-singular iff the corresponding quadric $Q_q\subset\mathbb{P}(V)$ has just one singular point $v(q)\in\mathbb{P}(V)$ and prove that T_qS consists of all quadrics passing through $v(q)$.

[3] i.e. the set of all tangency points $p\neq q$ where S touched by the tangent lines drawn from q

Plane curves: singularities, intersections, and Plücker relations[1]

GEOMETRICAL TANGENTS AT SINGULARITY. Let $C \subset \mathbb{P}_2$ be a curve given by an equation $F(x) = 0$ of degree d and $p \in C$ be a (singular) point of multiplicity $m \geqslant 2$. Then all the polars $C_p^{(\nu)}$ (which would be given by equations[2] $\widetilde{F}(p^{d-\nu}, x^\nu) = 0$) are trivial for $0 \leqslant \nu \leqslant (m-1)$ and m-th degree polar $C_p^{(m)}$ (given by $\widetilde{F}(p^{d-m}, x^m) = 0$) is non trivial but singular: its Taylor expansion near p

$$\widetilde{F}(p^{d-m}, (p+tq)^m) = \sum_{\mu=1}^{m} t^\mu \binom{m}{\mu} \widetilde{F}(p^{d-m+\mu}, q^\mu) = t^m \widetilde{F}(p^{d-m}, q^m)$$

contains just one term and a line (p,q) either is a component of $C_p^{(m)}$ (when $\widetilde{F}(p^{d-m}, q^m) = 0$) or intersects $C_p^{(m)}$ only at p with multiplicity m (when $F(p^{d-m}, q^m) \neq 0$). So, $C_p^{(m)}$ splits into union of m lines (pq_i), where q_i are the roots of $\widetilde{F}(p^{d-m}, q^m) = 0$ considered as degree m equation on q, where q runs through any fixed line $\ell \not\ni p$. (Of course, some of (pq_i) may coincide when the roots became multiple.) The lines (pq_i) are called *geometrical* tangent lines to C at p.

Geometrically, generic line (pq), through p, intersects C at p with multiplicity m, because the Taylor expansion $F(p+tq) = \binom{d}{m} \cdot t^m \cdot \widetilde{F}(p^{d-m}, q^m) + \cdots$ starts with non-zero m-th degree term. The geometric tangents (pq_i) are the lines whose intersection multiplicity with C at p jumps w.r.t. the generic value. Algebraically, this means that $\widetilde{F}(p^{d-m}, x^m) = \xi_1(x)\,\xi_2(x) \cdots \xi_m(x)$ is the product of m linear forms $\xi_1, \xi_2, \dots, \xi_s$ whose zeros are the geometric tangents (pq_i) (again, some of them may coincide).

Example 10.1: the simplest singularities. Given a curve $C \subset \mathbb{P}_2$, an m-typle point $p \in C$ is called *an m-typle node* (or *an m-typle selfintersection*) if there are m distinct geometrical tangents $\ell_1, \ell_2, \dots, \ell_m$ to C through p. Geometrically, this means that C has m mutually transversal branches through p. The difference $(\ell_i, C)_p - m - 1$ is called *an order of the contact* between ℓ_i and the corresponding branch of C. A node is called *ordinary* if all the geometrical tangents have the second order contacts with its branches, that is $(\ell_i, C)_p = m + 1\ \forall i$.

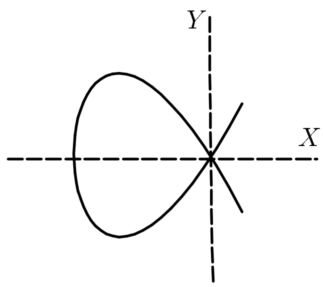

Fig. 10.1. The node $y^2 = x^2(x+1)$.

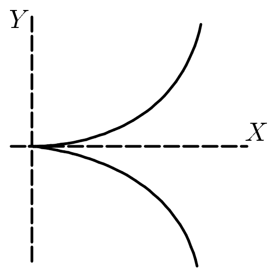

Fig. 10.2. The cusp $y^2 = x^3$.

A double point $p \in C$ is called *a cusp* (or *a selfcontact*) if the quadratic polar of p is a double line ℓ. Geometrically, this means that C has two branches which do touch each other at p. The unique geometrical tangent ℓ at p is called *a cuspidal tangent*. A local intersection number $(\ell, C)_p$ measures *an order of the selfcontact* for C at p; clearly, $(\ell, C)_p \geqslant 3$. A cusp is called *ordinary*, if $(\ell, C)_p = 3$ is minimal possible.

We say that C has *only the simplest singularities*, if the singular points of C are exhausted by ordinary double nodes and ordinary cusps. Two cubic curves with the simplest singularities are shown on the figs. 10.1–10.2. For higher degree curves, the neighborhood of the simple singularity looks similarly[3].

Exercice 10.2. Show that irreducible cubic curve has at most one (automatically simple) singularity.
HINT. A line ℓ has to be a component of a cubic C as soon as $(\ell, C) \geqslant 4$.

Example 10.3: how much is to put a singularity on a curve? Given a point $p \in \mathbb{P}_2$, then the polar map

$$S^d V^* \xrightarrow{F \mapsto \partial^{d-m} F / \partial p^{d-m}} S^m V^*$$

[1] WARNING!!! In this lecture we suppose that the ground field k is algebraically closed and has zero characteristic.

[2] recall that $\widetilde{F}(p^{d-\nu}, x^\nu) = \mathrm{pl}(F)(\underbrace{p, p, \dots, p}_{d-\nu}, \underbrace{x, x, \dots, x}_{\nu}) = \dfrac{\nu!}{d!} \dfrac{\partial^{(d-\nu)} F}{\partial p^{(d-\nu)}}(x) = \dfrac{(d-\nu)!}{d!} \dfrac{\partial^\nu F}{\partial x^\nu}(p)$

[3] one can show that any smooth curve in $\mathbb{P}_n$ admits a plane projection that has only the simplest singularities

is a linear epimorphism. Hence the curves whose m-degree polar coincides with a given collection of lines through p form a projective subspace of codimension $\dim(S^m V^*) - 1 = m(m+3)/2$. For example, 5 parameters in a curve equation are fixed by assuming that this curve has at a given point a cusp with a given cuspidal tangency. However, these restrictions, if come from several distinct points, are not independent, in general.

<u>Affine neighborhood of a singularity.</u> Practical computation of geometric tangents usually becomes simpler in affine chart with the origin in the singular point in question. Let C have an affine equation $f(x,y) = 0$ in such a chart U. Write it as $\sum\limits_{\nu>0} f_\nu(x,y) = 0$, where each $f_\nu(x,y)$ is homogeneous of degree ν, and consider a line $\ell_{\alpha:\beta}$ given parametrically as[4] $x = \alpha t\,,\ y = \beta t$. Then $C \cap \ell$ is given by the following equation on t:

$$f_m(\alpha,\beta)\,t^m + f_{m+1}(\alpha,\beta)\,t^{m+1} + \ \cdots\ + f_d(\alpha,\beta)\,t^d = 0$$

where m is the degree of lowest non trivial homogeneous component of f and each $f_\nu(\alpha,\beta)$ is actually nothing but the ν-th degree polar of p evaluated at $q = (\alpha:\beta)$. Thus, the multiplicity of p coincides with the degree of lowest non trivial homogeneous component of f and the directions $(\alpha:\beta)$ of the geometrical tangents through p are the roots of this component, i.e. satisfy the equation $\varphi_m(\alpha,\beta) = 0$.

<u>Example 10.4</u>: analyzing singularities. Taking $x = \alpha t$, $y = \beta t$ in the nodal cubic equation $y^2 - x^2 - x^3 = 0$, we get the lowest term $(\beta+\alpha)(\beta-\alpha)\,t^2$, which vanishes for $(\alpha:\beta) = (1:\pm 1)$; so, there are two distinct geometrical tangents $x = \pm y$. Local intersection number $(\ell, C)_p = 3$ for each tangent line ℓ, i.e. each tangent has the second order contact with its branch. The cuspidal cubic on the 10.2 has the lowest term $\beta^2\,t^2$. So, the second polar is a the double line $x = 0$ with local intersection 3 with C at the origin.

As an advanced example, consider a quartic given by the polynomial $F(t) = t_0^4 - t_0^3 t_1 + t_0^2 t_2^2 - t_1^2 t_2^2$. Its singularities are $t \in \mathbb{P}_2$ where all partial derivatives

$$\partial F/\partial t_0 = 4\,t_0^3 - 3\,t_0^2 t_2 - 2\,t_0 t_2^2\,, \quad \partial F/\partial t_1 = -2\,t_1 t_2^2\,, \quad \partial F/\partial t_2 = -t_0^3 + 2\,t_0^2 t_2 - 2\,t_1^2 t_2$$

vanish simultaneously. It happens at two points $a = (0:0:1)$ and $b = (0:1:0)$. Take an affine chart with $x = t_0/t_2\,,\ y = t_1/t_2$ near a. Then $F = 0$ turns into $x^2 - y^2 - x^3 + x^4 = 0$ with two simple geometrical tangents $x = \pm y$ at the origin. Since a local intersection number equals 3 for each tangency, a is an ordinary node. Taking a chart with $x = t_0/t_1\,,\ y = t_2/t_1$ near b, we get the equation $y^2 - x^4 + x^3 y - x^2 y^2$ whose geometrical tangent is a double line $\ell = \{x = 0\}$ with $(\ell, C)_p = 4$. So, b is the non ordinary cusp, where C has a selfcontact of order 4.

<u>Blow up.</u> Geometrically, the substitution $x = \alpha t$, $y = \beta t$ lifts C from $\mathbb{P}_2$ to a surface $\Gamma \subset \mathbb{P}_1 \times \mathbb{P}_2$ called *a blow up* of $p \in \mathbb{P}_2$. It is described as follows. Identify a pencil of lines through p with any fixed line $\mathbb{P}_1 = (ab) \not\ni p$ and consider the incidence graph $\Gamma \stackrel{\text{def}}{=} \{(\ell, q) \in \mathbb{P}_1 \times \mathbb{P}_2 \mid \ell \ni q\}$. It is an algebraic surface in $\mathbb{P}_1 \times \mathbb{P}_2$: if we put $p = (1:0:0)$, $a = (0:1:0)$, $b = (0:0:1)$, take $q = \alpha a + \beta b$, and consider $((\alpha:\beta), (x_0:x_1:x_2))$ as coordinates on $\mathbb{P}_1 \times \mathbb{P}_2$, then $(x_0:x_1:x_2) \in (pq)$ is equivalent to the quadratic relation $\alpha x_2 = \beta x_1$.

Projection $\Gamma \xrightarrow{\ \sigma\ } \mathbb{P}_2$ is bijective outside p, but $\sigma^{-1}(p) \simeq \mathbb{P}_1$ is the pencil of lines through p on $\mathbb{P}_2$ (see fig. 10.3). A map $(\,(\alpha:\beta)\,,\,t) \longmapsto (\,(\alpha:\beta)\,,\,(1:\alpha t:\beta t)\,) \in \mathbb{P}_1 \times \mathbb{P}_2$ gives a rational parameterization for some affine neighborhood of this exceptional fiber in Γ. Full preimage $\sigma^{-1}(C)$, of a curve $C \subset \mathbb{P}_2$ passing through p, consists of 2 components: $\sigma^{-1}(p) \simeq \mathbb{P}_1$ and and some curve whose equation (in terms of parameters $(\alpha:\beta;t)$ on Γ) is a result of the substitution $x = \alpha t$, $y = \beta t$ in the affine equation for C.

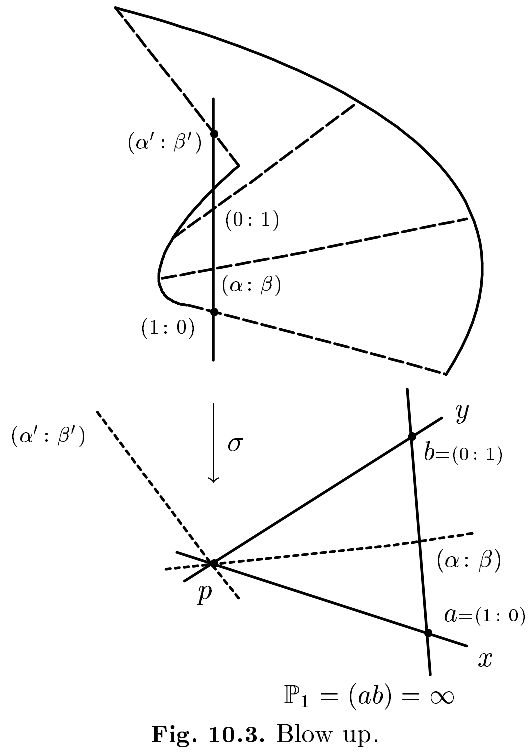

Fig. 10.3. Blow up.

<u>Local intersection multiplicity of two plane curves.</u> Consider two curves $C_1, C_2 \subset \mathbb{P}_2$ given by homogeneous equations $F = 0$, $G = 0$ of degrees n, m without common divisors. Let $u \in C_1 \cap C_2$. We fix two points p, v such that the line (pu) satisfies the conditions:

$$(pu) \cap C_1 \cap C_2 = \{u\} \qquad \& \qquad v \notin (pu) \qquad (28)$$

[4] as in the beginning of this lecture, line $\ell_{\alpha:\beta}$ has a form $(p + tq)$, where $q = (\alpha:\beta) \in U_\infty \simeq \mathbb{P}_1$ is runing through the infinite projective line of the chart U

and will use the triple $\{p,\, u,\, v\}$ as a basis for $\mathbb{P}_2$. Let a point $q(x) = u + x\,v$ tend to u (as $x \to 0$) along the line (pu). We restrict the both curves onto a varying line $\ell(x) = (p, q(x))$ and write $\alpha_0(x), \dots, \alpha_n(x)$ for the points $C_1 \cap \ell(x)$ and $\beta_0(x), \dots, \beta_m(x)$ for the points $C_2 \cap \ell(x)$ (see fig. 10.4). These points are the roots of two homogeneous polynomials $f_x(t_0, t_1) = F(t_0 p + t_1 q(x))$, $g_x(t_0, t_1) = G(t_0 p + t_1 q(x))$ in $t = (t_0 : t_1)$ with coefficients depending on the parameter x. When x varies, the pints $\alpha_\nu(x)$, $\beta_\mu(x)$ draw the branches of C_1 and C_2. Let $\beta_1(x), \dots, \beta_r(x)\,;\, \alpha_1(x), \dots, \alpha_s(x)$ be all the branches that come to u as $x \to 0$.

Natural geometric definition[5] of local intersection multiplicity $(C_1, C_2)_u$ puts it to be equal to the sum of orders of rs infinitesimals $\alpha_i(x) - \beta_j(x)$ w. r. t. x as $x \to 0$. Since the sum of orders of $\alpha_i(x) - \beta_j(x)$ coincides with the order of zero of the resultant R_{f_x, g_x} (considered as a polynomial in x) at $x = 0$, we can treat $(C_1, C_2)_u$ pure algebraically as the multiplicity of x in the prime factorization of R_{f_x, g_x} in $k[x]$. Such the definition is correct (i. e. does not depend on a choice of p, v satisfying (28)), because of the following

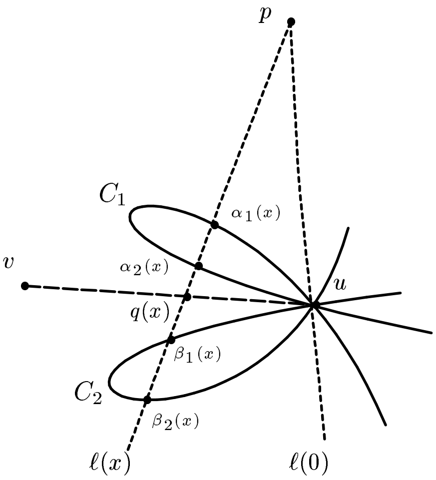

Fig. 10.4. Zeuthen's rule.

<u>Lemma 10.5</u>. *Consider the resultant $R_{f,g}(p, q) \in k[p, q]$, of two binary forms $f(t_0, t_1) = F(t_0 p + t_1 q)$, $g(t_0, t_1) = G(t_0 p + t_1 q)$, as a polynomial in $p = (p_0, p_1, p_2)$, $q = (q_0, q_1, q_2)$. Then its irreducible factorization in $k[p, q]$ has a form*[6]:

$$R_{f,g}(p, q) = const \cdot \prod_{w \in C_1 \cap C_2} \det \begin{pmatrix} w_0 & w_1 & w_2 \\ p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix}^{m_w} \tag{29}$$

where the multiplicities m_w can be calculated by the Zeuthen rule with any choice of p, v satisfying (28).

Proof. Denote the determinants in the right side of (29) by $D_w(p, q)$. Geometrically, $D_w(p, q) = 0$ is an irreducible quadric, which consists of all pairs $p, q \in \mathbb{P}_2$ whose joining line (pq) belongs to the pencil of lines through w. If $w \in C_1 \cap C_2$, then $R(p, q)$ vanishes along the quadric $D_w(p, q) = 0$. Hence, by Hilbert's Nullstellensatz, any such $D_w(p, q)$ divides the resultant. Wise verse, if $R(p, q) = 0$, then the restrictions of F, G onto the line (p, q) have a common root, i. e. (p, q) pass through some $w \in C_1 \cap C_2$ and $\prod\limits_{w \in C_1 \cap C_2} D_w(p, q)$ vanishes at this (p, q). So, this product vanishes everywhere along $V(R(p, q))$ and, again by Hilbert, $R(p, q)$ should divide some power of $\prod\limits_{w \in C_1 \cap C_2} D_w(p, q)$. To check the Zeuthen rule, fix p and $q = q(x)$ as on the fig. 10.4 and note that in (29) only $D_w(p, q(x))$ with $w = u$ vanishes at $x = 0$, because of (28); and this one $D_u(p, q(x)) = \det(u, p, u + x\,v) = x\,\det(u, p, v)$ is proportional to x. $\square$

<u>Intersection theory of plane curves.</u> It follows immediately from the Zeuthen rule, that the local intersection multiplicities are distributive w. r. t. the curve branches, that is if C_1 has b_1 branches passing through u and C_2 has b_2 ones, then $(C_1, C_2)_u$ is the sum of $b_1 b_2$ mutual intersection indices between the branches[7]. On the other side, since each $D_w(p, q)$ in (29) is bilinear in (p, q) and $R_{f,g}(p, q)$ has bidegree (mn, mn), we get *the Bézout theorem*:

<u>Theorem 10.6</u>. $\sum\limits_{w \in C_1 \cap C_2} (C_1, C_2)_w = \deg C_1 \cdot \deg C_2$ *for any two plane projective curves without common components.* $\square$

<u>Example 10.7</u>: proper tangents and class. A tangent lane is called *proper*, if its tangency point is smooth. A number of proper tangents to C passing through a generic point $q \in \mathbb{P}_2$ is called *a class* of C and denoted by $c = c(C)$. If $\deg C = d$, then by Corollary 9.5 the tangents coming form a point $q \in \mathbb{P}_2 \setminus Sing(C)$ touch C at the points of $C \cap C_q^{(d-1)}$, where $C_q^{(d-1)}$ is $(d-1)$-th degree polar of q. If C is irreducible, then $C \cap C_q^{(d-1)}$ consists of $d(d-1)$ points[8] counted with multiplicities. Besides the proper tangency points, $C \cap C_q^{(d-1)}$ contains also all singular points of C, because each line trough a singularity is (non proper) tangent. So, class of irreducible curve satisfies inequality $c \leqslant d(d-1)$, which turns to equality iff C is smooth.

<u>Example 10.8</u>: inflections. A smooth point $p \in C$ is called *an inflection*, if $(C, T_p C)_p \geqslant 3$. An inflection is called *ordinary* (or *simple*), if this number equals 3. If $p \in C$ is an inflection, then the quadratic polar $C_p^{(2)}$ of p has the zero restriction onto $\ell = T_p C$, i. e. ℓ is a component of $C_p^{(2)}$. Note that p is a smooth point of the conic $C_p^{(2)}$, because $C_p^{(2)}$ and C have the same linear polar w. r. t. p and p is smooth on C. So, $p \in C$ is an inflection iff $C_p^{(2)} = \ell \cup \ell'$ with $\ell \cap \ell' \neq p$. The points $q \in \mathbb{P}_2$

[5]known as *Zeuthen's rule*

[6]geometrically, $R_{f,g}(p, q) = 0$ defines in the space of lines (pq) a figure whose irreducible components are pencils of lines centered at the intersection points and the multiplicities of these components are predicted by the Zeuthen rule

[7]in particular, $(n_1 C_1 + n_2 C_2, D) = n_1(C_1, D) + n_2(C_2, D)$, where $m_1 C_1 + m_2 C_2$ is a curve given by equation $F_1^{m_1} F_2^{m_2} = 0$ and $F_1 = 0$, $F_2 = 0$ are the equations for C_1 and C_2

[8]Note that this is not true in positive characteristic: for example, if char $(k) = 2$, then *all* tangents to the smooth conic $x_0^2 = x_1 x_2$ pass through one common point.

with degenerate quadratic polar $C_q^{(2)}$ form a curve H_C, which is called *the Hessian* of C. It is defined by the equation $\det C_q^{(2)} = 0$, which has degree $3(d-2)$ in q, where $d = \deg C$. Hence, an irreducible curve of degree $d \geqslant 3$ has at most $3d(d-2)$ inflections, which are contained in $C \cap H_C$. Again, this intersection contains also all singular points[9] of C.

Example 10.9: affine localization. Let us restrict the picture 10.4 onto affine chart where (pv) is the infinity, (uv) is the x-axis, and (up) is the y-axis. Then the line pencil through p turns to the family of vertical lines $x = \text{const}$. Consider affine equations $f(x,y) = 0$, $g(x,y) = 0$ for C_1, C_2 as (nonhomogeneous) polynomials in y with the coefficients in $k[x]$. Their resultant $R_{f,g}(x)$ is a polynomial in x and vanishes at $x = 0$. The multiplicity of this zero root coincides with $(C_1, C_2)_{(0,0)}$. If there are known some explicit analytic expressions of all the branches $y = \alpha_i(x)$ and $y = \beta_j(x)$ through x (even not algebraic, say several starting terms of the (formal fractional) power series expansions are OK), then $(C_1, C_2)_{(0,0)}$ usually can be also computed explicitly by looking at either the orders of $\alpha_i(x) - \beta_j(x)$ or the order of the resultant.

Dual curve Let $C \subset \mathbb{P}_2$ be an irreducible curve given by an equation $F(x) = 0$ of degree d. For any smooth $p \in C$ its tangent $\tau_p = T_pC$ defines a point $\tau_p^* = \operatorname{Ann} \tau_p \in \mathbb{P}^\times$ on the dual plane $\mathbb{P}_2^\times$. When p varies along C, τ_p^* also is running through some curve $C^\times \subset \mathbb{P}_2^\times$ called *a dual curve* for C. The degree of dual curve, i.e. the number of its intersection points with a generic line $\vartheta = q^\times \subset \mathbb{P}_2^x$, is nothing but the number of proper tangents to C living in a generic pencil of lines (centered at a generic point $q \in \mathbb{P}_2$). Thus $\deg(C^\times) = c(C)$.

Claim 10.10. *$C^{\times\times} = C$ (in particular, $\deg(C) = c(C^\times)$).*

Proof. A tangency $\vartheta = T_{\tau_1^*}C^\times \subset \mathbb{P}_2^\times$, at a smooth point $\tau_1^* \in C^\times$, is a limit of secant lines $\sigma = (\tau_1^*, \tau_2^*)$ as $\tau_2^* \to \tau_1^*$ (see figs. 10.5–10.6). The secant σ represents a pencil of lines through $\sigma^* = \tau_1 \cap \tau_2 \in \mathbb{P}_2$. Clearly, $\sigma^* \to p_1$ as $p_2 \to p_1$. $\square$

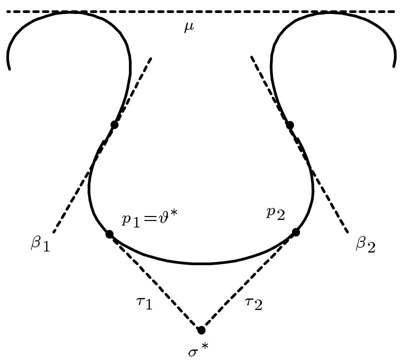

Fig. 10.5. A curve $C \subset \mathbb{P}_2$.

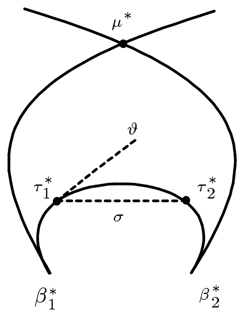

Fig. 10.6. The dual curve $C^\times \subset \mathbb{P}_2^\times$.

The figs. 10.5–10.6 also show that selfcontacts (cusps) turns into inflections and selfintersections — to *multiple tangents*[10] under the duality $C \longleftrightarrow C^\times$. In particular, if C has at most the simple singularities, then special proper tangents of $C^\times$ are exhausted by simple inflections and double tangents.

Plücker identities. Let C be a curve of class c and degree d with singularities exhausted by δ ordinary selfintersections of multiplicities $m_1, m_2, \dots, m_n$ and $\varkappa$ ordinary cusps. Then

$$c = d\,(d-1) - 3\,\varkappa - \sum_{\nu=1}^{\delta} m_\nu (m_{\nu-1}) \tag{30}$$

If we assume, in addition, that C has only ordinary inflections, then their number $\iota = \iota(C)$ equals

$$\iota = 3\,d\,(d-2) - 8\,\varkappa - 3\,\sum_{\nu=1}^{\delta} m_\nu (m_{\nu-1}) \tag{31}$$

This formulas are known as *the Plücker identities*. We shell prove them in the next lecture using a geometrical approach traced back to Chasles, Cayley and Brill[11]

Exercice 10.11. Let $q \notin C$ lie neither on an inflection tangency nor on a geometric tangency through a singular point of C; we write $C_q^{(d-1)}$ for $(d-1)$-th degree polar of q with respect to C. Deduce (30) from the equality $(C, C_q^{(d-1)}) = d(d-1)$ by proving that $(C, C_q^{(d-1)})_p$ equals 1 for smooth p, equals 3, if p is an ordinary cusp, and equals $m(m-1)$, if p is an ordinary m-typle selfintersection.

[9] In fact, even for singular curves one can write precise equations between degree, class, number of inflections and some data describing singularities; we'll do this below for curves with simplest singularities.

[10] a proper tangency is called *multiple*, if it touch the curve in several distinct points

[11] We follow the book: J. G. Semple, L. Roth, *Introduction to algebraic geometry*, Oxford, 1949.

HINT. In the first case p is smooth on $C_q^{(d-1)}$ and $T_p C_q^{(d-1)} \neq T_p C$; in the second case p is smooth on $C_q^{(d-1)}$ again, but $T_p C_q^{(d-1)}$ coincides with the cuspidal tangency; in the third case p is an $(m-1)$-typle point on $C_q^{(d-1)}$, but each geometrical tangency of C at p is transversal to $C_q^{(d-1)}$, that is, intersects it with multiplicity $(m-1)$. Now, use the Zeuthen rule.

If both C and $C^{\times}$ have at most the simple singularities, then the Plücker relations written for the both curves turn into

$$c = d\,(d-1) - 3\,\varkappa - 2\,\delta \qquad \iota = 3\,d\,(d-2) - 8\,\varkappa - 6\,\delta \qquad d = c\,(c-1) - 3\,\iota - 2\,\beta \qquad \varkappa = 3\,c\,(c-2) - 8\,\iota - 6\,\beta$$

where β is a number of bitangents to C. Any three of d, c, $\varkappa$, δ, β, ι can be found from these equations as soon as the other three are known.

Plane curves: Correspondences and Chasles - Cayley - Brill formula[1]

<u>BLOW UP OF $\Delta \subset \mathbb{P}_2 \times \mathbb{P}_2$.</u> Identify $\mathbb{P}_2^\times$ with the set of lines on $\mathbb{P}_2$ and consider the incidence graph

$$\mathcal{B} \stackrel{\text{def}}{=} \{(p,q,\ell) \mid p,q \in \ell\} \subset \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^\times$$

It is given by two quadratic equations $\sum \vartheta_\nu x_\nu = \sum \vartheta_\nu y_\nu = 0$ on $(x,y,\vartheta) \in \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^\times$. Topologically, $\mathcal{B}$ is a 4-dimensional compact manifold. A projection $\mathcal{B} \xrightarrow{\sigma} \mathbb{P}_2 \times \mathbb{P}_2$ is bijective outside the diagonal $\Delta \stackrel{\text{def}}{=} \{(p,p)\} \subset \mathbb{P}_2 \times \mathbb{P}_2$. Each fiber $\sigma^{-1}(p,p) = \{(p,p,\ell) \mid \ell \ni p\}$ over $(p,p) \in \Delta$ is naturally identified with the line pencil through p on $\mathbb{P}_2$ and 3-dimensional submanifold $E \stackrel{\text{def}}{=} \sigma^{-1}(\Delta) \subset \mathcal{B}$ is called *an exceptional divisor*. Let us denote the projections of $\mathcal{B}$ onto consequent factors of $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^\times$ by π_1, π_2, π_3 and write $A_1 = \pi_1^{-1}(\lambda)$, $A_2 = \pi_2^{-1}(\lambda)$, $M = \pi_3^{-1}(\lambda)$ for the full preimages of a generic line λ living on these planes. Topologically, A_1, A_2, and M are 3-dimensional cycles on $\mathcal{B}$ and their homology classes don't depend on the choice of the line λ in each plane. Any 1-parametric algebraic family of «pointed» lines $(pq) \subset \mathbb{P}_2$ can be pictured by an algebraic curve $\Gamma \subset \mathcal{B}$. Topological location of such a curve is described by a triple of numbers:

$$\begin{aligned}
\alpha_1 &= \#(\Gamma \cap A_1) && \text{— a number of } p\text{-points in } \Gamma = \{(p,q,\ell)\} \text{ laying on a generic line } \lambda \subset \mathbb{P}_2;\\
\alpha_2 &= \#(\Gamma \cap A_2) && \text{— a number of } q\text{-points in } \Gamma = \{(p,q,\ell)\} \text{ laying on a generic line } \lambda \subset \mathbb{P}_2;\\
\mu &= \#(\Gamma \cap M) && \text{— a number of lines } \ell \text{ in } \Gamma = \{(p,q,\ell)\} \text{ passing through a generic point}^2\ \lambda^\times \in \mathbb{P}_2.
\end{aligned}$$

Strongly speaking, we should use the topological intersection indices instead of «the numbers of points». But for all Γ we will consider below there is an open dense set of lines[3] such that all corresponding A_1, A_2, M intersect Γ transversally in a finite constant number of points. We will always suppose that α_1, α_2, μ are calculated using A_1, A_2, M taken from these open dense sets[4]. The triples $(p,q,\ell) \in \Gamma$ with $p = q$, i.e. the intersection points $\Gamma \cap E$, are called *exceptional*. Typically, Γ has a finite number of exceptional points. Our goal is to equip the exceptional points with appropriate multiplicities and express the number $\xi(\Gamma)$ of «exceptional points counted with multiplicities» through α_1, α_2, μ in the following three examples.

<u>Example 11.1:</u> join family. Let $C_1, C_2 \subset \mathbb{P}_2$ be two curves of degrees d_1, d_2 without common components. Fix any point $u \in \mathbb{P}_2$ outside the both curves and all the lines joining pairs of their intersection points. Then

$$\Gamma = \{(p,q,\ell) \in \mathcal{B} \mid p \in C_1,\ q \in C_2,\ \ell \ni u\}$$

is a curve in $\mathcal{B}$ given by an obvious triple of algebraic equations. Its exceptional points are $(p,p,(pu))$ with $p \in C_1 \cap C_2$, that is $\xi(\Gamma)$ counts the intersections of C_1 and C_2. Further, $\alpha_1 = \alpha_2 = d_1 d_2$, because a generic line λ intersect, say C_1, in d_1 distinct points $p_1, p_2, \dots, p_{d_1}$ and for each of them $C_2 \cap (up_\nu)$ consist of d_2 points. Finally, $\mu = d_1 d_2$ too, because there is a unique line $\ell = (u\lambda^\times)$ passing through a given $\lambda^\times$ and this line contains $d_1 d_2$ distinct pairs (p_i, q_j) with $p_i \in C_1 \cap \ell$, $q_j \in C_2 \cap \ell$, if $\lambda^\times$ is general enough.

<u>Example 11.2:</u> secant family. In the above example, let $C_1 = C_2 = C$ be the same irreducible curve of degree d given by an equation $F(x) = 0$ and u be a point outside the curve, all its singular tangents, and all lines joining pairs of distinct singularities. Then a closure of $\{(p,q,\ell) \in \mathcal{B} \mid p \neq q,\ p,q \in C,\ \ell \ni u\}$ is a curve given by the equations $F(x) = F(y) = \vartheta(u) = 0$ in $(x,y,\vartheta) \in \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^x$. Exceptional points of this curve are $(p,p,(pu))$ such that $\mathsf{mult}(C,(pu))_p \geqslant 2$, i.e. either $(pu) = T_pC$ or $p \in Sing(C)$. So, ξ counts singular points of C and proper tangents coming from c to C. By the same reasons as above, $\alpha_1 = \alpha_2 = \mu = d\,(d-1)$ in this case.

<u>Example 11.3:</u> tangent family. Let C be as above and Γ be a closure of $\{(p,q,\ell) \in \mathcal{B} \mid p \neq q,\ p,q \in C,\ \ell = T_pC\}$ (it is given by the equations $F(x) = F(y) = \widetilde{F}(x^{d-1}, y) = 0$). Exceptional points of Γ are (p,p,ℓ) such that $\mathsf{mult}(C,\ell)_p \geqslant 3$, i.e. either inflection tangents at smooth p or geometric tangents at singular p. So, $\xi(\Gamma)$ counts inflections and singularities. Since a simple tangent at a smooth point p intersect the curve in $(d-2)$ more points q, we have $\alpha_1 = d\,(d-2)$. Clearly, $\mu = c\,(d-2)$, because there are c proper tangents to C through generic $\lambda^\times \in \mathbb{P}_2$ by the definition of class. Finally, $\alpha_2 = d\,(c-2)$, because a generic line intersects C in d distinct smooth points $q_1, q_2, \dots, q_d$ and for each of them there

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[1] WARNING!!! In this lecture we suppose that the ground field k is algebraically closed and has zero characteristic.

[2] here $\lambda \subset \mathbb{P}_2^\times$ is the pencil of lines through a point $\lambda^\times \in \mathbb{P}_2$

[3] in the space of all lines

[4] it will be an exercise for readers, to check the existence of such open sets of lines in each of examples below

are $(c-2)$ proper tangents (q_jp) touching C at some $p \neq q_j$: when a point $q \notin C$ tends to some $q_j \in C$, precisely 2 of c tangents through q turn to $T_{q_j}C$ (see fig. 11.1; other arguments will appear in Example 11.8 and Exercice 11.9).

A correspondence on $\mathbb{P}_1$ is called *algebraic of type m-n*, if the pairs of corresponding points $(p,q) \in \mathbb{P}_1 \times \mathbb{P}_1$ form an algebraic curve $\gamma \subset \mathbb{P}_1 \times \mathbb{P}_1$ given by an irreducible bihomogeneous polynomial $g(t',t'')$ of bidegree (m,n) in $(t',t'') = \big((t'_0 : t'_1), (t''_0 : t''_1) \big)$. This curve is called *a graph* of the correspondence. So, *images* of a given point $p \in \mathbb{P}_1$ are presented by the equation $g(p,t) = 0$ in $t \in \mathbb{P}_1$ and *preimages* — by the equation $g(t,p) = 0$. Since these equations have respectively m and n ordinary distinct roots for almost all[5] p, a generic point has n distinct images and m distinct preimages under an m-n correspondence.

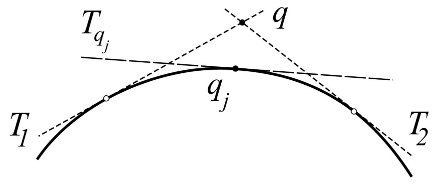

Fig. 11.1. Two proper tangent lines disappear as $q \to q_j$.

A point $e \in \mathbb{P}_1$ is called *a fixed point* of the correspondence, if it corresponds to itself[6], that is, if $g(e,e) = 0$. So, the set of all fixed points is $\gamma \cap \Delta$, where $\Delta = \{(t,t)\} \subset \mathbb{P}_1 \times \mathbb{P}_1$ is the diagonal. Since $g(t,t)$ is homogeneous of degree $m+n$ in $t = (t_0, t_1)$, any algebraic m-n correspondence has $m+n$ fixed points counted with multiplicities, where the multiplicity of a fixed point e means the local intersection multiplicity $(\gamma, \Delta)_{(e,e)}$, of the curve $g(t',t'') = 0$ and the line $t' = t''$ at the point $(e,e) \in \gamma$. In particular, it can be calculated by the Zeuthen rule applied in any affine chart $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}_2 \ni (e,e)$ as it was explained in **lecture 10**.

Claim 11.4. *Let $U \subset \mathbb{P}_1$ be an affine chart with the origin at a fixed point e of a correspondence γ, x be an affine coordinate on U, and $y_1(x)$, $y_2(x)$, ... , $y_m(x)$ be all γ-images of x which tend to 0 as $x \to 0$. Then $(\gamma, \Delta)_{(0,0)}$ equals the sum of orders of infinitesimals $y_i(x) - x$ with respect to x.*

Proof. Let (x,y) be affine coordinates on $\mathbb{A}^2 = U \times U$ (see fig. 11.2). Since the both lines $x = 0$ and $y = 0$ contain just one intersection point $(0,0) \in \gamma \cap \Delta$ we can use the line pencil $x = \text{const}$ parameterized by the x-axis to calculate $(\gamma, \Delta)_{(0,0)}$ as the sum of orders of infinitesimals $\alpha_i(x) - \beta_j(x)$ where α_i and β_j run trough the intersections of a vertical line $x = \text{const}$ respectively with γ and with Δ. So, $\alpha_i(x) = y_i(x)$ and there is just one $\beta(x) = x$. $\square$

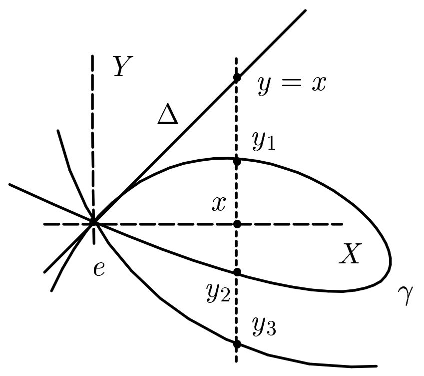

Fig. 11.2. Fixed point.

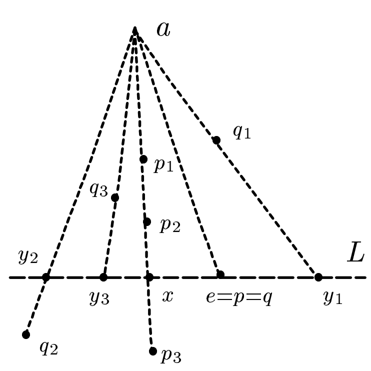

Fig. 11.3. Exceptional point.

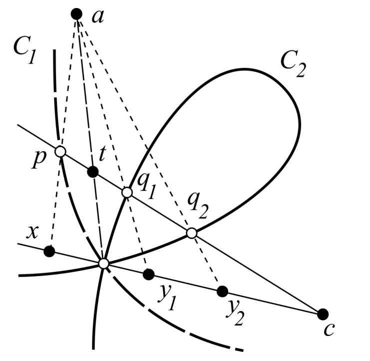

Fig. 11.4. Intersection point.

Exceptional point multiplicities. A curve $\Gamma \subset \mathcal{B}$, of pointed lines (p,q,ℓ), defines an algebraic correspondence on $\mathbb{P}_1$ as follows. Fix a point $a \in \mathbb{P}_2$ such that it lies on precisely μ distinct lines ℓ of Γ and a generic line through a contains exactly α_1 distinct p-points and exactly α_2 distinct q-points of Γ. Then consider the pencil of lines through a as $\mathbb{P}_1$ in question and say that $(ap) \longleftrightarrow (aq)$ iff $(p,q,\ell) \in \Gamma$ for some[7] ℓ. This is an algebraic α_1-α_2 correspondence, because a generic point has α_2 images and α_1 preimages, certainly. A line through a corresponds to itself under γ_Γ precisely in two cases: either it belongs to Γ, i. e. contains 2 points $p \neq q$ such that $(p,q,\ell) \in \Gamma$, or it pass through an exceptional point e such that $(e,e,\ell) \in \Gamma$ for some ℓ.

Let us define the multiplicity of an exceptional point $(e,e,\ell) \in \Gamma$ as a multiplicity of the corresponding fixed point (ae) of the correspondence γ_Γ. By Claim 11.4, it can be calculated geometrically as follows. Parameterize the pencil of lines through a by some line $L \not\ni a$, which pass through an exceptional point e (see fig 11.3), and fix on L an affine coordinate x centered at e. Let (ay_1), (ay_2), ... , (ay_m), where $y_\nu \in L$, be all lines corresponding to (ax) and tending to (ae) as $x \to e$. Then the multiplicity of (pe) equals the sum of orders of infinitesimals $y_i(x) - x$ w. r. t. x as $x \to 0$.

Claim 11.5. *A total number of exceptional points counted with multiplicities equals*[8] $\xi(\Gamma) = \alpha_1 + \alpha_2 - \mu$.

[5] that is, for all p outside some finite subset on $\mathbb{P}_1$ where the both discriminants vanish
[6] of course, besides itself, a fixed point may have several other (pre) images as well
[7] of course, $\ell = (pq)$, if $p \neq q$
[8] this formula is known as *the Chasles - Cayley - Brill formula*

Proof. Since $\deg(\gamma_\Gamma) = (\alpha_1, \alpha_2)$, it has $\alpha_1 + \alpha_2$ fixed points. By the choice of a, each line (ap) such that $(p, q, (ap)) \in \Gamma$ for some $q \neq p$, $q \in (ap)$ has multiplicity 1 as a fixed point for γ_Γ. The residuary contribution of exceptional fixed points equals $\xi(\Gamma)$. $\square$

Example 11.6: exceptional point multiplicities in join family (continuation of Example 11.1). In this case the multiplicity of an exceptional point $(e, e, (ce)) \in \Gamma$ coincides with the local intersection number $(C_1, C_2)_e$ (see fig. 11.4). Indeed, use the line pencil centered at c to compute $(C_1, C_2)_e$ by the Zeuthen rule as it was explained in the previous lecture. If we take a outside all geometric tangents to the both curves at e and such that e is the only intersection point of C_1 and C_2 on (ae), then $(C_1, C_2)_e$ is a sum of orders $p - q_j$ w.r.t. t as $t \to 0$. But it is clear from fig. 11.4 that $p - q_j$ is like $x - y_j$ and t is like x as soon the both lines (ce) and (ae) do not touch the branches of C_1, C_2 at e. So, $\xi(\Gamma)$ is the sum of local intersection numbers of C_1 and C_2, i.e. $(C_1, C_2) = \xi(\Gamma) = \alpha_1 + \alpha_2 - \mu = d_1 d_2 + d_1 d_2 - d_1 d_2 = d_1 d_2$. We get the Bézout theorem.

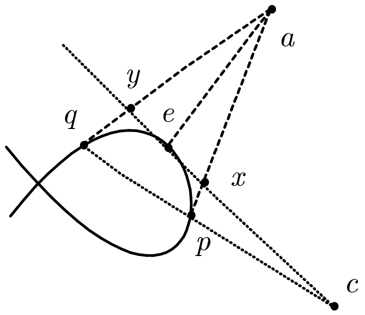

Fig. 11.5.

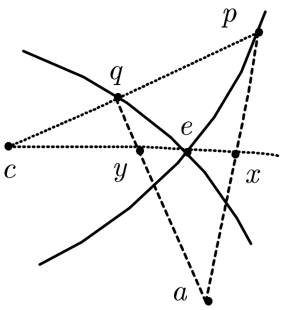

Fig. 11.6.

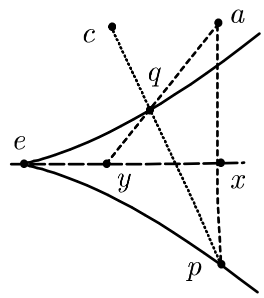

Fig. 11.7.

Example 11.7: exceptional point multiplicities in secant family (continuation of Example 11.2). There are 3 types of exceptional points here. A proper tangency $(e, e, (ce))$ (see fig. 11.5) has multiplicity 1, because any x closed to e has a unique image y coming to e when $x \to e$ and $y - x$ is like $x - e$. If e is an ordinary m-type selfintersection, then a line through c closed to (ce) contains m p-points running through m branches of C; each such p-point produces $(m-1)$ q-points coming from other $(m-1)$ branches; so, there are $m(m-1)$ differences $y - x$ and each of them is like $(x - e)$ (see fig. 11.6) as soon (ce) does not touch any branch. If e is an ordinary cusp (see fig. 11.7, where the line pencil through a is parameterized by the cuspidal tangent), then any x closed to e produces two p-points (intersections of (ax) with two branches of C) and each of them has just one q-point (intersection of (cp) with the other branch of C); it is easy to see from fig. 11.7 that $(x - y) \sim (p - q) \sim (x - e)^{3/2}$ as $x \to e$. So, the cuspidal point contributes $2 \cdot (3/2) = 3$. Hence,

$$c + \sum_{\nu}^{\delta} m_\nu (m_\nu - 1) + 3\varkappa = \xi(\Gamma) = \alpha_1 + \alpha_2 - \mu = d(d-1) + d(d-1) - d(d-1) = d(d-1)$$

by the Chasles - Cayley - Brill formula. This gives the first Plücker identity.

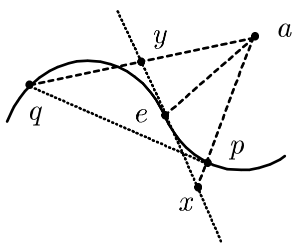

Fig. 11.8.

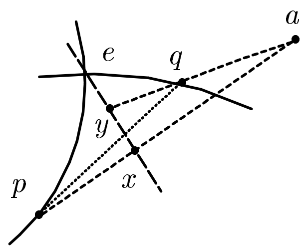

Fig. 11.9.

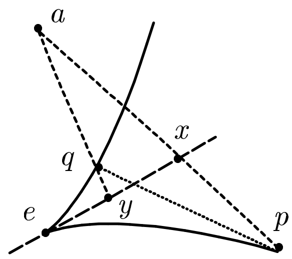

Fig. 11.10.

Example 11.8: exceptional point multiplicities in tangent family (continuation of Example 11.3). The figs. 11.8–11.10 show that each difference $(y - x)$ is like $(x - e)$ for all three types of exceptional points. An inflection point (fig. 11.8) produces a unique difference and has a multiplicity 1; each of m branches through an m-typle selfintersection (fig. 11.9) produces $(m-1)$ differences, so the multiplicity equals $m(m-1)$ here; a cusp (fig. 11.10) produces 2 differences and has multiplicity 2. Hence,

$$\iota + \sum_{\nu}^{\delta} m_\nu (m_\nu - 1) + 2\varkappa = \xi(\Gamma) = \alpha_1 + \alpha_2 - \mu = d(d-2) + d(c-2) - c(d-2) = d^2 - 4d + 2c$$

by the Chasles - Cayley - Brill formula. Since $c = d(d-1) - \sum m_\nu (m_\nu - 1) - 3\varkappa$, we get the second Plücker identity $\iota = 3d(d-2) - 3\sum m_\nu (m_\nu - 1) - 8\varkappa$.

Exercice 11.9. Let c be a smooth point of C in Example 11.2 and Example 11.7. Check that a multiplicity of an exceptional point $(c, c, T_c C) \in \Gamma$ equals 2 (so, we have really $\alpha_2 = d(c-2)$ in Example 11.3).

Affine algebraic – geometric dictionary

<u>Affine varieties: their ideals and coordinate algebras.</u> Let $X = V(J) \subset \mathbb{A}^n$ be an affine algebraic variety given by some ideal $J =\subset k[x_1, x_2, \dots, x_n]$ of polynomial equations. We write $I(X)$ for the ideal of all polynomials vanishing along X. If k is algebraically closed, then by Hilbert's Nullstellensatz

$$I(X) = \sqrt{J} \overset{\text{def}}{=} \{\, f \in k[x_1, x_2, \dots, x_n] \mid f^n \in J \text{ for some } n \in \mathbb{N}\,\}$$

is *the radical* of J. Clearly, $V(J_1) \subset V(J_2) \iff \sqrt{J_1} \supset \sqrt{J_2}$.

Exercice 12.1. Let $O \in \mathbb{A}_n$ be the origin; prove that $I(\{O\})$ is maximal ideal generated by x_i.

So, finitely generated commutative k-algebra $k[X] \overset{\text{def}}{=} k[x_1, x_2, \dots, x_n]/I(X)$ is *reduced*, i.e. does not contain nilpotent elements: $a^n = 0 \Rightarrow a = 0$. This algebra is called *coordinate* (or *structure*) algebra of the affine algebraic variety $X \subset \mathbb{A}^n$. Geometrically, $k[X]$ consists of functions $X \xrightarrow{v \mapsto f(v)} k$ obtained by restricting the polynomials $f \in k[x_1, x_2, \dots, x_n]$ onto $X \subset \mathbb{A}^n$; such the functions are called *regular* algebraic functions on X.

<u>Claim 12.2</u>. *Each reduced finitely generated algebra A over an algebraically closed field can be realized as $A = k[X]$ for some affine algebraic variety X.*

Proof. Since $A = k[x_1, x_2, \dots, x_n]/I$ is reduced, $f^n \in I \Rightarrow f \in I$ for any $f \in k[x_1, x_2, \dots, x_n]$. By Hilbert, this means that $X = V(I) \subset \mathbb{A}^n$ has $I(X) = I$ and $A = k[x_1, x_2, \dots, x_n]/I(X) = k[X]$. □

<u>Points.</u> Given a point $p \in X$, the evaluation map $\mathrm{ev}_p : k[X] \xrightarrow{f \mapsto f(p)} k$ coincides with the factorization $k[X] \xrightarrow{f \mapsto f \,(\mathrm{mod}\, \mathfrak{m}_p)} k[X]/\mathfrak{m}_p$, where $\mathfrak{m}_p \overset{\text{def}}{=} \{f \in k[X] \mid f(p) = 0\}$. Hence, $\mathfrak{m}_p = \ker(\mathrm{ev}_p)$ is a proper maximal ideal in $k[X]$. It is called *a maximal ideal* of p.

<u>Claim 12.3</u>. *If k is algebraically closed, then the correspondences $p \longleftrightarrow \mathrm{ev}_p \longleftrightarrow \mathfrak{m}_p$ establish bijections between the points of X, the homomorphisms $k[X] \longrightarrow k$ identical on k, and the proper maximal ideals in $k[X]$.*

Proof. The second correspondence is obviously bijective. Clearly, $p \neq q \Rightarrow \mathfrak{m}_p \neq \mathfrak{m}_q$, because there is, say, a linear form $\mathbb{A}^n \xrightarrow{\ell} k$ such that $\ell \in \mathfrak{m}_p$ but $\ell \notin \mathfrak{m}_q$. Finally, consider a proper maximal ideal $\mathfrak{m} \subset k[X]$ and write $k[X]$ as $k[x_1, x_2, \dots, x_n]/I(X)$. A full preimage $\widetilde{\mathfrak{m}} \subset k[x_1, x_2, \dots, x_n]$ of $\mathfrak{m}$ is a proper maximal ideal, because of $\widetilde{\mathfrak{m}} \supset I(X)$ and $k[x_1, x_2, \dots, x_n]/\widetilde{\mathfrak{m}} = k[X]/\mathfrak{m} = k$. Hence, $V(\widetilde{\mathfrak{m}}) \subset \mathbb{A}^n$ is nonempty and is contained in X, i.e. there is a point $p \in X$ such that $f(p) = 0$ for any $f \in \mathfrak{m}$. So, $\mathfrak{m} \subset \mathfrak{m}_p$ and $\mathfrak{m} = \mathfrak{m}_p$, because $\mathfrak{m}$ is maximal. □

<u>Algebraic varieties via spectra.</u> A set of all proper maximal ideals in a given k-algebra A is called *a maximal spectrum* of A and is denoted by $Spec_m(A)$. We can treat an affine algebraic variety over an algebraically closed field k pure algebraically as a maximal spectrum $Spec_m(A)$ of an arbitrary finitely generated reduced k-algebra A whose elements $f \in A$ are considered as the functions $Spec_m(A) \xrightarrow{\mathfrak{m} \mapsto f \,(\mathrm{mod}\, \mathfrak{m})} k$.

<u>Regular morphisms of algebraic varieties.</u> Any map of sets $X \xrightarrow{\varphi} Y$ produces *a pull back homomorphism* φ^* from the algebra off all k-valued functions on Y to the one on X; it sends $Y \xrightarrow{f} k$ to the composition

$$\varphi^* f \overset{\text{def}}{=} f \circ \varphi \,:\, X \xrightarrow{\varphi} Y \xrightarrow{f} k\,.$$

A map $X \xrightarrow{\varphi} Y$ between affine algebraic varieties is called *a regular morphism* of algebraic varieties, if φ^* sends the regular algebraic functions on Y to the regular algebraic functions on X, i.e. induces a well defined homomorphism of coordinate algebras $k[Y] \xrightarrow{\varphi^*} k[X]$.

<u>Claim 12.4</u>. *Let A, B be finitely generated reduced algebras over any algebraically closed field k. Then each homomorphism $B \xrightarrow{\psi} A$ such that $\psi(1) = 1$ is a pull back homomorphism: $\psi = \varphi^*$ for a unique regular map $Spec_m(A) \xrightarrow{\varphi} Spec_m(B)$; this map sends a maximal ideal $\mathfrak{m} \subset A$ to its full preimage ${\varphi^*}^{-1}(\mathfrak{m}) \subset B$.*

Proof. Let $Spec_m(A) \xrightarrow{\varphi} Spec_m(B)$ be a regular morphism, $p \in Spec_m(A)$ be a point, and $f \in B$ be a function on $Spec_m(B)$. Then $f(\varphi(p)) = 0 \iff \varphi^* f(p) = 0$, i.e. $f \in \mathfrak{m}_{\varphi(p)} \iff \varphi^*(f) \in \mathfrak{m}_p$. So, if $B \xrightarrow{\psi} A$ is a pull back of some $Spec_m(A) \xrightarrow{\varphi} Spec_m(B)$, then φ has to send $\mathfrak{m}_p \longmapsto \psi^{-1}(\mathfrak{m}_p)$ for each $p \in Spec_m(A)$. On the other hand, $\psi^{-1}(\mathfrak{m}) \subset B$ is proper maximal ideal for any proper maximal $\mathfrak{m} \subset A$, because of $\varphi^*(1) = 1 \Rightarrow B/\psi^{-1}(\mathfrak{m}) = \mathrm{im}\,(\psi)/(\mathfrak{m} \cap \mathrm{im}\,(\psi)) \simeq k$.

So, $\varphi : Spec_m(A) \xrightarrow{\mathfrak{m}_p \mapsto \psi^{-1}(\mathfrak{m}_p)} Spec_m(B)$ is well defined map of sets. To compute its pull back homomorphism, note that for any $b \in B$, $p \in Spec_m(A)$ we have

$$\varphi^* b(p) = b(\varphi(p)) = b \;\left(\operatorname{mod} \mathfrak{m}_{\varphi(p)}\right) = b \;\left(\operatorname{mod} \psi^{-1}(\mathfrak{m}_p)\right) = \psi(b) \;(\operatorname{mod} \mathfrak{m}_p) = \psi b(p) \,.$$

So, $\varphi^* = \psi$ as required. □

<u>Geometric schemes.</u> Let A be an arbitrary algebra over a field k. The set of all its nilpotents $\mathfrak{n}(A) \subset A$ is an ideal, which is contained in any proper maximal ideal $\mathfrak{m} \subset A$, because $A/\mathfrak{m} = k$ has no nilpotent elements. So, a factor algebra $A_{\text{red}} \stackrel{\text{def}}{=} A/\mathfrak{n}$ is reduced and has the same maximal spectrum $X = Spec_m A = Spec_m A_{\text{red}}$. If k is algebraically closed, then the intersection of all maximal ideals in A_{red} is zero, because it consists of all functions vanishing everywhere on the affine algebraic variety X. Hence, the intersection of all maximal ideals in A coincides with $\mathfrak{n}(A)$.

Exercice 12.5. Show that in general situation, where A is an arbitrary commutative ring, $\mathfrak{n}(A)$ coincides with the intersection of all proper *prime*[1] ideals $\mathfrak{p} \subset A$

A pair $(A, Spec_m A)$, where A is an arbitrary finitely generated algebra over algebraically closed field, is called *an affine geometrical scheme*. Affine algebraic variety $X = Spec_m A = Spec_m A_{\text{red}}$ is called *a support* of this scheme. Intuitively, the scheme differs from X by allowing some «infinitezimals», that is «nilpotent functions», whose «numerical values« vanish everywhere on X. Usually, these nilpotent elements encode some «multiplicities» on X.

<u>Example 12.6</u>: an intersection of affine algebraic varieties $X, Y \subset \mathbb{A}^n$ is defined as $V(I(X) + I(Y))$, i.e. by the union of all equations for X and Y. If the intersection is non-transversal, a factor algebra

$$A = k[x_1, x_2, \dots, x_n]/(I(X) + I(Y))$$

is not reduced: say, if $X = V(x)$, $Y = V(x^2 - y) \subset \mathbb{A}_2 = Spec_m(k[x,y])$, then $k[x,y]/(x, y^2 - x) \simeq k[y]/(y^2)$ has quadratic nilpotent y.

Of course, geometrically, the intersection points are exhausted by an algebraic variety $Spec_m A_{\text{red}}$ (for $A = k[y]/(y^2)$ it consists of unique point $Spec_m(k)$). But a significant information will be lost under the replacement of A by A_{red}. If we want, say, to develop an intersection multiplicities technique, then we have to treat intersections as *schemes* rather than as algebraic varieties and investigate the difference between A and A_{red}.

By the definition, a *regular morphism* $(A, Spec_m A) \xrightarrow{(\varphi^*, \varphi)} (B, Spec_m B)$ of schemes is a pair that consists of an algebra homomorphism $B \xrightarrow{A}$ and a regular map $Spec_m(A) \xrightarrow{\varphi} Spec_m(B)$ of supporting algebraic varieties such that $\varphi^* f(p) = f(\varphi(p))$ for all $f \in B$, $p \in Spec_m(A)$. Note that now φ^* can not be uniquely recovered from φ, because the last one does not seen nilpotent infinitesimals!

<u>A direct product of affine algebraic varieties.</u> Let A, B be finitely generated k-algebras. Then the tensor product of algebras[2] $A \otimes B$ is a finitely generated k-algebra with the multiplication

$$(A \otimes B) \times (A \otimes B) \xrightarrow{(a \otimes b, \alpha \otimes \beta) \mapsto a\alpha \otimes b\beta} A \otimes B \,.$$

<u>Claim 12.7</u>. *A set-theoretical product of affine algebraic varieties $X = Spec_m A$, $Y = Spec_m B$ has a natural structure of an affine algebraic variety w.r.t. which the projections $X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$ are regular. Namely, $X \times Y = Spec_m(A \otimes B)$. Moreover, it has the following universal property: for any two regular maps of affine algebraic varieties: $X \xleftarrow{\varphi} Z \xrightarrow{\psi} Y$ there exists a unique regular map $Z \xrightarrow{\varphi \times \psi} X \times Y$ fitting into commutative diagram*

$$\begin{tikzcd}
& X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X & & Y \\
& Z \arrow[ul, "\varphi"] \arrow[ur, "\psi"'] \arrow[uu, "\varphi \times \psi"] &
\end{tikzcd}$$

Proof. Let us *define* $X \times Y$ as $Spec_m(A \otimes B)$ and the projection maps π_X, π_Y as the regular maps whose pull-back homomorphisms are the canonical algebra inclusions

$$\alpha : A \xhookrightarrow{a \mapsto a \otimes 1} A \otimes B \xhookleftarrow{1 \otimes b \leftarrow b} B : \beta \,,$$

[1] an ideal $\mathfrak{p} \subset A$ is called *prime*, if $A/\mathfrak{p}$ has no zero divisors

[2] as a vector space over k it coincides with the tensor product of vector spaces A, B and consists of all *finite* sums $\sum a_\nu \otimes b_\nu$ with $a_\nu \in A$, $b_\nu \in B$; for example, $k[x] \otimes k[y]$ is naturally isomorphic to $k[x, y]$

Then the universal property from the claim is nothing but the universal property characterizing the tensor product of algebras: if $Z = Spec_m(C)$, then for any two algebra homomorphisms $A \xrightarrow{\varphi^*} C \xleftarrow{\psi^*} B$ there is a unique homomorphism $A \otimes B \xrightarrow{\varphi^* \otimes \psi^*} C$ such that $(\varphi^* \otimes \psi^*) \circ \alpha = \varphi^*$ and $(\varphi^* \otimes \psi^*) \circ \beta = \psi^*$.

Exercice 12.8. Deduce this universality of the tensor product of algebras w.r.t. the algebra homomorphisms from the universality of the tensor product of vector spaces w.r.t. bilinear maps (discussed in «Tensor Guide»).

In particular, on the level of points, which are nothing but the regular morphisms from $Z = Spec_m(k)$ (or equivalently, algebra surjections onto k), we have a set-theoretical bijection

$$Spec_m(A) \times Spec_m(B) \simeq \mathrm{Hom}(Spec_m(k), X) \times \mathrm{Hom}(Spec_m(k), Y) \simeq \mathrm{Hom}(Spec_m(k), X \times Y) \simeq Spec_m(A \otimes B).$$

It remains to show that $A \otimes B$ is reduced, i.e. we have defined an affine variety but not only a scheme. Let us write $f \in A \otimes B$ as $\sum a_\nu \otimes b_\nu$ with linearly independent (over k) functions $b_\nu \in B$. If our f produces a zero function on $X \times Y$, then $f(p,q) = 0 \ \forall (p,q) \in X \times Y \Rightarrow \sum a_\nu(p) \cdot b_\nu = 0$ in $k[Y] \Rightarrow$ all $a_\nu(p) = 0 \quad \forall p \in X \quad \Rightarrow$ all $a_\nu = 0$ in $A \Rightarrow f = 0$ in $A \otimes B$. □

ZARISKI TOPOLOGY. Any affine algebraic variety $X = Spec_m A$ admits a canonical topology whose closed sets are zeros of ideals $I \subset A$, i.e. have a form $V(I) = \{x \in X \mid f(x) = 0 \quad \forall f \in I\}$.

Exercice 12.9. Check that $V(I)$ satisfy the closed set properties, namely, $\varnothing = V(1)$; $X = V(0)$; $\bigcap V(I_\nu) = V(\sum I_\nu)$, where $\sum I_\nu$ consists of all finite sums $\sum f_\nu$ with $f_\nu \in I_\nu$; $V(I) \cup V(J) = V(IJ)$, where IJ is an ideal spanned by all products ab with $a \in I$, $b \in J$.

Since any ideal $I \subset k[X]$ is finitely generated, each closed set may be presented as a finite intersection of hypersurfaces: $V(I) = V(f_1, f_2, \dots, f_m) = \bigcap V(f_\nu)$. So, any Zariski open set is a finite union of *principal* open sets $\mathscr{D}(f) \stackrel{\text{def}}{=} X \setminus V(f) = \{x \in X \mid f(x) \neq 0\}$. Zariski topology has a pure algebraic nature: its properties are far enough from the metric topology standards, because Zariski neighborhoods express rather some divisibility conditions than any distance relations.

Example 12.10: irreducible closed sets. A topological space X is called *reducible*, if $X = X_1 \cup X_2$ for some proper closed subsets $X_1, X_2 \subset X$. It's a vapid notion for the usual metric topology, where everything is reducible. In Zariski topology, the reducibility means an existence of non zero $f_1, f_2 \in k[X]$ such that f_1 vanishes along X_1 and f_2 vanishes along X_2. Since $f_1 f_2$ vanishes everywhere, $f_1 f_2 = 0$ in $k[X]$. So, X is reducible iff $k[X]$ has zero divisors. For example, a hypersurface $\{g(x) = 0\} \subset \mathbb{A}^n$ is irreducible iff g is a power of an irreducible polynomial. Irreducible algebraic sets generalize the prime numbers in arithmetics.

Claim 12.11. *Any affine algebraic variety admits a unique finite decomposition $X = \bigcup X_i$, where $X_i \subset X$ are irreducible proper closed subsets such that $X_i \not\subset X_j \ \forall\, i \neq j$ (they are called irreducible components of X).*

Proof. A decomposition is constructed step by step. If X is reducible, the first step takes $X = Z_1 \cup Z_2$, where $Z_{1,2}$ are proper closed subsets. Let $X = \bigcup Z_\nu$ after n steps. If each Z_ν is irreducible, then for each ν and any irreducible closed subset $Y \subset X$ either $Y \cap Z_\nu = \varnothing$ or $Y \subset Z_\nu$, because of $Y = \bigcup (Z_\nu \cap Y)$. So, if we take away all Z_ν contained in some other Z_μ, then we get the required decomposition and it is unique. If there are some reducible Z_ν after n steps, $(n+1)$-th step replaces each of them by an union of two proper closed subsets. If this procedure would never stop, then it produces an infinite chain of strictly embedded subsets $X \supset Y_1 \supset Y_2 \supset \dots$, i.e. an infinite chain of strictly increasing ideals $(0) \subset I_1 \subset I_2 \subset \dots$, which does not exist in the Noetherian algebra $k[X]$. □

Example 12.12: «big» open sets. Zariski topology is week and non Hausdorf. For example, $Z \subset \mathbb{A}^1$ is Zariski closed iff Z is finite. If X is irreducible, then any non empty open $U_1, U_2 \subset X$ have a nonempty intersection, because of $X = (X \setminus U_1) \cup (X \setminus U_2)$ in the contrary case.

Exercice 12.13. Prove that $f = g$ in $k[X]$, if X is irreducible and $f|_U = g|_U$ over some open non-empty $U \subset X$.
HINT. If $U = \mathscr{D}(h)$, then $X = V(h) \cup V(f - g)$.

Example 12.14: Zariski topology on $X \times Y$ is finer than the product of Zariski topologies on X and Y, because the closed $Z \subset X \times Y$ are not exhausted by the products of closed subsets on X, Y. For example, if $X = Y = \mathbb{A}^1$, then any curve, say a hyperbola $V(xy - 1)$, is closed in Zariski topology on $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, whereas the products of closed sets on $\mathbb{A}^1$ are exhausted by finite unions of isolated points and coordinate lines.

Claim 12.15. *A regular morphism $X \xrightarrow{\varphi} Y$ of algebraic varieties is continuous in Zariski topology.*

Proof. A preimage $\varphi^{-1}(Z)$ of a closed $Z = V(I) \subset Y$ consists of all $x \in X$ such that $0 = f(\varphi(x)) = {\varphi^*}^{-1} f(x)$ for all $f \in I$. So, it coincides with the zero set of an ideal ${\varphi^*}^{-1}(I) \subset k[X]$. □

MORPHISM DECOMPOSITION. Let $X \xrightarrow{\varphi} Y$ be a regular morphism of affine algebraic varieties. Then, k-algebra homomorphism $k[Y] \xrightarrow{\varphi^*} k[X]$ can be decomposed as $k[Y] \twoheadrightarrow \mathrm{im}(\varphi^*) \hookrightarrow k[X]$. Since $k[X]$ is reduced, $\mathrm{im}(\varphi^*) \subset k[X]$ is reduced too, i.e. there is an affine algebraic variety $Z = Spec_m \, \mathrm{im}(\varphi^*)$ such that $X \xrightarrow{\varphi} Y$ is decomposed as $X \xrightarrow{\varphi_1} Z \xrightarrow{\varphi_2} Y$, where $k[Y] \xrightarrow{\varphi_2^*} k[Z]$ is surjective, $k[Z] \xrightarrow{\varphi_1^*} k[X]$ is injective. The injectivity of φ_1^* means that there are no non zero functions $f \in k[Z]$ vanishing along $\varphi_1(X)$, i.e. $\varphi_1(X) \subset Z$ is dense. The surjectivity of φ_2^* means that φ_2 induces an isomorphism between Z and a closed subset $V(\ker \varphi_2^*) \subset Y$ given by the ideal $\ker(\varphi_2^*) = \ker(\varphi) \subset k[Y]$ (which is automatically radical). In other words, the Zariski closure $Z = \overline{\varphi(X)} \subset Y$ is an affine algebraic variety and the maps $X \longrightarrow \overline{\varphi(X)} \hookrightarrow Y$ are regular morphisms.

<u>DOMINANT MORPHISMS, CLOSED EMBEDDINGS, AND FINITE MORPHISMS.</u> If X is irreducible and the algebra homomorphism $k[Y] \xrightarrow{\varphi^*} k[X]$ is injective, then the corresponding morphism $X \xrightarrow{\varphi} Y$ is called *dominant*. Geometrically, this means that $\overline{\varphi(X)} = Y$. If X is reducible, then φ is called *dominant* when its restriction onto *each* irreducible component of X is dominant. A morphism $X \xrightarrow{\varphi} Y$ is called *a closed embedding*, if $k[Y] \xrightarrow{\varphi^*} k[X]$ is surjective. This means that φ identifies X with $V(\ker \varphi^*) \subset Y$.

Exercice 12.16. Show that any dominant morphism of irreducible affine varieties $X \xrightarrow{\varphi} Y$ can be decomposed as

$$X \xhookrightarrow{\psi} Y \times \mathbb{A}^m \xrightarrow{\pi} Y \,, \tag{32}$$

where ψ is a closed embedding and π is the natural projection along $\mathbb{A}^m$.

HINT. Let $A = k[X]$, $B = k[Y]$. Since A is finitely generated B-algebra via the inclusion $B \xhookrightarrow{\varphi^*} A$, there is an epimorphism of B-algebras $B[x_1, x_2, \dots, x_m] \xrightarrow{\psi} A$ for some m.

Given a regular morphism $X \xrightarrow{\varphi} Y$, then the coordinate algebra $k[X]$ can be considered as an algebra over $\varphi^*(k[Y]) = k[\overline{\varphi(X)}] \subset k[X]$. A morphism φ is called *finite*, if $k[X]$ is integer over $\varphi^*(k[Y])$. Since $k[X]$ is finitely generated as algebra over $\varphi^*(k[Y])$ (even over k), finiteness of φ means that $k[X]$ is finitely generated also as $\varphi^*(k[Y])$-*module*, i.e. there are some $f_1, f_2, \dots, f_m \in k[X]$ such that any $h \in k[X]$ can be written as $h = \sum \varphi^*(g_i)\, f_i$ for some $g_i \in k[Y]$.

<u>Claim 12.17</u>. *Let $X \xrightarrow{\varphi} Y$ be a finite morphism of affine algebraic varieties. Then $\varphi(Z)$ is closed for any closed $Z \subset X$ and induced morphism $Z \xrightarrow{\varphi|_Z} \varphi(Z)$ is finite. Moreover, if X is irreducible and $Z \neq X$, then $\varphi(Z) \neq Y$.*

Proof. Let $I = I(Z) \subset k[X]$ be the ideal of $Z \subset X$. Then $Z \xrightarrow{\varphi|_Z} Y$ has $\varphi_z^* : k[Y] \xrightarrow{\varphi^*} k[X] \longrightarrow k[X]/I$. Since $k[X]$ is finitely generated as $\varphi^*(k[Y])$-module, $k[Z] = k[X]/I$ is finitely generated as a module over

$$k[\,\overline{\varphi(Z)}\,] = \varphi|_z^*(k[Y]) = \varphi^*(k[Y])/(I \cap \varphi^*(k[Y]))\,,$$

i.e. $Z \longrightarrow \overline{\varphi(Z)}$ is a finite morphism. To prove that $\varphi(Z) = \overline{\varphi(Z)}$, we can restrict ourself onto irreducible components of Z, i.e. suppose that Z is irreducible. Let $B = k[Z]$, $A = k[\,\overline{\varphi(Z)}\,] \subset B$, and $f_1, f_2, \dots, f_m$ generate B as A-module. Since $\varphi|_z$ takes a maximal ideal $\mathfrak{m}_p \subset B$ to the maximal ideal $\mathfrak{m}_p \cap A \subset A$, a point $q \in Spec_m(A)$ does not belong to $Spec_m(B)$ iff its maximal ideal $\mathfrak{m}_q \subset A$ generates non proper ideal in B, that is $\mathfrak{m}_q \cdot B = B$. In this case we can write $f_i = \sum \beta_{i\nu} f_\nu$ for some $\beta_{i\nu} \in \mathfrak{m}_q$; that is the zero homomorphism of A-modules: $B \longrightarrow 0$, which takes each f_i to zero, can be presented in terms of the generator system $\{f_\nu\}$ by the matrix $E - (\beta_{\nu i})$. Hence, the multiplication by $\det(E - (\beta_{ij}))$ annihilates B. Since there no zero divisors in $B = k[Z]$, we get $\det(E - (\beta_{ij})) = 0$. But $\det(E - (\beta_{ij})) = 1 + \beta$ where $\beta \in \mathfrak{m}_q$. So, $1 \in \mathfrak{m}_q$ and $\mathfrak{m}_q \subset A$ is non proper as well.

To prove that $\varphi(Z) \neq Y$ for $Z \subsetneq X$, let us take a non zero function $f \in k[X]$ vanishing along Z and write the integer equation of the lowest possible degree for f over $\varphi^*(k[Y])$ as

$$f^m + \varphi^*(g_1)\, f^{m-1} + \cdots + \varphi^*(g_{m-1})\, f + \varphi^*(g_m) = 0\,.$$

Computing its left side at any $z \in Z$, we get $\varphi^*(g_m)(z) = 0$, that is $g_m(\varphi(z)) = 0$. So, if $\varphi(Z) = Y$, then $g_m \equiv 0$ along Y, i.e. $\varphi^*(g_m) = 0$ in $k[X]$. Since $k[X]$ has no zero divisors, the minimal equation above is divisible by f. Contradiction. $\square$

<u>NORMAL ALGEBRAIC VARIETIES.</u> If Y is irreducible, then $k[Y]$ has no zero divisors and its quotient field is well defined; it is called *the field of rational functions* on Y and is denoted $k(Y)$. Irreducible Y is called *normal*, if all rational functions $f \in k(Y)$ integer over $k[Y]$ belong to $k[Y]$. For example, affine spaces $\mathbb{A}^n$ are normal (as well as any X with factorial $k[X]$).

<u>Claim 12.18</u>. *Let $X \xrightarrow{\varphi} Y$ be a surjective finite morphism. If Y is normal, then $\varphi(U)$ is open for any open $U \subset X$ (in particular each irreducible component of X is surjectively mapped onto Y).*

Proof. We can suppose that $U = \mathscr{D}(f)$ is principal. Let $p \in \mathscr{D}(f)$; we have to find $a \in k[Y]$ such that $\varphi(p) \in \mathscr{D}(a) \subset \varphi(\mathscr{D}(f))$. To this aim consider a map

$$\psi = \varphi \times f \,:\, X \xrightarrow{p \mapsto (\varphi(p), f(p))} Y \times \mathbb{A}^1\,.$$

It is regular and finite: $k[Y] \subset k[X]$ (via φ^*), $k[Y \times \mathbb{A}^1] = k[Y][t] \xrightarrow{\psi^*} k[X]$ is the evaluation $t \longmapsto f$, and $k[X]$ is finitely generated even as $k[Y]$-module. Now let us extend $k[X]$ to the $k(Y)$-algebra $B = k(Y) \underset{k[Y]}{\otimes} k[X]$, which consists of all fractions b/a, $b \in k[X]$, $0 \neq a \in k[Y]$ with the usual equivalence $b'/a' = b''/a'' \iff b''a' - b'a''$ divides zero in $k[X]$. Since f is integer over $k[Y]$, f is algebraic over $k(Y)$, i.e. the kernel of evaluation map $k(Y)[t] \xrightarrow{t \mapsto f} B$ is the principal ideal $(\mu_f) \subset k(Y)[t]$ spanned by the minimal polynomial $\mu_f(t) = t^m + a_1 t^{m-1} + \cdots + a_{m-1}t + a_m \in k(Y)[t]$ of f over $k(Y)$.

Exercice 12.19. Show that that μ_f has coefficients in $k[Y]$, i.e. $\mu_f \in k[Y \times \mathbb{A}^1]$.

HINT. Let $g(f) = 0$ be an *integer* equation for f over $k[Y]$. Then $g(t) = \mu_f(t) \cdot h(t)$ in $k(Y)[t]$ for some monic h. Now use the Gauss – Kronecker – Dedekind lemma (see Corollary 8.10) and normality of Y.

So, $\ker \psi^* = (\mu_f)$ and $\overline{\operatorname{im} \psi} = Spec_m\left(k[Y \times \mathbb{A}^1]/(\mu_f)\right) = V(\mu_f)$. In other words, ψ gives a finite surjection of X onto the hypersurface $V(\mu_f) \subset Y \times \mathbb{A}^1$ and φ is obtained by projecting this hypersurface onto Y. In particular, the preimages of given $y \in Y$ are surjectively mapped by f onto the roots of a polynomial

$$\mu_f(y;t) = t^m + a_1(y)\, t^{m-1} + \cdots + a_m(y) \in k[t]$$

obtained by evaluating the coefficients of μ_f at $y \in Y$. Those preimages which belong to $\mathscr{D}(f)$ correspond to non-zero roots. Since $f(p) \neq 0$, the polynomial $\mu_f(\varphi(p);t)$ has a non zero root, i.e. $\mu_f(\varphi(p);t) \neq t^m$ and there is a non zero coefficient $a_i(\varphi(p)) \neq 0$. Then $\mu_f(y;t)$ has a non zero root over each $y \in \mathscr{D}(a_i) \subset Y$, i.e. $\varphi(\mathscr{D}(f)) \supset \mathscr{D}(a_i) \ni \varphi(p)$ as required.

What about irreducible components, consider the irreducible decomposition $X = \cup X_\nu$. Then

$$U_i = X \setminus \bigcup_{\nu \neq i} X_\nu = X_i \setminus \bigcup_{\nu \neq i} (X_i \cap X_\nu)$$

is open in X and dense in X_i. Since $\varphi(U_i)$ is non empty open subset of Y, $\varphi(X_i) = \overline{\varphi(U_i)} = Y$. $\square$

Algebraic manifolds

<u>LOCALIZATION.</u> Let $U \subset X$ be an open subset of an affine algebraic variety and $u \in U$. A function $U \xrightarrow{f} k$ is called *regular* at u, if there are $p, q \in k[X]$ such that $q(u) \neq 0$ and $f(x) = p(x)/q(x)\ \forall\, x \in \mathscr{D}(q) \cap U$. Functions $U \xrightarrow{f} k$ which are regular at any $u \in U$ form a commutative ring denoted by $\mathscr{O}_X(U)$ or by $\Gamma(U, \mathscr{O}_X)$. It is called *a ring of local regular functions* on $U \subset X$.

<u>Claim 13.1</u>. *Let X be irreducible and $h \in k[X]$. Then any $f \in \mathscr{O}_X(\mathscr{D}(h))$ can be written as $f(x) = r(x)/h^d(x)$ for appropriate $r \in k[X]$, $d \in \mathbb{N}$ (in particular, taking $h \equiv 1$, we get $\mathscr{O}_X(X) = k[X]$).*

Proof. If $f \in \mathscr{O}_X(\mathscr{D}(h))$, then $\forall\, u \in \mathscr{D}(h)$ there are $p_u, q_u \in k[X]$ such that $q_u(u) \neq 0$ and $f(x) = p_u(x)/q_u(x)\quad \forall\, x \in \mathscr{D}(q_u) \cap \mathscr{D}(h)$. So, $\bigcap\limits_{u \in U} V(q_u) \subset V(h)$ and, by Hilbert, there are some $u_1, u_2, \dots, u_m \in \mathscr{D}(h)$ such that $h^d = \sum q_{u_\nu} g_\nu$ for appropriate $g_1, g_2, \dots, g_m \in k[X]$ and $d \in \mathbb{N}$. At the same time, $f(x)\, q_{u_\nu}(x) = p_{u_\nu}(x)\quad \forall\, \nu$ and $\forall\, x \in \mathscr{D}(h)$ including $\mathscr{D}(h) \cap V(q_{u_\nu})$. Indeed, let $q_{u_\nu}(w) = 0$ for some $w \in \mathscr{D}(h)$; rewriting $f = p_{u_\nu}/q_{u_\nu}$ as p_w/q_w with $q_w(w) \neq 0$ we get $p_{u_\nu}(x)\, q_w(x) = q_{u_\nu}(x)\, p_w(x)\quad \forall\, x \in \mathscr{D}(h \cdot q_{u_\nu} \cdot q_w)$ and hence, by Exercice 12.13, for any $x \in X$ at all; in particular, $p_{u_\nu}(w) = q_{u_\nu}(w)\, p_w(w)/q_w(w) = 0$. So, $f\, h^d = \sum f\, q_{u_\nu}\, g_\nu = \sum p_{u_\nu}\, g_\nu \in k[X]$. □

<u>Corollary 13.2</u>. *Each $\mathscr{D}(f) = Spec_m\, k[X][f^{-1}]$ is an affine algebraic variety and inclusion $\mathscr{D}(f) \hookrightarrow X$ is a regular map whose pull back is $k[X] \hookrightarrow k[X][f^{-1}]$.* □

<u>STRUCTURE SHEAF.</u> The correspondence $\mathscr{O}_X : U \longmapsto \mathscr{O}_X(U)$ is called *a structure sheaf* of an *affine algebraic manifold* X. If $U = \bigcup W_i$ is an union of open sets, then $U \xrightarrow{f} k$ is regular iff each its restriction $f|_{W_i}$ is regular on W_i. Conversely, a collection of functions $W_i \xrightarrow{f_i} k$ such that $f_i \equiv f_j$ on $W_i \cap W_j$ gives a unique regular function $f \in \mathscr{O}_X(\cup W_i)$ whose restriction onto W_i is f_i for all i. Although Corollary 13.2 shows that open sets are *locally* affine, general open $U \subset X$ is *not* an affine algebraic variety and there is no bijection between $Spec_m \mathscr{O}_X(U)$ and U.

Exercice 13.3. Let $U = \mathbb{A}^n \setminus O$ be the complement to the origin. Show that $\mathscr{O}_{\mathbb{A}^n}(U) = k[\mathbb{A}^n]$ for $n \geqslant 2$.
HINT. Use the covering $U = \bigcup \mathscr{D}(x_i)$ and Claim 13.1.

<u>ALGEBRAIC MANIFOLDS.</u> Let X be a topological space. An open subset $U \subset X$ is called *an algebraic affine chart* on X, if there exists an affine algebraic variety X_U and *a homeomorphism* $X_U \xrightarrow{\varphi_U} U$. Two algebraic charts $X_U \xrightarrow{\varphi_U} U$ and $X_W \xrightarrow{\varphi_W} W$ on X are called *compatible*, if their *transition map* $\varphi_{WU} = \varphi_W \circ \varphi_U^{-1}$, which identifies $\varphi_U^{-1}(U \cap W) \subset X_U$ with $\varphi_W^{-1}(U \cap W) \subset X_W$, is a regular isomorphism of algebraic open sets, i.e. its pull back $\Gamma\left(\varphi_W^{-1}(U \cap W), \mathscr{O}_{X_W}\right) \xrightarrow{\varphi_{WU}^*} \Gamma\left(\varphi_U^{-1}(U \cap W), \mathscr{O}_{X_U}\right)$ is a well defined isomorphism of k-algebras. A (finite) open covering $X = \bigcup U_\nu$ by mutually compatible algebraic charts is called a (finite) *algebraic atlas* on X. Two algebraic atlases are called *equivalent*, if their union is an algebraic atlas too. A topological space X equipped with a class of equivalent (finite) algebraic atlases is called *an algebraic manifold* (of finite type).

Exercice 13.4. Check that projective spaces and Grassmannians are algebraic manifolds of finite type as well as any zero set of a collection of multihomogeneous polynomials on $\mathbb{P}_{n_1} \times \mathbb{P}_{n_2} \times \cdots \times \mathbb{P}_{n_m}$.

<u>Example 13.5</u>: a direct product $X \times Y$ of algebraic manifolds X, Y is an algebraic manifold too. Its atlas consists of all pairwise products $U \times W$, where $U \subset X$, $W \subset X$ are affine algebraic charts on X, Y.

<u>Example 13.6</u>: separability. The standard atlas on $\mathbb{P}_1$ consists of two charts $\varphi_i : \mathbb{A}_1 \xrightarrow{\sim} U_i \subset \mathbb{P}_1$, $i = 0, 1$, and $\varphi_0^{-1}(U_0 \cap U_1) = \varphi_1^{-1}(U_0 \cap U_1) = \{t \in \mathbb{A}^1 \mid t \neq 0\}$ is the complement to the origin. The charts $U_{0,1} = \mathbb{A}^1$ are glued together along $\mathbb{A}^1 \setminus \{O\}$ via transition map $\varphi_{01} : t \mapsto 1/t$. If we replace it by the identity map $\widetilde{\varphi}_{01} : t \mapsto t$, then we get an other manifold called «$\mathbb{A}^1$ with doubled origin», which looks like ——————:——————. Such the pathology is known as a *nonseparability*. It has appeared, because the last gluing rule is not «complete» and could be extended from $\mathbb{A}^1 \setminus \{O\}$ to a larger set. This can be formalized as follows. Two inclusions $U_0 \hookleftarrow U_0 \cap U_1 \hookrightarrow U_1$ give an embedding $U_0 \cap U_1 \hookrightarrow U_0 \times U_1$. In the case of $\mathbb{P}_1$, this is an inclusion $(\mathbb{A}^1 \setminus O) \hookrightarrow \mathbb{A}^2$ given as $x = t$; $y = 1/t$; it identifies $U_0 \cap U_1$ with *a closed* subset $V(xy - 1) \subset \mathbb{A}^2 = U_0 \times U_1$. In the second case, the embedding $U_0 \cap U_1 \hookrightarrow U_0 \times U_1 = \mathbb{A}^2$ is given as $x = t$; $y = t$ and has a *nonclosed* image $\Delta \setminus \{(0,0)\}$, where $\Delta = V(x - y) \subset \mathbb{A}^2$ is the diagonal. An algebraic manifold X is called *separable*, if an image of the canonical embedding $U \cap W \hookrightarrow U \times W$ is closed for any two affine charts $U, W \subset X$. Since this image is nothing more than an intersection of the diagonal $\Delta \subset X \times X$ with the affine chart $U \times W$ on $X \times X$, a manifold X is separable iff the diagonal $\Delta \subset X \times X$ is closed. For example, $\mathbb{A}^n$ and $\mathbb{P}_n$ are separable, because the diagonals in $\mathbb{A}^n \times \mathbb{A}^n$ and in $\mathbb{P}_n \times \mathbb{P}_n$ are given, accordingly, by the equations $x_i = y_i$ and $x_i y_j = x_j y_i$.

<u>REGULAR FUNCTIONS AND MORPHISMS.</u> Let $U \subset X$ be an open set. A function $U \xrightarrow{f} k$ is called *regular*, if each point $u \in U$ has an affine neighborhood $X_W \xrightarrow{\varphi_W} W \ni u$ such that $f \circ \varphi \in \mathscr{O}_{X_W}\left(\varphi^{-1}(U \cap W)\right)$. In other

words, a local function on X is regular, if it induces a local regular functions on each affine chart. Regular functions $U \longrightarrow k$ form a commutative ring $\mathscr{O}_X(U)$; a correspondence $U \longmapsto \mathscr{O}_X(U)$ is called *a structure sheaf* on X. More generally, a map of algebraic manifolds $X \xrightarrow{\varphi} Y$ is called *a regular morphism*, if its pull back is well defined homomorphism of k-algebras: $\mathscr{O}_Y(U) \xrightarrow{\varphi^*} \mathscr{O}_X(\varphi^{-1}U)$ for any open $U \subset Y$. For example, a set of regular morphisms $X \longrightarrow \mathbb{A}^1$ coincides with $\mathscr{O}_X(X)$.

Rational maps. A regular[1] morphism $U \xrightarrow{\varphi} Y$, which is defined only on some open dense $U \subset X$, is called *a rational map* from X to Y. One should be careful in composing rational maps, because an image of the first map may be completely outside the domain where the second is defined.

Example 13.7: a projection $\mathbb{A}^{n+1} \xrightarrow{\pi} \mathbb{P}_n$ sending a point $A \in \mathbb{A}^n$ to the line $(OA) \in \mathbb{P}_n$ is a rational surjection defined on $U = \mathbb{A}^n \setminus O$. In terms of the standard affine chart $\mathbb{A}^n \xrightarrow{\varphi_i} U_i = \{(t_0, t_1, \dots, t_n) \in \mathbb{P}_n \mid t_i = 1\}$, the pull back homomorphism $\mathscr{O}_{\mathbb{P}_n}(U_i) \xrightarrow{\pi^*} \mathscr{O}_{\mathbb{A}^{n+1}}(\pi^{-1}(U_i))$ sends $f(x_1, x_2, \dots, x_n) \in k[x_1, x_2, \dots, x_n] = \mathscr{O}_{\mathbb{P}_n}(U_i)$ to the rational function $\widetilde{f}(t_0, t_1, \dots, t_n) = f(t_0/t_i, \dots, t_{i-1}/t_i, t_{i+1}/t_i, \dots, t_n/t_i) \in \mathscr{O}_{\mathbb{A}^{n+1}}(\mathscr{D}(t_i)) = \mathscr{O}_{\mathbb{A}^{n+1}}(\pi^{-1}(U_i))$.

Closed submanifolds. Any closed subset $Z \subset X$ of an algebraic manifold X has natural structure of algebraic manifold. Namely, for any affine chart U the intersection $Z \cap U$ is a closed subset of U, that is an affine algebraic set $Spec_m\,(\mathscr{O}_X(U)/\mathscr{I}_Z(U))$, where $\mathscr{I}_Z(U) = \{f \in \mathscr{O}_X(U) \mid f|_Z \equiv 0\}$ is the ideal of $Z \cap U$ on U. The correspondence $U \longmapsto \mathscr{I}_Z(U)$ is called *the ideal sheaf* of the closed submanifold $Z \subset X$. This is a subsheaf of the structure sheaf. It consists of all local regular functions vanishing along Z. A regular morphism of arbitrary algebraic manifolds $X \xrightarrow{\varphi} Y$ is called *a closed embedding*, if $\varphi(X) \subset Y$ is a closed submanifold and φ gives an isomorphism between X an $\varphi(X)$. One can say that an algebraic manifold X is affine iff it admits a closed embedding into affine space. Similarly, a manifold X is called *projective*, if it admits a closed embedding $X \hookrightarrow \mathbb{P}_m$ for some m.

Example 13.8: closed submanifold $X \subset Y$ is separable as soon Y is, because the diagonal in $X \times X$ is a preimage of the diagonal in $Y \times Y$ under an embedding $X \times X \hookrightarrow Y \times Y$. In particular, any affine or projective manifold is separable and has a finite type.

Example 13.9: graph of morphism. Let $X \xrightarrow{\varphi} Y$ be a regular morphism. A preimage of the diagonal $\Delta \subset Y \times Y$ under an induced morphism $X \times Y \xrightarrow{\varphi \times \mathrm{Id}_Y} Y \times Y$ is called *a graph* of φ and is denoted by Γ_φ. Geometrically, $\Gamma_\varphi = \{(x, f(x)\} \subset X \times Y$. It is closed, if Y is separable. For example, a graph of a regular morphism of affine manifolds $Spec_m\,(A) \xrightarrow{\varphi} Spec_m\,(B)$ is given in $A \otimes B$ by the equation system $1 \otimes f = \varphi^*(f) \otimes 1$, where f runs through B and $B \xrightarrow{\varphi^*} A$ is the pull back of φ.

Example 13.10: family of closed submanifolds. Any regular morphism $X \xrightarrow{\pi} Y$ may be considered as a family of closed submanifolds $X_y = \pi^{-1}(y) \subset X$ parameterized by the points $y \in Y$. If $X \xrightarrow{\pi} Y$, $X' \xrightarrow{\pi'} Y$ are two families with the same base, then a regular morphism $X \xrightarrow{\varphi} X'$ is called *a morphism of families* (or a morphism *over* Y), if it sends X_y to X'_y for any $y \in Y$, that is if $\pi = \pi' \circ \varphi$. A family $X \xrightarrow{\pi} Y$ is called *constant* or *trivial*, if it is isomorphic over Y to a direct product $X_0 \times Y \xrightarrow{\pi_Y} Y$ for some manifold X_0.

Example 13.11: blow up a point $p \in \mathbb{P}_n$. Lines passing through a given point $p \in \mathbb{P}_n$ form a projective space $E \simeq \mathbb{P}_{n-1}$, which can be identified with any hyperplane $H \subset \mathbb{P}_n$ such that $p \notin H$. An incidence graph $\mathcal{B}_p = \{(\ell, q) \in E \times \mathbb{P}_n \mid q \in \ell\}$ is called *a blow up* of $p \in \mathbb{P}_n$. If $n \geqslant 2$, then the projection $\sigma_p : \mathcal{B}_p \twoheadrightarrow \mathbb{P}_n$ is bijective outside $q = p$, but a preimage $\sigma_p^{-1}(p) \subset \mathcal{B}_p$ coincides with E; it is called *an exceptional divisor*. The second projection $\varrho_E : \mathcal{B}_p \twoheadrightarrow E$ fibers $\mathcal{B}_p$ as *a line bundle* over E whose fiber at a point $\ell \in E$ coincides with the line $\ell \subset \mathbb{P}_n$ itself; this line bundle is called *a tautological line bundle* over E. If we fix homogeneous coordinates $(t_0 : t_1 : \dots : t_n)$ on $\mathbb{P}_n$ such that $p = (1 : 0 : \cdots : 0)$ and identify E with the hyperplane $H = \{(0 : q_1 : \cdots : q_n)\} \subset \mathbb{P}_n$, then $(q, t) \in \mathcal{B}_p$ iff $\mathrm{rk}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & q_1 & \cdots & q_n \\ t_0 & t_1 & \cdots & t_n \end{pmatrix} = 3$, that is $q_i t_j = q_j t_i$ for all $1 \leqslant i < j \leqslant n$. So, $\mathcal{B}_p$ is closed submanifold of $H \times \mathbb{P}_n$.

Closed morphisms. A regular morphism $X \xrightarrow{\varphi} Y$ is called *closed*, if $\varphi(Z) \subset Y$ is closed for any closed $Z \subset X$. Of course, any closed embedding is closed. Claim 12.17 says that any finite morphism of affine manifolds is closed. The projection $\mathbb{P}_m \times \mathbb{A}^n \longrightarrow \mathbb{A}^n$ is also closed by Example 8.27.

Corollary 13.12. *If X is a projective manifold, then the projection $X \times Y \longrightarrow Y$ is closed for any manifold Y.*

Proof. Taking an affine chart on Y, we can suppose that Y is affine, i. e. that $X \times Y$ is a closed submanifold of $\mathbb{P}_m \times \mathbb{A}^n$. Then our projection is the restriction of the closed map $\mathbb{P}_m \times \mathbb{A}^n \longrightarrow \mathbb{A}^n$ onto the closed subset $X \times Y \subset \mathbb{P}_m \times \mathbb{A}^n$. □

[1] as above, the regularity means that the pull back $\mathscr{O}_Y(W) \xrightarrow{\varphi^*} \mathscr{O}_X(\varphi^{-1}(W))$ is a well defined k-algebra homomorphism

Corollary 13.13. *If X is projective and Y is separable, then any morphism $X \xrightarrow{\varphi} Y$ is closed.*

Proof. Since Y is separable, the graph $\Gamma_\varphi \subset X \times Y$ is closed. $Z \times Y$ is also closed in $X \times Y$ for any closed $Z \subset X$. But $\varphi(Z)$ is the image of $\Gamma_\varphi \cap (Z \times Y)$ under the projection $X \times Y \longrightarrow Y$. □

Corollary 13.14. *If X is a connected projective manifold, then $\mathscr{O}_X(X) = k$. Moreover, each regular map from X to any affine manifold contracts X into one point.*

Proof. Let us identify $k = \mathbb{A}^1$ with an affine chart on $\mathbb{P}_1$ and consider a global regular function $X \xrightarrow{f} k$ as a regular morphism $X \xrightarrow{f} \mathbb{P}_1$. Since $f(X) \subsetneq \mathbb{P}_1$ is closed and connected, it is one point. In particular, if $X \xrightarrow{\varphi} \mathbb{A}^n$ is regular, then each coordinate form $x_i \in k[\mathbb{A}^n]$ takes a constant value along $\varphi(X)$. □

Finite morphisms of manifolds. A regular morphism of arbitrary algebraic manifolds $X \xrightarrow{\varphi} Y$ is called *finite*, if $W = \varphi^{-1}(U)$ is an affine chart on X for any affine chart $U \subset Y$ and the restriction $W \xrightarrow{\varphi_W} U$ is a finite morphism of affine algebraic varieties. It follows from Claim 12.17 that any finite morphism is closed and a restriction of a finite morphism onto any closed submanifold $Z \subset X$ is a finite morphism as well. Moreover, if X is irreducible, then each proper closed $Z \subset X$ goes to a *proper* closed subset of Y.

Example 13.15: a projection of any projective manifold $X \subsetneq \mathbb{P}_n$ from any point $p \notin X$ onto any hyperplane $H \not\ni p$ is a finite morphism. To check this, let us fix the coordinates as in Example 13.11 and follow the notations of that example. Consider a standard affine chart on H, say $U_{q_n} = \{q = (0 : u_1 : \cdots : u_{n-1} : 1)\} \subset H$. Its preimage $Y = \pi_p^{-1}(U_{q_n}) \subset X$ under the projection from p lies inside the cone over U_{q_n} with the punctured vertex p (because $p \notin X$). The blow up maping σ_p identifies this punctured cone with the affine space $\mathbb{A}^n = U_{q_n} \times \mathbb{A}^1$ via the substitution $t = \vartheta p + q_u$, where $t = (t_0 : t_1 : \ \dots \ : t_n)$ is the homogeneous coordinate on $\mathbb{P}_n$, $p = (1 : 0 : \ \dots \ : 0)$, $q_u = (0 : u_1 : \cdots : u_{n-1} : 1) \in U_{q_n}$. If X is given by a system of homogeneous equations $f_\nu(t) = 0$, then Y is given in affine coordinates (u, t) by equations

$$f_\nu(\vartheta p + q_u) = \alpha_0^{(\nu)}(u)\,\vartheta^m + \alpha_1^{(\nu)}(u)\,\vartheta^{m-1} + \cdots + \alpha_m^{(\nu)}(u) = 0\,. \tag{33}$$

So, Y is affine and it remains to check that $k[Y] = k[u][\vartheta]/(f_\nu(\vartheta p + q_u))$ is a finitely generated $k[u]$-module. To this aim it is enough to find in the ideal spanned by equations (33) an integer equation with $\alpha_0(u) \equiv 1$, because the factorization through such an equation already gives a finitely generated $k[u]$-module. But this is possible, since the leading coefficients $\alpha_0^{(\nu)}(u)$ have no common zero in U_{q_n}. Indeed, if such a zero u_0 exists, then the homogeneous version of (33): $\alpha_0^{(\nu)}(u_0)\,\vartheta_0^m + \alpha_1^{(\nu)}(u_0)\,\vartheta_0^{m-1}\vartheta_1 + \cdots + \alpha_m^{(\nu)}(u_0)\vartheta_1^m = 0$ has the root $(\vartheta_0 : \vartheta_1) = (1 : 0)$. But these homogeneous equations, obtained by the substitution $t = \vartheta_0\, p + \vartheta_1\, q_u$, describe the intersection $X \cap (pq_{u_0})$ and this root gives the point p itself, but $p \notin X$!

Exercice 13.16. Check that a composition of finite morphisms is finite and prove that any projective manifold admits a finite surjective morphism onto a projective space.

Corollary 13.17. *Each affine manifold X admits a finite surjective morphism φ onto an affine space.*

Proof. Let $X \subsetneq \mathbb{A}^n$, where $\mathbb{A}^n$ is embedded in $\mathbb{P}_n$ as the standard chart U_0. We write H_∞ for $\mathbb{P}_n \setminus U_0$ and $\overline{X} \subset \mathbb{P}_n$ for the projective closure of X. A projection of $\overline{X}$ from any point $p \in H_\infty \setminus (\overline{X} \cap H_\infty)$ onto any hyperplane $L \not\ni p$ induces a finite morphism from $X = \overline{X} \setminus (\overline{X} \cap H_\infty)$ to $\mathbb{A}^{n-1} = L \setminus (L \cap H_\infty)$. If it is non surjective, we repeat this procedure. □

Dimension. Maximal $n \in \mathbb{N}$ such that there exists a chain of irreducible closed subsets:

$$\{x\} = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_{n-1} \subsetneq X_n \subset X\,. \tag{34}$$

is called *a dimension* of an algebraic manifold X at a given point $x \in X$; it is denoted by $\dim_x X$. Certainly, $X_n = X$, if X is irreducible. If X is reducible, $\dim_x X$ coincides with the maximal dimension of irreducible components passing through x.

Exercice 13.18. Show that $\dim_p X = \dim_p U$ for any affine chart $U \ni p$.

Hint. Let $X_1, X_2 \subset X$ be two closed irreducible subsets and $U \subset X$ be an open set such that both $X_1 \cap U$, $X_2 \cap U$ are nonempty. Then $X_1 = X_2 \iff X_1 \cap U = X_1 \cap U$, because of $X_i = \overline{X_i \cap U}$.

Claim 13.19. *Let $X \xrightarrow{\varphi} Y$ be a finite morphism of irreducible manifolds, $x \in X$, and $y = \varphi(x) \in Y$. Then $\dim_x X \leqslant \dim_y Y$ and the equality holds iff $\varphi(X) = Y$.*

Proof. By the above exercise we can suppose X, Y to be affine. Then Claim 12.17 implies that each chain (34) in X produces a chain $\cdots \subsetneq \varphi(X_i) \subsetneq \varphi(X_{i+1}) \subsetneq \cdots$ of closed irreducible submanifolds in Y. Wise versa, if $\varphi(X) = Y$, then given a chain (34) in Y, for each i we can choose an irreducible component of $\varphi^{-1}(X_i)$ mapped surjectively onto X_i. This gives a chain (34) in X. □

Corollary 13.20. $\dim_p \mathbb{A}^n = n$ *at any* $p \in \mathbb{A}^n$.

Proof. Clearly, $\dim \mathbb{A}^0 = 0$. Suppose inductively that $\dim \mathbb{A}^{n-1} = n - 1$. Since any proper closed $X \subset \mathbb{A}^n$ has a finite projection on $\mathbb{A}^{n-1}$, Claim 13.19 implies $\dim_p X \leqslant (n-1)$ for any p, i.e. $\dim_p \mathbb{A}^n \leqslant n$. On the other hand, there is a chain (34) of affine subspaces passing through p, so $\dim_p \mathbb{A}^n \geqslant n$. □

Corollary 13.21. *Let X be an irreducible affine manifold and $X \xrightarrow{\varphi} \mathbb{A}^m$ be a surjective finite morphism. Then $\dim_p X = m$ at each $p \in X$. In particular, m doesn't depend on a choice of φ and $\dim_p X$ is the same for all $p \in X$.* $\square$

Exercice 13.22. Prove that $\dim(X \times Y) = \dim X + \dim Y$ for any irreducible manifolds X, Y.

Exercice 13.23. Let $V(f) \subset \mathbb{A}^n$ be given by irreducible $f \in k[x_1, x_2, \dots, x_n]$. Show that $\dim V(f) = (n-1)$.
Hint. Find a surjective finite projection $V(f) \longrightarrow \mathbb{A}^{n-1}$ (comp. with Exercice 13.16 and Corollary 13.17).

Claim 13.24. *If X is irreducible, then $\dim_p V(f) = \dim_p(X) - 1$ for any non constant $f \in \mathcal{O}_X(X)$ $\forall p \in V(f)$.*

Proof. We can assume that X is affine. Fix some finite surjection $X \xrightarrow{\pi} \mathbb{A}^m$ and consider, as in the proof of Claim 12.18, the induced map $\varphi = \pi \times f : X \xrightarrow{x \mapsto (\pi(x), f(x))} \mathbb{A}^m \times \mathbb{A}^1$. It maps X finitely and surjectively onto affine hypersurface $V(\mu_f) \subset \mathbb{A}^m \times \mathbb{A}^1$ given by the minimal polynomial $\mu_f(u,t) = t^n + \alpha_1(u)\, t^{n-1} + \cdots + \alpha_n(u) \in k[u_1, u_2, \dots, u_m][t]$, of f over $k(\mathbb{A}^m)$, and $V(f) = \varphi^{-1}(H \cap V(\mu_f))$, where $H \subset \mathbb{A}^m \times \mathbb{A}^1$ is the hyperplane $t = 0$. Inside $H = \mathbb{A}^n$, the intersection $H \cap V(\mu_f)$ is a hypersurface given by the equation $\alpha_n(u) = 0$. So, we have a finite surjection $V(f) \xrightarrow{\varphi|_{V(f)}} V(a_n) \subset \mathbb{A}^n$ and can apply Exercice 13.23 and Claim 13.19. $\square$

Corollary 13.25. *For any X and $f \in k[X]$ the inequality $\dim_p V(f) \geqslant \dim_p(X) - 1$ holds at any $p \in V(f)$.* $\square$

Claim 13.26. *Let $X_1, X_2 \subset \mathbb{A}^n$ be closed submanifolds and $x \in X_1 \cap X_2$. Then $\dim_x(X_1 \cap X_2) \geqslant \dim_x(X_1) + \dim_x(X_2) - n$.*

Proof. Write $X_1 \xhookrightarrow{\varphi_1} \mathbb{A}^n$, $X_2 \xhookrightarrow{\varphi_2} \mathbb{A}^n$ for the closed embeddings. Then $X_1 \cap X_2$ is the preimage of the diagonal $\Delta \subset \mathbb{A}^n \times \mathbb{A}^n$ under the map $X_1 \times X_2 \xhookrightarrow{\varphi_1 \times \varphi_2} \mathbb{A}^n \times \mathbb{A}^n$. So, it is given inside $X_1 \times X_2$ by the pull backs of n equations $x_i = y_i$ defining Δ inside $\mathbb{A}^n \times \mathbb{A}^n$. It remains to apply Corollary 13.25. $\square$

Corollary 13.27. *If $\dim(X_1) + \dim(X_2) \geqslant n$ for some closed $X_1, X_2 \subset \mathbb{P}_n$, then $X_1 \cap X_2 \neq \varnothing$.*

Proof. Let $\mathbb{P}_n = \mathbb{P}(V)$. Consider in $\mathbb{A}^{n+1} = \mathbb{A}(V)$ affine cones[2] X_1', X_2'' consisting of lines passing through the origin $O \in \mathbb{A}^{n+1}$ and forming X_1 and X_2 By the previous claim $\dim_O(X_1' \cap X_2'') \geqslant \dim_O(X_1) + 1 + \dim_O(X_2) + 1 - n - 1 \geqslant 1$. So, $X_1' \cap X_2''$ is not exhausted by O. $\square$

Theorem 13.28. *Let $X \xrightarrow{\varphi} Y$ be a dominant morphism of irreducible manifolds. Then $\dim_x \varphi^{-1}(\varphi(x)) \geqslant \dim X - \dim Y$ for any $x \in X$ and there exists non empty open $U \subset Y$ such that $\dim \varphi^{-1}(y) = \dim X - \dim Y$ for all $y \in U$.*

Proof. Projecting finitely an affine chart containing $\varphi(x)$ onto $\mathbb{A}^m$, we reduce the first part of the theorem to the case $Y = \mathbb{A}^m = \mathit{Spec}_m\, k[u_1, u_2, \dots, u_m]$, $\varphi(x) = 0$. Then $\varphi^{-1}(0)$ is an intersection of m hypersurfaces $V(\varphi^*(u_i))$ in X and the required inequality follows inductively from Corollary 13.25. For the second part, we can suppose that $X = \mathit{Spec}_m A$, $Y = \mathit{Spec}_m B$ are affine and φ is the projection $Y \times \mathbb{A}^m \xrightarrow{\pi} Y$ restricted onto closed submanifold $X \subset Y \times \mathbb{A}^m$ (see the decomposition (32) from Exercice 12.16). We are going to apply Corollary 13.17 fiberwise over Y. Consider the closure $\overline{X} \subset Y \times \mathbb{P}_m$ and choose a hyperplane $H \subset \mathbb{P}_m$ and a point $p \in \mathbb{P}_m \setminus H$ such that the section $Y \times \{p\} \subset Y \times \mathbb{P}_m$ is not contained in $\overline{X}$. The fiberwise projection from p onto H is well defined over an open subset $U \subset Y$ complementary to $\overline{\pi}\Big(\big(Y \times \{p\}\big) \cap \overline{X}\Big)$, where $Y \times \mathbb{P}_m \xrightarrow{\overline{\pi}} Y$ is the projection along $\mathbb{P}_m$. So, replacing Y by its open subset, we get a finite morphism $X \longrightarrow Y \times \mathbb{A}^{m-1}$. After a number of such replacements we can suppose that φ is a finite surjection $X \twoheadrightarrow Y \times \mathbb{A}^n$ followed by the projection $Y \times \mathbb{A}^n \twoheadrightarrow Y$; it clearly has fibers of dimension $n = \dim X - \dim Y$. $\square$

Corollary 13.29. ([3]) *$\forall k \in \mathbb{Z}$ sets $X_k \stackrel{\text{def}}{=} \{x \in X \mid \dim_x \varphi^{-1}(\varphi(x)) \geqslant k\}$ are closed in X for any morphism $X \xrightarrow{\varphi} Y$ of algebraic manifolds.*

Proof. We can suppose that X, Y are irreducible. If $k \leqslant \dim(X) - \dim(Y)$, then $X_k = X$ by the above theorem. For $k > \dim(X) - \dim(Y)$ we can replace Y by $Y' = Y \setminus U$, where U is the same as in Theorem 13.28, and X — by $X' = \varphi^{-1}(Y')$. Clearly, $X_k \subset X'$ and we can repeat the arguments decreasing the dimensions of X, Y. $\square$

Exercice 13.30. Show that isolated points of fibers of a morphism $X \xrightarrow{\varphi} Y$ form an open subset in X.

Corollary 13.31. *If $X \xrightarrow{\varphi} Y$ is closed, than $Y_k \stackrel{\text{def}}{=} \{y \in Y \mid \dim \varphi^{-1}(y) \geqslant k\}$ is closed in Y $\forall k \in \mathbb{Z}$.* $\square$

Corollary 13.32. *Let $X \xrightarrow{\varphi} Y$ be a closed morphism with irreducible fibers of the same dimension. If Y is irreducible, then X is irreducible as well.*

Proof. Let $X = X' \cup X''$. Since each fiber $\varphi^{-1}(y)$ is irreducible, it completely belongs to one of X', X''. Applying Corollary 13.31 to the restriction $X' \xrightarrow{\varphi|_{X'}} Y$, we see that a set of all fibers completely contained in X' is mapped to some closed subset $Z' \subset Y$. Similarly, all fibers completely contained in X'' are also mapped to some closed $Z'' \subset Y$. So, $Y = Z' \cup Z''$ but both Z', Z'' should be proper as soon as X', X'' were proper. $\square$

[2] they have the same equations as X_1, X_2 but these equations are considered now as affine rather than homogeneous
[3] it is known as *Chevalley's theorem on semi-continuity*

Working example: lines on surfaces

Variety of lines on surfaces of given degree. We are going to analyze the set of lines laying on a surface $S \subset \mathbb{P}_3$ of a given degree d.

Exercice 14.1. Carry out the complete analysis for $d = 2$.

To this aim consider the space $\mathbb{P}_N = \mathbb{P}(S^dV^*)$, of surfaces of degree d in $\mathbb{P}_3 = \mathbb{P}(V)$, and identify the set of all lines in $\mathbb{P}(V)$ with the Plücker quadric $Q_P \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$. Let $\Gamma \overset{\text{def}}{=} \{\, (S,\ell) \in \mathbb{P}_N \times Q_P \mid \ell \subset S \,\} \subset \mathbb{P}_N \times Q_P$ be the incidence graph.

Claim 14.2. *Γ is closed submanifold of $\mathbb{P}_N \times Q_P$.*

Proof. 2-dimensional subspace spanned by $u, w \in V$ coincides with the image of the contraction map $V^* \longrightarrow V$, which sends $\xi \in V^*$ to $\langle \xi \,,\, u \wedge w \rangle$. So, the line $\ell = (uw)$ lies on a surface S given by $F = 0$ iff $F(\langle \xi \,,\, u \wedge w \rangle) \cong 0$ identically in $\xi \in V^*$. In coordinates, let e_ν form a base of V, ξ_ν be the coordinates of ξ w.r.t. the dual base of V^*, and $u \wedge w = \sum\limits_{\mu \neq \nu} p_{\mu\nu}\, e_\mu \wedge e_\nu$, where $p_{\mu\nu} = -p_{\nu\mu}$ are the associated Plücker coordinates on $\mathbb{P}_5 \supset Q$. Then $\langle \xi \,,\, u \wedge w \rangle = \sum\limits_i \xi_i \cdot (\sum_\nu p_{i\nu} e_\nu)$. Substitute this into F, expand the result through the monomials in ξ, and write down that all coefficients of this expansion vanish — this gives a system of polynomial equations on the coefficients of F and p_{ij} describing $\Gamma \subset \mathbb{P}_N \times Q_P \subset \mathbb{P}_N \times \mathbb{P}_5$. □

Claim 14.3. *Projection $\Gamma \xrightarrow{\pi_2} Q_P$ is surjective; all its fibers are projective spaces of dimension $d(d+1)(d+5)/6 - 1$.*

Proof. Let a line $\ell \subset \mathbb{P}(V)$ be given by $x_0 = x_1 = 0$. Then $S \supset \ell$ iff S has an equation $0 = x_2 \cdot F_2(x) + x_3 \cdot F_3(x)$, where $F_2, F_3 \in S^{d-1}V^*$ are arbitrary homogeneous polynomials. These equations form a vector space W, which coincides with the image of the linear operator $S^{d-1}V^* \oplus S^{d-1}V^* \xrightarrow{(f,g) \mapsto x_2 f + x_3 g} S^dV^*$ whose kernel consists of all (f,g) such that $x_2 f = -x_3 g$ that is possible iff $f = x_3 h$ and $g = -x_2 h$ for some $h \in S^{d-2}V^*$. Hence, the kernel is isomorphic to $S^{d-2}V^*$ and $\dim W = 2\,\dim(S^{d-1}V^*) - \dim(S^{d-1}V^*) = \Big(2\,d(d+1)(d+2) - (d-1)d(d+1)\Big)/6 = d(d+1)(d+5)/6$. □

Corollary 14.4. *Γ is an irreducible projective manifold of dimension $d(d+1)(d+5)/6 + 3$.*

Proof. This follows at once from Corollary 13.32 and Theorem 13.28. □

Claim 14.5. *A generic[1] surface $S_d \subset \mathbb{P}_3$ of degree $d \geqslant 4$ does not contain lines.*

Proof. By Corollary 13.13, the image of the projection $\Gamma \xrightarrow{\pi_1} \mathbb{P}_N$, that is the set of all surfaces containing some lines, is closed irreducible submanifold of $\mathbb{P}_N = \mathbb{P}(S^dV^*)$. By Theorem 13.28, its dimension equals $\dim\Gamma$ minus the minimal dimension of non-empty fibres of π_1. In particular, the image is proper, as soon as $\dim\Gamma < N$, i.e. if $d(d+1)(d+5)/6+3 < (d+1)(d+2)(d+3)/6$, which holds for all $d \geqslant 4$. □

Claim 14.6. *Each cubic surface $S_3 \subset \mathbb{P}_3$ contains lines; generically, this line set is finite.*

Proof. Taking in the previous proof $d = 3$, we get $\dim\Gamma = N = 19$ and π_1 is surjective, if it has a non-empty 0-dimensional fiber. Let us find, say, all the lines on a cubic C with affine equation $xyz = 1$. This affine piece does not contain the lines at all, because $x = x_0 + \alpha\, t\,,\ y = y_0 + \beta\, t\,,\ z = z_0 + \gamma\, t$ lies on C iff $\alpha\beta\gamma = 0$, $\alpha\beta z_0 + \beta\gamma x_0 + \gamma\alpha y_0 = 0$, and $\alpha y_0 z_0 + \beta x_0 z_0 + \gamma x_0 y_0 = 0$, but $x_0 y_0 z_0 = 1$, which leads to contradiction when we go from the left to the right: for example, $\alpha = 0 \Rightarrow \beta = 0$ or $\gamma = 0 \Rightarrow \beta = \gamma = 0$. To describe C at infinity, put $x = x_1/x_0$, $y = x_2/x_0$, $z = x_3/x_0$ and rewrite its equation as $x_1x_2x_3 = x_0^3$. Thus, $C \cap \{x_0 = 0\}$ consists of 3 lines: $x_i = x_0 = 0$, $i = 1, 2, 3$. □

Exercice 14.7*. Find all lines on the (smooth) Fermat cubic C_F, given by $\sum x_i^3 = 0$.

Hint. C_F is preserved by the permutations of the coordinates; up to such a permutation, a pair of linear equations for $\ell \subset C_F$ is reduced, by Gauss, to $x_0 = \alpha x_2 + \beta x_3$, $x_1 = \gamma x_2 + \delta x_3$; now substitute, show that $\alpha\beta\gamma\delta = 0$, and find them...

Lines on a smooth cubic. Now, let $S \subset \mathbb{P}_3$ be a smooth cubic surface with an equation $F(x) = 0$.

Claim 14.8. *Reducible plane section of S splits either into a line and a smooth conic or into a triple of distinct lines.*

Proof. We have to show that a plane section $\pi \cap S$ can not contain a double line component. If there is a double line $\ell \subset \pi \cap S$, take the coordinates where π is given by $x_2 = 0$ and ℓ by $x_2 = x_3 = 0$. Then $F(x) = x_2 Q(x) + x_3^2 L(x) = 0$ for some linear L and quadratic Q. Let a be an intersection point of ℓ with the quadric $Q(x) = 0$. Then $x_2(a) = x_3(a) = Q(a) = 0$ and all partial derivatives $\partial F/\partial x_i$ vanish at a, i.e. S is singular at a. □

Corollary 14.9. *A point of S can belong at most to 3 lines laying on S and these lines should be coplanar.*

Proof. Indeed, all lines passing through $p \in S$ and laying on S belong to $S \cap T_pS$. □

Claim 14.10. *Given $\ell \subset S$, there are precisely 5 distinct planes $\pi_1, \pi_2, \dots, \pi_5$ containing ℓ and intersecting S in a triple of lines; moreover, if $\pi_i \cap S = \ell \cup \ell_i \cup \ell_i'$, then $\ell_i \cap \ell_j = \ell_i \cap \ell_j' = \ell_i' \cap \ell_j' = \varnothing\ \forall\, i \neq j$ (in particular, S contains some skew lines) and any line on S skew to ℓ must intersect for each i precisely one of ℓ_i, ℓ_i'.*

[1] i.e. any from some dense Zariski open subset in the space of all hypersurfaces

Proof. Fix a base $\{e_0, e_1, e_2, e_3\}$ for V such that the line $\ell = (e_0e_1)$ (given by equations $x_2 = x_3 = 0$) lays on S. Then the equation $F(x) = 0$, defining S, can be written in this base as:

$$L_{00}(x_2, x_3) \cdot x_0^2 + 2\,L_{01}(x_2, x_3) \cdot x_0x_1 + L_{11}(x_2, x_3) \cdot x_1^2 + 2\,Q_0(x_2, x_3) \cdot x_0 + 2\,Q_1(x_2, x_3) \cdot x_1 + R(x_2, x_3) = 0 \qquad (35)$$

where $L_{ij}, Q_\nu, R \in k[x_2, x_3]$ are homogeneous of degrees $1, 2, 3$ respectively. Let us parameterize the plain pencil through ℓ by the points $e_\vartheta = \vartheta_2e_2 + \vartheta_3e_3 \in (e_2e_3)$ and write $(t_0 : t_1 : t_2)$ for homogeneous coordinates in the plane $\pi_\vartheta = (e_0e_1e_\vartheta)$ w. r. t. these 3 base points. The equation for the plane conic $(\pi_\vartheta \cap S) \setminus \ell$ is obtained from (35) by the substitution $x = (t_0 : t_1 : \vartheta_2t_3 : \vartheta_3t_3)$ and cancelling the common factor t_3. The resulting conic has the Gram matrix

$$G = \begin{pmatrix} L_{00}(\vartheta) & L_{01}(\vartheta) & Q_0(\vartheta) \\ L_{01}(\vartheta) & L_{11}(\vartheta) & Q_1(\vartheta) \\ Q_0(\vartheta) & Q_1(\vartheta) & R(\vartheta) \end{pmatrix}$$

whose determinant $D(\vartheta_2, \vartheta_3) = L_{00}(\vartheta)L_{11}(\vartheta)R(\vartheta) + 2\,L_{01}(\vartheta)Q_0(\vartheta)Q_1(\vartheta) - L_{11}(\vartheta)Q_0^2(\vartheta) - L_{00}(\vartheta)Q_1^2(\vartheta) - L_{01}(\vartheta)^2R(\vartheta) \in k[\vartheta_2, \vartheta_3]$ is homogeneous of degree 5 in $\vartheta = (\vartheta_2 : \vartheta_3)$, i. e. has 5 roots counted with multiplicities. We have to show that all these roots are simple. Each root corresponds to the splitting of the conic into a pair of lines, whose intersection point lies either on ℓ or outside ℓ.

In the first case, we can fix a base in such a way that two lines in question are $\ell' = (e_0e_2)$ and $\ell'' = \big(e_0\,(e_1 + e_2)\big)$ given by equations $x_3 = x_1 = 0$ and $x_3 = (x_1 - x_2) = 0$; this corresponds to the root $\vartheta = (1 : 0)$ and its multiplicity equals the highest power of ϑ_3 which divides $D(\vartheta_2, \vartheta_3)$. Since $\ell, \ell', \ell'' \subset S$, the equation (35) has a form $x_1x_2(x_1 - x_2) + x_3 \cdot q(x)$ with some quadratic $q(x)$, i. e. the only elements of G possibly not divisible by ϑ_3 are $L_{11} \equiv x_2 \pmod{\vartheta_3}$ and $Q_1 \equiv -x_2^2/2 \pmod{\vartheta_3}$. So, $D(\vartheta_2, \vartheta_3) \equiv -L_{00}Q_1^2 \pmod{\vartheta_3^2}$ and this term is of order 1 in t_3 as soon $x_1x_2^2$ and $x_0^2x_2$ do come in (35) with non zero coefficients. But the first is the only monomial that gives non zero contribution into $\partial F/\partial x_1$ computed at $e_2 \in S$ and the second — in $\partial F/\partial x_2$ at $e_0 \in S$; so, they do.

In the second case, we can fix a base in order to have $\ell' = (e_0e_2)$, $\ell'' = (e_1e_2)$ (given by $x_3 = x_1 = 0$ and $x_3 = x_0 = 0$), and the same root $\vartheta = (1 : 0)$. Now, (35) takes $x_0x_1x_2 + x_3 \cdot q(x)$ non zero modulo ϑ_3 entry of G is only $L_{01} \equiv x_2/2 \pmod{\vartheta_3}$. So, $D(\vartheta_2, \vartheta_3) \equiv -L_{01}^2R \pmod{\vartheta_3^2}$, which is of order 1 in t_3 if $x_2^2x_3$ and $x_0x_1x_2$ really do appear in (35). The second does, because otherwise F is divisible by x_3. The first is the only monomial that gives non zero contribution into $\partial F/\partial x_3$ computed at $e_2 \in S$.

The rest assertions follow immediately from Corollary 14.9, Claim 14.8 and remark that any line in $\mathbb{P}_3$ intersects any plane. ☐

Claim 14.11. *Any 4 mutually skew lines on S do not lay on a common quadric and always there exist either 1 or 2 (but no more!) lines on S intersecting all these 4 lines.*

Proof. Any line passing through 4 distinct points of S has to lay on S. If 4 lines on S belong to a line family ruling a quadric[2] then the second ruling line family of this quadric, that is the quadric itself, lays on S, i. e. S is reducible. Now it remains to apply Exercice 2.20. ☐

Configuration of 27 lines. Take 2 skew lines a, $b \subset S$ and construct 5 pairs of lines ℓ_i, ℓ'_i predicted by Claim 14.10 applied to $\ell = a$. Let us write ℓ_i for those which do meet b and ℓ'_i for those which do not. Then there are 5 more lines ℓ''_i coupled with ℓ_i by Claim 14.10 applied to $\ell = b$. Each ℓ''_i meets b but neither a nor ℓ_j with $j \neq i$; so, it intersects all ℓ'_j with $j \neq i$. Any line $c \subset S$, different from 17 just constructed, is skew to a, b but meets either ℓ_i or ℓ'_i for each i. By Claim 14.11, all lines meeting $\geqslant 4$ of ℓ_i's are exhausted by a, b. If c meets $\leqslant 2$ out of ℓ_i's, then, up to a permutation of indices, it meets ℓ'_1, ℓ'_2, ℓ'_3 and, say, either ℓ'_4 or ℓ_5; but in the both cases a and ℓ''_5 are the only such the lines. So, c has to intersect precisely 3 out of ℓ_i.

Claim 14.12. *Remaining lines $c \subset S$ are in 1–1 correspondence with 15 triples $\{i, j, k\} \subset \{1, 2, 3, 4, 5\}$.*

Proof. There is at most one line c intersecting a given triple of ℓ_i's — this is the second possible line besides a meeting all these ℓ_i's and the rest ℓ'_j's (all 5 are mutually skew). On the other hand, by Claim 14.10, for each i there are precisely 10 lines on S intersecting ℓ_i: 4 of them are a, b, ℓ'_i, ℓ''_i and other 6 have to intersect exactly 2 of the rest four ℓ_j's. So, we have 1–1 correspondence between these 6 lines and $6 = \binom{4}{2}$ choices of pairs of ℓ's. ☐

Corollary 14.13. *Each smooth cubic surface $S \subset \mathbb{P}_3$ contains precisely 27 lines; their incidence combinatorics is the same for all S.* ☐

Exercice 14.14. Let $\mathfrak{G} \subset \mathfrak{S}_{27}$ be a group of all permutations of the 27 lines preserving all the incidence relations between them; find the order of $\mathfrak{G}$. (Answer: $|\mathfrak{G}| = 51\,840 = 2^7 \cdot 3^4 \cdot 5$)

Exercice 14.15*. Consider the field of 4 elements: $\mathbb{F}_4 \overset{\text{def}}{=} \mathbb{F}_2[\omega]/(\omega^2 + \omega + 1)$, where $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Like for $\mathbb{R} \subset \mathbb{C}$, the extension $\mathbb{F}_2 \subset \mathbb{F}_4$ has a conjugation automorphism: $z \longmapsto \overline{z} \overset{\text{def}}{=} z^2$, which lives $\mathbb{F}_2$ fixed and permutes two roots of $\omega^2 + \omega + 1$. Show that a factor group of the *unitary* 4×4 - matrices with entries in $\mathbb{F}_4$ modulo the scalar matrices is canonically embedded as a (normal) subgroup of index 2 into the group $\mathfrak{G}$ from Exercice 14.14.

Hint. This group preserves the Fermat cubic C_F (see Exercice 14.7) whose equation over $\mathbb{F}_4$ turns to Hermitian form $\sum x_i\overline{x}_i$.

[2] comp. with Corollary 2.19

Abstract nonsense appendix

<u>Categories.</u> Let us use the notion «a category» without giving its formal definition precisely[1]. Informally, *a category* $\mathscr{C}$ consists of *objects*, which form a class[2] denoted by $\mathrm{Ob}\,\mathscr{C}$, and for each pair of objects $X, Y \in \mathrm{Ob}\,\mathscr{C}$ there is a set of *morphisms* $\mathrm{Hom}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y)$, which are distinct for distinct pairs X, Y (usually they are visualized as arrows $X \xrightarrow{\varphi} Y$). This data satisfy the following crucial properties:

- for any ordered triple of objects $X, Y, Z \in \mathrm{Ob}\,\mathscr{C}$ there is *a composition map*
$$\mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \xrightarrow{(\varphi,\psi)\mapsto\varphi\circ\psi} \mathrm{Hom}(X,Z)\ ,$$
which is associative, i. e. $(\chi\circ\varphi)\circ\psi = \chi\circ(\varphi\circ\psi)$ (as soon as they are defined);
- for any $X \in \mathrm{Ob}\,\mathscr{C}$ there is a unique *identity morphism* $\mathrm{Id}_X \in \mathrm{Hom}(X,X)$ that satisfies $\varphi\circ\mathrm{Id}_X = \varphi$ and $\mathrm{Id}_X\circ\psi = \psi$ for any morphisms $X \xrightarrow{\varphi} Y$, $Y \xrightarrow{\psi} X$ and any $Y \in \mathrm{Ob}\,\mathscr{C}$.

For example, the following categories should be well known: topological spaces and continuous maps as the morphisms; finitely generated k-algebras with unity and algebra homomorphisms preserving unity; affine algebraic varieties with regular maps.

More exotic examples: each partially ordered set can be considered as a category where $\mathrm{Hom}(X,Y)$ either contains just one arrow (if $X \leqslant Y$) or is empty (if X and Y are non comparable); each monoid[3] can be considered as a category with just one object X and $\mathrm{Hom}(X,X)$ equal to the monoid itself.

Two objects $X, Y \in \mathrm{Ob}\,\mathscr{C}$ are called *isomorphic*, if there are two arrows[4] $X \underset{\psi}{\overset{\varphi}{\rightleftarrows}} Y$ such that $\varphi\circ\psi = \mathrm{Id}_Y$, $\psi\circ\varphi = \mathrm{Id}_X$.

Given a category $\mathscr{C}$, one can always construct *an opposite category* $\mathscr{C}^{\circ}$ with the same objects: $\mathrm{Ob}\,\mathscr{C}^{\circ} = \mathrm{Ob}\,\mathscr{C}$ and inverted arrows: $\mathrm{Hom}_{\mathscr{C}^{\circ}}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(Y,X)$. The duality $\mathscr{C} \leftrightarrow \mathscr{C}^{\circ}$ is called *reversing of arrows*.

We have seen that the category of finitely generated k-algebras looks like the opposite category to the category of affine algebraic varieties over the same field k. To make this statement more precise we need a tool «for comparing» the categories.

<u>Functors</u> are «homomorphisms of categories». More precisely, *a covariant functor* $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is a map $\mathrm{Ob}\,\mathscr{C} \xrightarrow{X\mapsto F(X)} \mathrm{Ob}\,\mathscr{D}$ together with a collection of maps $\mathrm{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{\varphi\mapsto F(\varphi)} \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$ defined for each pair $X, Y \in \mathrm{Ob}\,\mathscr{C}$ and preserving the compositions, i. e. satisfying $F(\varphi\circ\psi) = F(\varphi)\circ F(\psi)$ as soon as $\varphi\circ\psi$ is defined (in particular, always $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$).

Dually, *a contravariant functor* $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is a covariant functor $\mathscr{C}^{\circ} \xrightarrow{F} \mathscr{D}$. In other words, it is an «antihomomorphism of categories», that is takes $\mathrm{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{\varphi\mapsto F(\varphi)} \mathrm{Hom}_{\mathscr{D}}(F(Y),F(X))$ for each pair $X, Y \in \mathrm{Ob}\,\mathscr{C}$ and satisfies $F(\varphi\circ\psi) = F(\psi)\circ F(\varphi)$.

For example, the dualization, which takes each vector space V over k to its dual V^* and each linear map $V \xrightarrow{\varphi} W$ to the dual map $W^* \xrightarrow{\varphi^*} V^*$, is a contravariant functor from the category of vector spaces and linear maps to itself. The double dualization gives then an example of a covariant functor.

For any $\mathscr{C}$ we always have *the identity functor* $\mathscr{C} \xrightarrow{\mathrm{Id}_{\mathscr{C}}} \mathscr{C}$, which acts identically on the objects and the arrows. An other trivial series of examples is given by *forgetful functors* acting from categories of sets equipped with an extra structure[5] and morphisms, which are maps of sets preserving this extra structure, to the category $\mathscr{S}et$, of all sets (without any structure). Such a functor also acts identically on objects and arrows — it just forgets the extra structure. Less trivial is

[1] like the notion «a set» is utilized usually during studying the calculus — and our reasons are quite similar: a deep settling down into logical casuistry is inevitable when speaking, say, about «the category of all sets»...

[2] say, class of sets, when category of sets is considered

[3] that is semigroup with unity

[4] called *inverse isomorphisms*

[5] it may be geometric: say, a topology, a differentiable manifold structure e. t. c. ; or algebraic: say, a structure of a group, ring, e. t. c.

Example 15.1: Hom-functors. Each $X \in \mathrm{Ob}\,\mathscr{C}$ produces two functors from $\mathscr{C}$ to category of sets. A covariant functor $\mathscr{C} \xrightarrow{h^X} \mathscr{S}et$ takes an object Y to $h^X(Y) \stackrel{\text{def}}{=} \mathrm{Hom}(X,Y)$ and an arrow $Y_1 \xrightarrow{\varphi} Y_2$ to the composition map $h^X(\varphi) : h^X(Y_1) = \mathrm{Hom}(X,Y_1) \xrightarrow{\psi \mapsto \varphi\circ\psi} \mathrm{Hom}(X,Y_2) = h^X(Y_2)$. A contravariant functor $\mathscr{C} \xrightarrow{h_X} \mathscr{S}et$ takes an object Y to $h_X(Y) \stackrel{\text{def}}{=} \mathrm{Hom}(Y,X)$ and an arrow $Y_1 \xrightarrow{\varphi} Y_2$ to the composition map

$$h_X(\varphi) \,:\, h_X(Y_2) = \mathrm{Hom}(Y_2,X) \xrightarrow{\psi\mapsto\psi\circ\varphi} \mathrm{Hom}(Y_1,X) = h_X(Y_1)\,.$$

Exercice 15.2. Show that in the category $\mathscr{M}od(K)$, of modules over commutative ring K with K-linear morphisms, the functor h^X takes any exact triple of modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ to an exact triple

$$0 \longrightarrow \mathrm{Hom}(X,A) \longrightarrow \mathrm{Hom}(X,B) \longrightarrow \mathrm{Hom}(X,C)$$

whose rightmost arrow is non surjective in general. Formulate and prove the similar property of the contravariant functor h_X.

NATURAL TRANSFORMATIONS. Given two (covariant) functors $F, G \,:\, \mathscr{C} \longrightarrow \mathscr{D}$, *a morphism of functors*[6] $F \xmapsto{f} G$ is a collection of arrows $F(X) \xrightarrow{f_X} G(X) \in \mathrm{Hom}_{\mathscr{D}}(F(X), G(X))$ (parameterized by $X \in \mathrm{Ob}\,\mathscr{C}$) such that for any morphism $X \xrightarrow{\varphi} Y$ in $\mathscr{C}$ we have the following commutative square[7] of morphisms in $\mathscr{D}$:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\varphi)} & F(Y) \\ {\scriptstyle f_X}\big\downarrow & & \big\downarrow{\scriptstyle f_Y} \\ G(Y) & \xrightarrow{G(\varphi)} & G(Y) \end{array} \tag{36}$$

For example, the canonical embeddings $V \xhookrightarrow{i_V} V^{**}$ of vector spaces into their double dual[8] present a natural transformation from the identical functor on the category of vector spaces to the double dualization functor.

Clearly, two natural transformations can be composed and the identity maps give an identity transformation from any functor to itself. So we have

Claim 15.3. *For any two categories $\mathscr{C}, \mathscr{D}$ all covariant functors $\mathscr{C} \xrightarrow{F} \mathscr{D}$ form a category $\mathscr{F}un\,(\mathscr{C},\mathscr{D})$ whose morphisms are natural transformations of functors.* □

EQUIVALENCE OF CATEGORIES. A functor $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is called *an equivalence* of categories, if there is a functor $\mathscr{D} \xrightarrow{G} \mathscr{C}$ (called *quasi-inverse* to F) such that the composition GF is isomorphic to $\mathrm{Id}_{\mathscr{C}}$ in category $\mathscr{F}un\,(\mathscr{C},\mathscr{C})$ and the composition FG is isomorphic to $\mathrm{Id}_{\mathscr{D}}$ in category $\mathscr{F}un\,(\mathscr{D},\mathscr{D})$. Note that our requirement «be isomorphic» to the identical functor is much weaker than an other possible request «coincide with» the identical functor.

For example, consider the category k^n that has a unique object — n-dimensional coordinate vector space over k and all linear maps $k^n \longrightarrow k^n$ as arrows. There is a natural functor $k^n \xrightarrow{I} \mathscr{V}ec_n(k)$ sending this category to the category of *all* n-dimensional vector space over k. This functor is an equivalence of categories: to construct (some) quasi-inverse $\mathscr{V}ec_n(k) \xrightarrow{G} k^n$ just fix a base in each vector space V (this identifies V with the standard coordinate vector space) and present all linear maps by their matrices (i. e. by linear transformations of the coordinate vector space). This example has the straightforward generalization:

Claim 15.4. *A functor $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is an equivalence of categories iff it is full*[9]*, faithful*[10]*, and each $Y \in \mathrm{Ob}\,\mathscr{D}$ is isomorphic to $F(X)$ for some $X \in \mathrm{Ob}\,\mathscr{D}$ (depending on Y).*

Proof. For any $Y \in \mathrm{Ob}\,\mathscr{D}$ fix some isomorphism $i_Y : Y \xrightarrow{\sim} F(X)$ existing by our assertion and write $G(Y)$ for such X (it depends on Y). For any arrow $Y_1 \xrightarrow{\varphi} Y_2$ define $G(Y_1) \xrightarrow{G(\varphi)} G(Y_2)$ as an arrow that corresponds to the arrow $i_{Y_2}\circ\varphi\circ i_{Y_1}^{-1} \,:\, F(G(Y_1)) \longrightarrow F(G(Y_2))$ under the assumed bijection[11]

$$\mathrm{Hom}_{\mathscr{C}}(G(Y_1), G(Y_2)) \xrightarrow{F} \mathrm{Hom}_{\mathscr{D}}(FG(Y_1), FG(Y_2))\,.$$

---

[6] also called *a natural transformation* of functors

[7] a diagram of morphisms in a category is called *commutative* if the compositions of arrows taken along different passes joining any two vertexes coincide

[8] sending a vector $v \in V$ to the corresponding evaluation functional $V^* \xrightarrow{\mathrm{ev}_v} k$

[9] F is called *full*, if all maps $\mathrm{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{\varphi\mapsto F(\varphi)} \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$ are surjective

[10] F is called *faithful*, if all maps $\mathrm{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{\varphi\mapsto F(\varphi)} \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$ are injective

[11] «full and faithful» in our assumptions means precisely this bijectivity

Now it remains to make an easy exercise below. $\square$

Exercice 15.5. Verify that $\mathscr{D} \xrightarrow{G} \mathscr{C}$ just constructed: a) is a functor; b) is quasi-inverse to F.

<u>Representable functors.</u> A contravariant functor $\mathscr{C} \xrightarrow{F} \mathscr{S}et$ (or a covariant functor $\mathscr{C}^\circ \xrightarrow{F} \mathscr{S}et$), is called *representable*, if it is isomorphic in the category $\mathscr{F}un\,(\mathscr{C}^\circ, \mathscr{S}et)$ to the functor $h_X : Y \longmapsto \mathrm{Hom}(Y,X)$ associated with some $X \in \mathrm{Ob}\,\mathscr{C}$ as it was described in Example 15.1. Such an object X is called *a representing object* for F. In fact, it is rather «the object» than «an object», because for any two isomorphisms $h_{X_1} \xleftarrow{f_1} F \xrightarrow{f_2} h_{X_2}$ in the category $\mathscr{F}un\,(\mathscr{C}^\circ, \mathscr{S}et)$ there exists *a unique isomorphism* $X_1 \xrightarrow{\sigma} X_2$ in the original category $\mathscr{C}$ such that for any $Y \in \mathrm{Ob}\,\mathscr{C}$ the morphism $h_{X_1}(Y) \xrightarrow{(f_2f_1^{-1})_Y} h_{X_2}(Y)$ (prescribed by the natural transformation $f_2f_1^{-1}$) coincides with the composition map $\mathrm{Hom}(Y,X_1) \xrightarrow{\psi\mapsto\sigma\circ\psi} \mathrm{Hom}(Y,X_2)$. This follows from slightly more abstract

<u>Claim 15.6</u>. *A full faithful covariant functor $\varrho : \mathscr{C} \longrightarrow \mathscr{F}un\,(\mathscr{C}^\circ, \mathscr{D})$ is defined by the prescriptions: for any object X $\varrho(X) = h_X$; for any arrow $X_1 \xrightarrow{\varphi} X_2$ $\varrho(\varphi)$ is a natural transformation $h_{X_1} \longmapsto h_{X_2}$ whose action $\varrho(\varphi)_Y$ at $Y \in \mathrm{Ob}\,\mathscr{C}$ is given by the map $h_{X_1}(Y) = \mathrm{Hom}(Y,X_1) \xrightarrow{\psi\mapsto\varphi\circ\psi} \mathrm{Hom}(Y,X_2) = h_{X_2}(Y)$.*

Proof. Write shortly $\mathscr{C}^\times$ for the category $\mathscr{F}un\,(\mathscr{C}^\circ, \mathscr{S}et)$. The following two steps are almost obvious:

Exercice 15.7. Check that $\varrho(\varphi)$ defined above really gives a natural transformation of functors $h_{X_1} \longmapsto h_{X_2}$ for any arrow $X_1 \xrightarrow{\varphi} X_2$.

Exercice 15.8. Check that $\varrho(\varphi_1\circ\varphi_2) = \varrho(\varphi_1)\circ\varrho(\varphi_2)$, i. e. we have really constructed a functor $\mathscr{C} \xrightarrow{\varrho} \mathscr{C}^\times$.

It remains to check that the map $\mathrm{Hom}_{\mathscr{C}}(X_1,X_2) \xrightarrow{\varrho} \mathrm{Hom}_{\mathscr{C}^\times}(h_{X_1},h_{X_2})$ is bijective for all $X_1,X_2 \in \mathrm{Ob}\,\mathscr{C}$. To construct the inverse map $\mathrm{Hom}_{\mathscr{C}^\times}(h_{X_1},h_{X_2}) \xrightarrow{\lambda} \mathrm{Hom}_{\mathscr{C}}(X_1,X_2)$, consider an arbitrary natural transformation $f : h_{X_1} \mapsto h_{X_2}$, which is an element of $\mathrm{Hom}_{\mathscr{C}^\times}(h_{X_1},h_{X_2})$. Its action at X_1 gives a map

$$h_{X_1}(X_1) = \mathrm{Hom}_{\mathscr{C}}(X_1,X_1) \xrightarrow{f_{X_1}} \mathrm{Hom}_{\mathscr{C}}(X_1,X_2) = h_{X_2}(X_1)\,.$$

We put $\lambda(f) \stackrel{\text{def}}{=} f_{X_1}(\mathrm{Id}_{X_1}) \in \mathrm{Hom}_{\mathscr{C}}(X_1,X_2)$, which is an arrow $X_1 \xrightarrow{\lambda(f)} X_2$ in $\mathscr{C}$. Then, clearly, $\lambda(\varrho(\varphi)) = \varphi$ for any morphism $X_1 \xrightarrow{\varphi} X_2$, because of $\varrho(\varphi)_{X_1}(\mathrm{Id}_{X_1}) = \varphi\circ\mathrm{Id}_{X_1} = \varphi$. Vice versa, we have $\varrho(\lambda(f)) = f$ for any $f : h_{X_1} \mapsto h_{X_2}$, because each arrow $Y \xrightarrow{\psi} X_1$ produces a commutative diagram of type (36):

$$\begin{tikzcd}
h_{X_1}(X_1) \arrow[r, "h_{X_1}(\psi)"] \arrow[d, "f_{X_1}"'] & h_{X_1}(Y) \arrow[d, "f_Y"] \\
h_{X_2}(X_1) \arrow[r, "h_{X_2}(\psi)"] & h_{X_2}(Y)
\end{tikzcd}$$

which says that right vertical map f_Y takes any $\psi \in h_{X_1}(Y) = \mathrm{Hom}_{\mathscr{C}}(Y,X_1)$ (considered as $h_{X_1}(\psi)$-image of $\mathrm{Id}_{X_1} \in h_{X_1}(X_1)$) to the $h_{X_2}(\psi)$-image of $f_{X_1}(\mathrm{Id}_{X_1}) = \lambda(f)$, i. e. to $\lambda(f)\circ\psi$, and this is precisely the action of $\varrho(\lambda(f))$ on ψ. $\square$

Exercice 15.9. A covariant functor $\mathscr{C} \xrightarrow{F} \mathscr{S}et$ is called *corepresentable*, if it is isomorphic in the category $\mathscr{F}un\,(\mathscr{C}, \mathscr{S}et)$ to the functor $h^X : Y \longmapsto \mathrm{Hom}(X,Y)$ associated with some $X \in \mathrm{Ob}\,\mathscr{C}$ (called *corepresentig object* for F). Show that a full faithful contravariant functor $\varrho^\circ : \mathscr{C} \longrightarrow \mathscr{F}un\,(\mathscr{C}, \mathscr{D})$ is defined by the prescriptions: for any object X $\varrho^\circ(X) = h^X$; for any arrow $X_1 \xrightarrow{\varphi} X_2$ $\varrho^\circ(\varphi)$ is a natural transformation $h^{X_2} \longmapsto h^{X_1}$ whose action $\varrho^\circ(\varphi)_Y$ at $Y \in \mathrm{Ob}\,\mathscr{C}$ is given by the map

$$h^{X_2}(Y) = \mathrm{Hom}(X_2,Y) \xrightarrow{\psi\mapsto\psi\circ\varphi} \mathrm{Hom}(X_1,Y) = h^{X_1}(Y)\,.$$

Hint. Just reverse all arrows in the above discussion.

<u>Defining objects by «universal properties».</u> The Claim 15.6 gives a strategy for reproducing several set-theoretical constructions (such as a direct product of sets e. t. c.) in a context of an arbitrary category $\mathscr{C}$. By the definition, a result of some «natural operation» with objects X_i is an object X such that for any Y a set $\mathrm{Hom}(Y,X)$ coincides with the result of the original set-theoretical version of this operation applied to the sets $\mathrm{Hom}(Y,X_i)$. More precisely, X should represent a functor, which takes Y to the result of the set-theoretical operation with $\mathrm{Hom}(Y,X_i)$'s. Such a definition is implicit and does not guarantee an existence of X — the functor in question can be not representable. But if a representing object exists, then it automatically satisfies some «universal properties» and is unique up to unique isomorphism preserving these properties.

<u>Example 15.10</u>: a product $A \times B$, of $A,B \in \mathrm{Ob}\,\mathscr{C}$, is an object representing a functor $Y \mapsto \mathrm{Hom}(Y,A) \times \mathrm{Hom}(Y,B)$ from $\mathscr{C}^\circ$ to $\mathscr{S}et$ (as soon as this functor is representable). In more details, for any Y we should have an isomorphism

$\beta_Y : \mathrm{Hom}(Y, A\times B) \xrightarrow{\sim} \mathrm{Hom}(Y,A)\times\mathrm{Hom}(Y,B)$ functorial w. r. t. arrows $Y_1 \longrightarrow Y_2$. Following the ideas from the proof of Claim 15.6, we can put here $Y = A\times B$ and write

$$\beta_{A\times B}(\mathrm{Id}_{A\times B}) \in \mathrm{Hom}(A\times B, A)\times\mathrm{Hom}(A\times B, B)$$

as (π_A, π_B) for appropriate arrows $A \xleftarrow{\pi_A} A\times B \xrightarrow{\pi_B} B$.

Exercice 15.11. Show that the triple $A \xleftarrow{\pi_A} A\times B \xrightarrow{\pi_B} B$ satisfies the following universal property: for any two morphisms $A \xleftarrow{\varphi} Y \xrightarrow{\psi} B$ there exists a unique morphism $Y \xrightarrow{\varphi\times\psi} A\times B$ such that $\varphi = \pi_A\circ(\varphi\times\psi)$ and $\psi = \pi_B\circ(\varphi\times\psi)$.

Exercice 15.12. Show that for any other triple $A \xleftarrow{\pi'_A} C \xrightarrow{\pi'_B} B$ satisfying the above universal property there exists a unique isomorphism $\gamma : C \xrightarrow{\sim} A\times B$ such that $\pi_A\circ\gamma = \pi'_A$ and $\pi_B\circ\gamma = \pi'_B$

Example 15.13: a coproduct $A\otimes B$, of $A, B\in \mathrm{Ob}\,\mathscr{C}$, is an object corepresenting a functor $Y \mapsto \mathrm{Hom}(A,Y)\times\mathrm{Hom}(B,Y)$ from $\mathscr{C}$ to $\mathscr{Set}$. Reversing arrows in the previous example, we can characterize it by the following universal property: there are two morphisms $A \xrightarrow{i_A} A\otimes B \xleftarrow{i_B} B$ such that for any two arrows $A \xrightarrow{\varphi} Y \xleftarrow{\psi} B$ in $\mathscr{C}$ there exists a unique morphism $A\otimes B \xrightarrow{\varphi\otimes\psi} Y$ such that $\varphi = (\varphi\otimes\psi)\circ i_A$ and $\psi = (\varphi\otimes\psi)\circ i_B$.

Exercice 15.14. Show that such the triple $A \xrightarrow{i_A} A\otimes B \xleftarrow{i_B} B$ (if exists) is unique up unique isomorphism commuting with i-arrows.

Exercice 15.15. Show that if one of two coproducts $A\otimes(B\otimes C)$, $(A\otimes B)\otimes C$ exists, then the other one exists as well and is isomorphic to the first. Prove the same statement also for the products.

Limits. The above definitions are just very particular examples of the following more abstract construction. Fix some category $\mathscr{N}$ (called *category of indexes* — one can treat $\mathscr{N}$ as a kind of «ordered set», where order is given by arrows, but what follows will be valid for *an arbitrary* $\mathscr{N}$ as well).

A functor $\mathscr{N} \xrightarrow{X} \mathscr{C}$ is nothing but a family of objects $X_\nu \in \mathrm{Ob}\,\mathscr{C}$ indexed by $i\in\mathrm{Ob}\,\mathscr{N}$ and morphisms $X_\nu \xrightarrow{\varphi_{\nu\mu}} X_\mu$ indexed by all arrows $i \longrightarrow j$ in $\mathscr{N}$. For example, if $\mathscr{N}$ really is a partially ordered set satisfying the extra condition $\forall\, i, j\ \exists\, k : k > i,\ k > j$, then a functor $\mathscr{N} \xrightarrow{X} \mathscr{C}$ is what is usually called *a direct spectrum* or *a direct system of morphisms*[12] in $\mathscr{C}$.

Further, the category $\mathscr{C}$, we are working in, can be tautologically sent into category $\mathscr{Fun}(\mathscr{N}, \mathscr{C})$ by a functor $X \mapsto \overline{X}$ that attaches a constant family each object $X\in\mathrm{Ob}\,\mathscr{C}$ to the $\overline{X}$ (whose $\overline{X}_\nu \equiv X$, $\varphi_{\nu\mu}\equiv \mathrm{Id}_X$ $\forall\nu$) and each arrow $X \xrightarrow{\psi} Y$ to the corresponding morphism of constant families[13].

Now, given an arbitrary family X_ν, $\varphi_{\nu\mu}$, that is a functor $\mathscr{N} \xrightarrow{X} \mathscr{C}$, its *projective limit* is an object $\varprojlim X_\nu \in \mathscr{C}$ representing a contravariant functor $Y \longmapsto \mathrm{Hom}_{\mathscr{Fun}(\mathscr{N},\mathscr{C})}(\overline{Y}, X)$ from $\mathscr{C}$ to $\mathscr{Set}$. Dually, *an inductive limit* $\varinjlim X_\nu$ corepresents a covariant functor $Y \longmapsto \mathrm{Hom}_{\mathscr{Fun}(\mathscr{N},\mathscr{C})}(X, \overline{Y})$. So, by the definition, we have an isomorphism $\mathrm{Hom}_{\mathscr{C}}(Y, \varprojlim X_\nu) = \mathrm{Hom}_{\mathscr{Fun}(\mathscr{N},\mathscr{C})}(\overline{Y}, X)$ functorial in Y. Applying it to $Y = \varprojlim X_\nu$, we get a natural transformation $\overline{\varprojlim X_\nu} \xrightarrow{\pi} X$ corresponding to $\mathrm{Id}_{\varprojlim X_\nu} \in \mathrm{Hom}_{\mathscr{C}}(\varprojlim X_\nu, \varprojlim X_\nu)$. This is a family of morphisms $\varprojlim X_\nu \xrightarrow{\pi_\nu} X_\nu$ such that $\pi_\mu = \varphi_{\nu\mu}\pi_\nu$ for all arrows $\varphi_{\nu\mu}$ in the family $\{X_\nu\}$. It satisfies the following universal property: for any object $Y\in\mathrm{Ob}\,\mathscr{C}$ equipped with a family of arrows[14] $Y \xrightarrow{\psi_\nu} X_\nu$ such that $\psi_\mu = \varphi_{\nu\mu}\psi_\nu$ there exists a unique morphism $Y \xrightarrow{\alpha} \varprojlim X_\nu$ such that $\psi_\nu = \pi_\nu\circ\alpha$ $\forall\nu$.

Exercice 15.16. Show that projective limit is uniquely characterized by this universal property (up to unique isomorphism commuting with π_ν's).

Exercice 15.17. Show that inductive limit $\varinjlim X_\nu$ is equipped with canonical maps $X_\nu \xrightarrow{i_\nu} \varinjlim X_\nu$ and satisfies the following universal property: given an object $Y\in\mathrm{Ob}\,\mathscr{C}$ with a family of arrows $X_\nu \xrightarrow{\psi_\nu} Y$ such that $\psi_\mu = \psi_\nu\varphi_{\mu\nu}$ (which give a natural transformation $X \overset{\psi}{\longmapsto} \overline{Y}$ in $\mathscr{Fun}(\mathscr{N},\mathscr{C})$), then there exists a unique morphism $\varinjlim X_\nu \xrightarrow{\alpha} Y$ such that $\psi_\nu = \alpha\circ\pi_\nu$ $\forall\nu$.

Exercice 15.18. Let $\mathscr{N}$ be an arbitrary partially ordered set (considered as a category). Show that any family of $\mathscr{N}$-indexed sets $\mathscr{N} \xrightarrow{X} \mathscr{Set}$ has $\varinjlim X$.

[12] dually, a functor $\mathscr{N}^{\circ} \xrightarrow{X} \mathscr{C}$ is called in this case *a inverse spectrum* or *a inverse system of morphisms*

[13] clearly $\overline{X} \overset{\psi}{\mapsto} \overline{Y}$ is a natural transformation of functors

[14] that is for any natural transformation $\overline{Y} \overset{\psi}{\longmapsto} X$ in $\mathscr{Fun}(\mathscr{N},\mathscr{C})$

HINT. A right queue of X is a sequence of elements $x_\nu \in X_\nu$ indexed by some $S \subset \operatorname{Ob}\mathcal{N}$ such that all $\mu > \nu$ belong to S as soon as $\nu \in S$ and $\varphi_{\mu\nu}(x_\mu) = x_\nu \ \forall\, \mu,\nu \in S$. Two right queues $\{x_\alpha\}$, $\{y_\beta\}$ are called equivalent, if $\forall\, x_\alpha,\, y_\beta\ \exists\, \gamma > \alpha,\beta$: $\varphi_{\alpha\gamma}(x_\alpha) = \varphi_{\beta\gamma}(y_\beta)$. Check that a set of all equivalence classes of right queues satisfies the universal properties defining $\varinjlim X$.

Exercice 15.19. Let $\mathcal{N} = \mathbb{N}$ be the set of all positive integers with the standard order. Find $\varprojlim A_n$ and $\varinjlim A_n$ of Abelian groups $A_n = \mathbb{Z}/p^n\mathbb{Z}$ w.r.t. an inverse system of canonical factorizations $\psi_{nm} : \mathbb{Z}/p^n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z}$ ($\forall\, m < n$) and w.r.t. a direct system of standard inclusions $\varphi_{mn} : \mathbb{Z}/p^m\mathbb{Z} \xhookrightarrow{[1]\mapsto[p^{n-m}]} \mathbb{Z}/p^n\mathbb{Z}$ (again $\forall\, m < n$).

HINT. $\varprojlim A_n = \mathbb{Z}_p$ is the set of all p-adic integers and $\varinjlim A_n \subset \mathbb{Q}/\mathbb{Z}$ consists of (mod $\mathbb{Z}$)-classes of fractions z/p^ℓ whose denominator is a power of p (so called p-rational numbers).

Exercice 15.20. Let $\mathcal{N} = \mathbb{N}$ as above but with the partial ordering prescribed by the divisibility. Find $\varprojlim A_n$ and $\varinjlim A_n$ of $A_n = \mathbb{Z}/n\mathbb{Z}$ w.r.t. an inverse system of factorizations $\psi_{nm} : \mathbb{Z}/n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}$ ($\forall\, m|n$) and w.r.t. a direct system of inclusions $\varphi_{mn} : \mathbb{Z}/m\mathbb{Z} \xhookrightarrow{[1]\mapsto[n/m]} \mathbb{Z}/n\mathbb{Z}$ (again $\forall\, m|n$).

HINT. $\varinjlim A_n = \mathbb{Q}/\mathbb{Z}$ and $\varprojlim A_n = \prod\limits_p \mathbb{Z}_p$ is the product of all rings of p-adic integer numbers (so called *completion* of $\mathbb{Z}$).

Example 15.21: fibered products (also called Cartesian squares, or coamalgams) are defined in an arbitrary category $\mathcal{C}$ as projective limits w.r.t. the category of indexes $\mathcal{N} = \{\bullet \longrightarrow \bullet \longleftarrow \bullet\}$ (3 objects and 2 non-identical arrows). Any functor $\mathcal{N} \longrightarrow \mathcal{C}$ is a diagram $X \xrightarrow{\xi} B \xleftarrow{\eta} Y$ in $\mathcal{C}$. Its projective limit is denoted by $X \underset{B}{\times} Y$ and called *a fibered product of* X, Y *over* B. It comes with the following commutative square (called *a Cartesian square*)

$$\begin{tikzcd} & X \underset{B}{\times} Y \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[dr, "\xi"'] & & Y \arrow[dl, "\eta"] \\ & B & \end{tikzcd} \tag{37}$$

which is universal in the following sense: for any other commutative square

$$\begin{tikzcd} & Z \arrow[dl, "\varphi'"'] \arrow[dr, "\psi'"] & \\ X \arrow[dr, "\xi"'] & & Y \arrow[dl, "\eta"] \\ & B & \end{tikzcd}$$

there exists a unique morphism $Z \xrightarrow{\varphi'\times\psi'} X \underset{B}{\times} Y$: $\varphi' = \varphi\circ(\varphi' \times \psi')$, $\psi' = \psi\circ(\varphi' \times \psi')$. Upper part of diagram (37) is uniquely (up to unique isomorphism commuting with φ, ψ) defined by this universality.

Example 15.22: amalgams (also called co-Cartesian squares, or coproducts) are inductive limits w.r.t. the index category $\mathcal{N}^\circ = \{\bullet \longleftarrow \bullet \longrightarrow \bullet\}$. Their expanded definition is obtained from the previous one by reversing the arrows: *an amalgam* of a diagram $X \xleftarrow{\xi} B \xrightarrow{\eta} Y$ is an universal (*co-Cartesian*) commutative square

$$\begin{tikzcd} & X \underset{B}{\otimes} Y & \\ X \arrow[ur, "\varphi"] & & Y \arrow[ul, "\psi"'] \\ & B \arrow[ul, "\xi"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

such that for any other commutative square

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "\varphi'"] & & Y \arrow[ul, "\psi'"'] \\ & B \arrow[ul, "\xi"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

there exists a unique morphism $X \underset{B}{\otimes} Y \xrightarrow{\varphi'\otimes\psi'} Z$ satisfying $\varphi' = (\varphi'\otimes\psi')\circ\varphi$, $\psi' = (\varphi'\otimes\psi')\circ\psi$.

<u>ABELIAN CATEGORIES.</u> Categories appearing in commutative algebra and geometry typically have extra structures on their morphisms $\mathrm{Hom}(X,Y)$: usually we can add morphisms, form their kernels, images e. t. c. A category $\mathscr{C}$ is called *additive*, if it satisfies the following properties:

- bifunctor $X,Y \longmapsto \mathrm{Hom}(X,Y)$ takes its values in the category of Abelian groups $\mathscr{A}b$ instead of $\mathscr{S}et$, i.e. $\mathrm{Hom}(X,Y)$ is an Abelian group $\forall\, X,Y \in \mathrm{Ob}\,\mathscr{C}$ and the composition
$$\mathrm{Hom}(Y,Z)\times\mathrm{Hom}(X,Y) \xrightarrow{(\varphi,\psi)\mapsto\varphi\circ\psi} \mathrm{Hom}(X,Z)\ ,$$
is bilinear: $(\varphi_1+\varphi_2)\circ(\psi_1+\psi_2) = f_1\circ\psi_1 + f_1\circ\psi_2 + f_2\circ\psi_1 + f_2\circ\psi_2$;

- there is *a zero object* $0\in\mathrm{Ob}\,\mathscr{C}$ such that $\mathrm{Hom}(0,0)=0$ is the zero group;

  Exercice 15.23. Deduce from the previous property that $\mathrm{Hom}(X,0)=\mathrm{Hom}(0,X)=0\ \forall\,X\in\mathrm{Ob}\,\mathscr{C}$ and 0 is defined by this property up to unique isomorphism (namely, the zero morphism $0\xrightarrow{0}0'$).

- for any pair of objects A,B there exist a diagram[15]:
$$A \underset{\pi_A}{\overset{i_A}{\rightleftarrows}} A\oplus B \underset{\pi_B}{\overset{i_B}{\leftrightarrows}} B \tag{38}$$
such that $\pi_b\circ i_A=0$, $\pi_A\circ i_B=0$, $\pi_A\circ i_A=\mathrm{Id}_A$, $\pi_B\circ i_B=\mathrm{Id}_B$ and $i_A\circ\pi_A+i_B\circ\pi_B=\mathrm{Id}_{A\oplus B}$.

  Exercice 15.24. Show that $A\oplus B$ is defined by this property up to unique isomorphism commuting with i's and π's.

One can emulate all natural constructions known for Abelian groups in a context of an arbitrary additive category $\mathscr{C}$. For example, define *a kernel* of an arrow $A\xrightarrow{\varphi}B$ in $\mathscr{C}$ as an object representing a functor
$$C \longmapsto \ker\Big(\mathrm{Hom}(C,A)\xrightarrow{\gamma\mapsto\varphi\circ\gamma}\mathrm{Hom}(C,B)\Big)$$
from $\mathscr{C}$ to $\mathscr{A}b$. If exists, the representing object $\ker(\varphi)$ comes with canonical map[16] $\ker(\varphi)\xrightarrow{\varkappa}A$ satisfying $\varphi\circ\varkappa=0$ and the following universality: for any arrow $C\xrightarrow{\gamma}A$ such that $\varphi\circ\gamma=0$ there exists a unique morphism $C\xrightarrow{\psi}\ker(\varphi)$ such that $\varkappa\circ\psi=\gamma$. This property fixes the kernel up to unique isomorphism commuting with $\varkappa$. Reversing arrows, we define *a cokernel* of $A\xrightarrow{\varphi}B$ as a universal morphism $B\xrightarrow{\chi}\mathrm{coker}\,(\varphi)$ such that $\chi\circ\varphi=0$ and for any arrow $B\xrightarrow{\gamma}C$ such that $\gamma\circ\varphi=0$ there exists a unique morphism $\mathrm{coker}\,(\varphi)\xrightarrow{\psi}C$ such that $\psi\circ\chi=\gamma$. Again, $\mathrm{coker}\,(\varphi)$ is uniquely defined by this property (up to unique isomorphism commuting with χ).

Exercice 15.25. Show that in the direct sum diagram (38) the arrow $A\xrightarrow{i_A}A\oplus B$ gives the kernel of the arrow $A\oplus B\xrightarrow{\pi_B}B$ and the arrow $B\xrightarrow{i_B}A\oplus B$ gives the kernel of the arrow $A\oplus B\xrightarrow{\pi_A}A$.

From the main theorem about homomorphisms of groups we expect two ways in which an image of arrow $A\xrightarrow{\varphi}B$ can be defined. Namely, $\mathrm{im}\,\varphi$ should be isomorphic to both: the kernel of $B\xrightarrow{\chi}\mathrm{coker}\,(\varphi)$ and the cokernel of $\ker(\varphi)\xrightarrow{\varkappa}A$. An additive category $\mathscr{C}$ is called *Abelian*, if it satisfies

- each arrow $A\xrightarrow{\varphi}B$ has kernel $\ker(\varphi)$, cokernel $\mathrm{coker}\,(\varphi)$ and is decomposed as
$$\begin{tikzcd} & \mathrm{im}\,(\varphi) \arrow[dr, "\beta"] & \\ A \arrow[ur, "\alpha"] \arrow[rr, "\varphi"] & & B \end{tikzcd}$$
where $\mathrm{im}\,\varphi\simeq\mathrm{coker}\,\Big(\ker(\varphi)\xrightarrow{\varkappa}A\Big)\simeq\ker\Big(B\xrightarrow{\chi}\mathrm{coker}\,(\varphi)\Big)$.

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[15]its middle term $A\oplus B$ is called *a direct sum* of A, B and all the diagram is called *a splitted exact triple*

[16]image of $\mathrm{Id}_{\ker\varphi}$ under the canonical identification of $\mathrm{Hom}(\ker\varphi,\ker\varphi)$ with $\ker\Big(\mathrm{Hom}(\ker\varphi,A)\xrightarrow{\gamma\mapsto\varphi\circ\gamma}\mathrm{Hom}(\ker\varphi,B)\Big)$

A morphism φ in Abelian category is called *surjective* (or *an epimorphism*), if $\operatorname{coker}\varphi = 0$. If $\ker\varphi = 0$, then φ is called *injective* (or *an monomorphism*).

Exercice 15.26. Show that in Abelian category:

a) $\ker(\varphi) \xrightarrow{\varkappa} A$ is injective and $B \xrightarrow{\chi} \operatorname{coker}(\varphi)$ is surjective for any arrow $A \xrightarrow{\varphi} B$;
b) φ is an isomorphism iff it is simultaneously surjective and injective.

Exercice 15.27. Check that in any additive category all squares

$$\begin{tikzcd} A \oplus B \arrow[r, shift left, "\pi_B"] \arrow[d, shift left, "\pi_A"] & B \arrow[l, shift left, "i_B"] \arrow[d, shift left, "0"] \\ A \arrow[u, shift left, "i_A"] \arrow[r, shift left, "0"] & 0 \arrow[l, shift left, "0"] \arrow[u, shift left, "0"] \end{tikzcd}$$

(coming from (38)) are simultaneously Cartesian and co-Cartesian.

Claim 15.28. *In any Abelian category there exist all fibered products and amalgams.*

Proof. To complete an arbitrary triple $X \xrightarrow{\xi} B \xleftarrow{\eta} Y$ to Cartesian square, write $K \xrightarrow{\varkappa} X \oplus Y$ for the kernel of morphism $\delta = \xi\circ\pi_X - \eta\circ\pi_Y \,:\, X \oplus Y \longrightarrow B$. Then a square

$$\begin{tikzcd} & K \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[dr, "\xi"'] & & Y \arrow[dl, "\eta"] \\ & B & \end{tikzcd}$$

where $\varphi = \pi_X\circ\varkappa$, $\psi = \pi_Y\circ\varkappa$ is commutative (because $\xi\varphi - \eta\psi = \delta\varkappa = 0$) and universal (because for any other triple $X \xleftarrow{\varphi'} Z \xrightarrow{\psi'} Y$ such that $\xi\varphi' = \eta\psi'$ only the canonical map[17] $\zeta = \varphi' \oplus \psi' : Z \longrightarrow X \oplus Y$ satisfies $\pi_X\zeta = \varphi'$, $\pi_Y\zeta = \psi'$ and can be lifted to an arrow $Z \xrightarrow{\zeta'} K$, since of $\delta\zeta = \xi\varphi' - \eta\psi' = 0$). □

Exercice 15.29. Show that a diagram $X \xleftarrow{\xi} B \xrightarrow{\eta} Y$ is completed to co-Cartesian square by a cokernel $X \oplus Y \xrightarrow{\chi} Q$ of a morphism $\delta = i_X\circ\xi - i_Y\circ\eta \,:\, B \longrightarrow X \oplus Y$.

Exercice 15.30. Show that for any fibered product (37) in Abelian category:

a) ξ is surjective $\Rightarrow$ ψ is surjective;
b) $K \xrightarrow{\varkappa} X \underset{B}{\times} Y$ is the kernel of $\varphi \Rightarrow K \xrightarrow{\psi\circ\varkappa} Y$ is the kernel of η.

LANGAUGE OF ELEMENTS. Let $\mathscr{C}$ be an Abelian category. Define *an element* a of $A \in \operatorname{Ob}\mathscr{C}$ as a class of equivalent arrows $X \xrightarrow{\alpha} A$ w.r.t. relation:

$$\left(X_1 \xrightarrow{\alpha_1} A\right) \sim \left(X_2 \xrightarrow{\alpha_2} A\right) \quad \Longleftrightarrow \quad \exists \text{ surjections } X_1 \twoheadleftarrow{\pi_1} Z \xrightarrow{\pi_2} X_2 \;:\; \alpha_1\pi_1 = \alpha_2\pi_2 \,.$$

Exercice 15.31. Show that this relation is transitive.

HINT. Use Claim 15.28 and Exercice 15.30.

Write $|A|$ for a set of all elements of $A \in \operatorname{Ob}\mathscr{C}$. Element $0 \in |A|$ represented by the zero morphism $0 \xrightarrow{0} A$ is called *the zero element*. If an element a is represented by $X \xrightarrow{\alpha} A$, then its *opposite* $-a$ is an element $X \xrightarrow{-\alpha} A$. Given two elements a, b represented by $X \xrightarrow{\alpha} A$, $Y \xrightarrow{\beta} A$, define their *sum* $a + b$ by a morphism $X \oplus Y \xrightarrow{\alpha\pi_X + \beta\pi_Y} A$.

Exercice 15.32. Show that all these definitions are correct and provide $|A|$ with a structure of an Abelian group.

Exercice 15.33*. Verify that in category $\mathscr{Ab}$, of Abelian groups, $|A|$ is isomorphic to A as an Abelian group.

For a morphism $A \xrightarrow{\varphi} B$ in Abelian category $\mathscr{C}$ define a map $|A| \xrightarrow{\varphi} |B|$ by taking an element a presented by $X \xrightarrow{h} A$ to an element $\varphi(a)$ presented by the composition $X \xrightarrow{\varphi\circ h} B$.

Exercice 15.34. Verify that this definition is correct and $|A| \xrightarrow{\varphi} |B|$ is a homomorphism of Abelian groups.

Exercice 15.35. Verify the following rules of diagram searching:

a) $A \xrightarrow{\varphi} B$ is mono $\Longleftrightarrow$ $(\forall\, a \in |A| \;\, \varphi(a) = 0 \Rightarrow a = 0) \Longleftrightarrow (\forall\, a_1, a_2 \in |A| \;\, \varphi(a_1) = \varphi(a_2) \Rightarrow a_1 = a_2)$;

[17] predicted by Exercice 15.27

b) $A \xrightarrow{\varphi} B$ is epi $\iff (\forall\, b \in |B|\ \exists\, a \in |A| : \varphi(a) = b)$; c) $\varphi = 0 \iff \varphi(a) = 0\ \forall\, a$;

d) $\operatorname{im}(A \xrightarrow{\varphi} B) = \ker(B \xrightarrow{\psi} C \iff (\psi\varphi = 0\ \&\ \forall\, b \in |B| : \psi(b) = 0\ \exists\, a \in |A| : \varphi(a) = \beta)$

Exercice 15.36*. Try to prove the following

Theorem 15.37. *$\mathscr{C} \xhookrightarrow{A \mapsto |A|} \mathscr{A}b$ is a faithful functor from an arbitrary Abelian category $\mathscr{C}$ into category of Abelian groups.* □

This section shows that one can work with objects of *an arbitrary* Abelian category[18] as if they are Abelian groups. In particular, we can use diagram searching and other «elementary business», say in the category of Abelian sheaves on a manifold.

[18]which is an abstract category just satisfying four axioms listed in the previous section

Vector bundles

FIBERED PRODUCTS. Given two families $Y_1 \xrightarrow{\pi_1} X$, $Y_2 \xrightarrow{\pi_2} X$ of algebraic manifolds over X (comp. with Example 13.10), then their *fibered product* over X is $Y_1 \underset{X}{\times} Y_2 \overset{\text{def}}{=} \{(y_1, y_2) \in Y_1 \times Y_2 \,|\, \varphi_1(y_1) = \varphi_2(y_2)\}$. In fact, it is an algebraic manifold. If $X = Spec_m\, K$, $Y_i = Spec_m\, A_i$, where K, A_1, A_2 are (finitely generated reduced) k-algebras, then the pull-backs $K \xrightarrow{\pi_i^*} A_i$ equip A_i with K-algebra structure and $Y_1 \underset{X}{\times} Y_2 = Spec_m\, A_1 \underset{K}{\otimes} A_2$, where $A_1 \underset{K}{\otimes} A_2$ is *the tensor product* of K-algebras A_i over K, that is the quotient algebra of $A_1 \otimes A_2$ by an ideal spanned by all differences $(\varkappa a_1) \otimes a_2 - a_1 \otimes (\varkappa a_2)$, where $\varkappa \in K$, $a_i \in A_i$.

Exercice 16.1. Write $A_i \xrightarrow{\alpha_i^*} A_1 \underset{K}{\otimes} A_2$ for two K-algebra homomorphisms sending $a_1 \in A_1$ and $a_2 \in A_2$ to $a_1 \otimes 1$ and $1 \otimes a_2$ respectively. Show that for any K-algebra B and any two homomorphisms of K-algebras $A_i \xrightarrow{g_i^*} B$ there exists a unique homomorphism of K-algebras $A_1 \underset{K}{\otimes} A_2 \xrightarrow{g_1^* \otimes g_2^*} B$ such that $g_i^* = (g_1^* \otimes g_2^*) \circ \alpha_i^*$ for both $i = 1, 2$. Show also that this universality determinate the triple $(\alpha_1^*, \alpha_2^*, A_1 \underset{K}{\otimes} A_2)$ uniquely up to unique isomorphism commuting with α_i^*'s.

HINT. This is completely similar to Exercice 12.8.

So, $Y_1 \underset{X}{\times} Y_2 \subset Y_1 \times Y_2$ is a closed submanifold equipped with two projections $Y_1 \underset{X}{\times} Y_2 \xrightarrow{\alpha_i} Y_i$ and satisfying the following important universal property[1]: for any family $Z \xrightarrow{f} X$ and any two morphisms of X-families $Z \xrightarrow{g_i}$ there exists a unique morphism of X-families $Z \xrightarrow{g_1 \times g_2} Y_1 \underset{X}{\times} Y_2$ such that $g_i = \alpha_i \circ (g_1 \times g_2)$ for both $i = 1, 2$. It is very important that k-algebra $A_1 \underset{K}{\otimes} A_2$ can be *non reduced* even if all three algebras in question are reduced (see Example 16.3 below). In this case $Y_1 \underset{X}{\times} Y_2$ is always considered as *geometric scheme* canonically equipped with the structure algebra $k[Y_1] \underset{k[X]}{\otimes} k[Y_2]$ rather than its underlaying manifold $Spec_m \left(k[Y_1] \underset{k[X]}{\otimes} k[Y_2] \right)_{\text{red}}$.

Example 16.2: base change. Any family $Y \xrightarrow{\pi} X$ can be lifted along any morphism[2] $X' \xrightarrow{f} X$ to the family $Y \underset{X}{\times} X' \xrightarrow{f^*(\pi)} X'$ fitting into commutative diagram

$$\begin{tikzcd} Y \underset{X}{\times} X' \arrow[r] \arrow[d, "f^*(\pi)"'] & Y \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

This procedure is called *a base change*. Algebraically, it is known as *extension of scalars*. For example, given $\mathbb{R}$-algebra (or just a vector space) V, then its complexification is nothing but $\mathbb{C} \underset{\mathbb{R}}{\times} V$.

Example 16.3: scheme restrictions and scheme preimages. Given a closed embedding $Z \overset{\varphi}{\hookrightarrow} X$ and an arbitrary family (i. e. a regular map) $Y \xrightarrow{f} X$, then the base change $Y \underset{X}{\times} Z \xrightarrow{f^*(\varphi)} Z$, of f along φ, is called *a scheme restriction* of the family Y onto the closed submanifold Z and the base change $Z \underset{X}{\times} Y \overset{f^*(\varphi)}{\hookrightarrow} Y$, of f along φ, is called *a scheme preimage* of the closed submanifold $Z \subset X$ under the morphism $Y \xrightarrow{f} X$. If X is affine and Z is given by an ideal $I \subset k[X]$, then geometrically $Z \underset{X}{\times} Y \overset{f^*(\varphi)}{\hookrightarrow} Y$ is a closed embedding of $Spec_m\, ()_{\text{red}}$ into Y, which identifies $f^{-1}(Z)$ with the zero set of the ideal $(f^*)^{-1}(I)$. But the structure algebra $(k[X]/I) \underset{k[X]}{\otimes} k[Y]$ is in general non reduced. For example, a scheme preimage of the cuspidal cubic $Z \subset \mathbb{A}_2$ given by equation $y^2 = x^3$ along the map $\mathbb{A}_1 \xrightarrow{t \mapsto (t,t^2)} \mathbb{A}_2$ (whose image is the parabola $y = x^2$) consists of two points $t = 0$ and $t = 1$ but is equipped with non reduced structure algebra $k[t] \underset{k[x,y]}{\otimes} \big(k[x,y]/(y^2 - x^3)\big) = k[t]/(t^4 - t^3)$ (we have used that x and y act on $k[t]$ as t and t^2).

[1] note that it is stronger than the set theoretical definition of $Y_1 \underset{X}{\times} Y_2$ given before and obtained by the specialization of the universality in question to $Z = \pi_1^{-1}(x) \times \pi_2^{-1}(x)$ for $x \in X$

[2] mathematically, «family» and «morphism» mean the same; we use different words just to outline the different roles of these maps but it is extremely important that these roles are completely symmetric

ALGEBRAIC VECTOR BUNDLE over an algebraic manifold X is an algebraic family of vector spaces over X, i.e. a regular map of algebraic manifolds $E \xrightarrow{\pi} X$ whose fiber $\pi^{-1}(x)$ over any $x \in X$ has a structure of a vector space over k and this structure algebraically depends on x, that is three fiberwise operations: picking the zero: $X \xrightarrow{x \mapsto [0]_x} E$; adding vectors: $E \underset{X}{\times} E \xrightarrow{([u]_x,[v]_x) \mapsto [u+v]_x} E$; and multiplying vectors by constants $I \underset{X}{\times} E \xrightarrow{([\lambda]_x,[v]_x) \mapsto [\lambda v]_x} E$ are the regular morphisms of algebraic manifolds commuting with the projections onto X (we have written above $I = X \times \mathbb{A}_1$ for the trivial line bundle, whose fiber is k, and $[v]_x$ for elements of the fiber $\pi^{-1}(x)$). Two vector bundles $E_1 \xrightarrow{\pi_1} X$, $E_2 \xrightarrow{\pi_2} X$ are called *isomorphic*, if there is an isomorphism of algebraic varieties $E_1 \xrightarrow{\varphi} E_2$ such that $\pi_2 \circ \varphi = \pi_1$ and $\forall\, x \in X$ the restriction $\pi_1^{-1}(x) \xrightarrow{\varphi|_{\pi_1^{-1}(x)}} \pi_2^{-1}(x)$ is linear isomorphism of vector spaces. A vector bundle is called *trivial* of rank d, if it is isomorphic to the direct product $X \times \mathbb{A}_d$ with the standard vector space structure on $\mathbb{A}_d = k^{\oplus d}$, which does not depend on $x \in X$.

A regular map $X \xrightarrow{s} E$ is called *a section*, if $\pi \circ s = \mathrm{Id}_X$, that is $s(x) \in \pi^{-1}(x)\ \forall x$. Each vector bundle has canonical *zero section*, which picks the zero at each fiber. A vector bundle $E \xrightarrow{\pi} X$ is trivial of rank d iff there are d regular sections $X \xrightarrow{s_i} E$ such that $\{s_1(x), \dots, s_d(x)\}$ form a base of $\pi^{-1}(x) \quad \forall\, x \in X$. Indeed, the fiberwise coordinate functions on E w.r.t. these basic vectors give the required isomorphism $E \xrightarrow{\sim} X \times \mathbb{A}_d$.

LOCALLY TRIVIAL VECTOR BUNDLE OF RANK d is a vector bundle $E \xrightarrow{\pi} X$ such that $\forall\, x \in X \ \ \exists$ open $U \ni x$ where the restricted bundle $\pi^{-1}(U) \longrightarrow U$ is trivial of rank d, i.e. has d basic sections $(s_1^{(U)}, s_2^{(U)}, \dots, s_d^{(U)}) : U \longrightarrow \pi^{-1}(U)$. If there are two such trivializations $(s_1^{(U)}, s_2^{(U)}, \dots, s_d^{(U)})$ and $(s_1^{(V)}, s_2^{(V)}, \dots, s_d^{(V)})$ defined, respectively, over some open U, V, then over each $x \in U \cap V$ these two basises are expressed through each other as $(s_1^{(U)}, s_2^{(U)}, \dots, s_d^{(U)}) = (s_1^{(V)}, s_2^{(V)}, \dots, s_d^{(V)}) \cdot \varphi_{VU}$, where $\varphi_{VU} = \varphi_{VU}(x)$ is a non degenerate $d \times d$ - matrix whose entries are regular functions on $U \cap V$ (i-th column of φ_{VU} contains the coordinates of $s_i^{(U)}(x)$ w.r.t. the base $\{s_1^{(V)}(x), \dots, s_d^{(V)}(x)\}$). So, we get the regular maps $U \cap V \xrightarrow{\varphi_{VU}} GL_d(k)$ called *transition functions* between two trivializations. They clearly satisfy the conditions $\varphi_{VU}\varphi_{UW} = \varphi_{VW}$, which hold over any triple intersection $U \cap V \cap W$, and $\varphi_{UV} = \varphi_{VU}^{-1}$, certainly. If we change local basis over each open set by some other, say $(\tilde{s}_1^{(U)}, \tilde{s}_2^{(U)}, \dots, \tilde{s}_d^{(U)}) = (s_1^{(V)}, s_2^{(V)}, \dots, s_d^{(V)}) \cdot \psi_U$ where $\psi_U = \psi_U(x)$ is any non degenerate $d \times d$ matrix whose entries are regular functions on the whole of U, then the transition functions also will be changed by $\tilde{\varphi}_{VU} = \psi_V^{-1} \varphi_{VU} \psi_U$

$GL_d(k)$-VALUED ČHECH'S 1-COCYCLE ON X associated with an open covering $X = \cup U_\nu$ is a series of regular maps $U_\alpha \cap U_\beta \xrightarrow{\varphi_{\alpha\beta}} GL_d(k)$ defined for any ordered pair of indexes (α, β) and such that $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$ over $U_\alpha \cap U_\beta$ for all α, β and $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$ over $U_\alpha \cap U_\beta \cap U_\gamma$ for all α, β, γ. Restricting $\varphi_{\alpha\beta}$ onto smaller open sets, we can produce from such the cocycle a cocycle associated with any finer[3] covering inscribed in the initial one. Two cocycles are called *equivalent* or *(co)homologous*, if there exist some common refinement $X = \cup U_\nu$ of their initial open coverings and some regular maps $U_\nu \xrightarrow{\psi_\nu} GL_d(k)$ such that the functions $\varphi_{\alpha\beta}$, $\tilde{\varphi}_{\alpha\beta}$, induced by these cocycles on the refinement, satisfy over each $U_\alpha \cap U_\beta$ the equation $\tilde{\varphi}_{\alpha\beta} = \psi_\alpha^{-1} \varphi_{\alpha\beta} \psi_\beta$. An equivalence class of Čhech 1-cocycles is called *a first Čech cohomology*; the set of these cohomologies is denoted by $H^1(X, GL_d(k))$.

Claim 16.4. *Isomorphism classes of locally trivial vector bundles of rank d are in 1–1 correspondence with the first Čech cohomologies $\varphi_{\alpha\beta} \in H^1(X, GL_d(k))$.*

Proof. Given a Čech cocycle $\varphi_{\alpha\beta}$, construct E as a manifold whose affine charts are $U_\alpha \times \mathbb{A}_d$ glued along $(U_\alpha \cap U_\beta) \times \mathbb{A}_d$ by the rule $U_\alpha \times \mathbb{A}_d \ni (x, v) \longmapsto (x, \varphi(\alpha\beta)(x) \cdot v) \in U_\beta \times \mathbb{A}_d$, where $v \in \mathbb{A}_d$ is a column vector. The cocycle conditions imply that these chard form an atlas and linearity of $\varphi(\alpha\beta)(x)$ for each x implies that the vector space structures of fibers are correctly glued together. Conversely, we have seen in the previous section that for a given vector bundle the transition functions between its trivializations form a Čech cocycle, which is changed by a homologous one under a changing of the trivialization or (what is the same) under a a fiberwise linear isomorphism of the bundle. □

Example 16.5: a tautological vector bundle $S \longrightarrow \mathbb{P}(V)$ is a rank 1 vector subbundle $S \subset \mathbb{P}(V) \times V$ whose fiber over $v \in \mathbb{P}(V)$ is the 1-dimensional subspace of V spanned by v. Over any affine chart $U_\alpha = \{v \in \mathbb{P}(V) \mid \alpha(v) \neq 0\}$, where $\alpha \in V^*$, it can be trivialized by the section $s^{(\alpha)}(v) = (v, v/\alpha(v)) \in \mathbb{P}(V) \times V$, which is a well defined regular function $U_\alpha \xrightarrow{s^{(\alpha)}} S \subset \mathbb{P}(V) \times V$. Since $s^{(\alpha)}(v) = s^{(\beta)}(v) \cdot (\beta(v)/\alpha(v))$ over each $v \in U_\alpha \cap U_\beta$, the transition functions between these trivializations are $\varphi_{\beta\alpha}(v) = \beta(v)/\alpha(v)$, which are well defined regular maps $U_\alpha \cap U_\beta \longrightarrow GL_1(k) = k^*$.

Example 16.6: a tautological vector bundle $S \longrightarrow \mathrm{Gr}(m, V)$ over the Grassmannian, whose points are m-dimensional subspaces $W \subset V$, is a rank m vector subbundle $S \subset \mathrm{Gr}(m, V) \times V$ whose fiber over $W \in \mathrm{Gr}(m, V)$ is the m-dimensional

[3] a covering $X = \cup W_\nu$ is called *finer* than a covering $X = \cup U_\nu$, if $\forall \nu\ \exists \mu \ : \ W_\nu \subset U_\mu$)

subspace $W \subset V$ itself. If we fix a base $\{e_1, e_2, \dots, e_n\}$ in V and for each $I = (i_1 < i_2 < \cdots < i_m) \subset (1, 2, \dots, n)$ consider the standard affine chard $U_I \subset \mathrm{Gr}(m, V)$, which consists of all W projected isomorphically onto the linear span of $\{e_{i1}, e_{i2}, \dots, e_{im}\}$, then S is trivialized over U_I by m sections $s^{(I)}_\nu(W) \subset W$ which form the unique base of W such that the coordinates (written in columns) of the basic vectors form $m \times n$ matrix $M_I(W)$ containing the identity $m \times m$ - submatrix in the rows I. Since for any $W \in U_I \cap U_J$ we have $M_I(W) = M_J(W) \cdot \varphi_{JI}(W)$, where $\varphi_{JI}(W)$ is the inverse matrix for the $m \times m$ submatrix of M_J situated in the rows I, the transition functions between two trivializations $s^{(I)}_\nu(W)$ and $s^{(J)}_\nu(W)$ are given by the maps $W \longmapsto \varphi_{JI}(W) \in GL_m(k)$. Clearly, these are regular maps well defined everywhere in $U_I \cap U_J$.

Linear constructions with vector bundles. Given two locally trivial vector bundles E, F of ranks r, s presented by Čech cocycles $\varphi_{\alpha\beta}$, $\psi_{\alpha\beta}$ over the same open covering $X = \cup U_\nu$, one can form their fiberwise direct sum $E \oplus F$ and tensor product $E \otimes F$ which have ranks $r + s$, rs and are presented by the cocycles $\varphi_{\alpha\beta} \oplus \psi_{\alpha\beta}$ and $\varphi_{\alpha\beta} \otimes \psi_{\alpha\beta}$ (direct sum and tensor product of linear operators). Similarly one can make other tensor constructions, say fiberwise exterior or symmetric powers $\Lambda^m E$, $S^m E$ of a given locally trivial vector bundle E.

Pull back. Given a regular map $X \xrightarrow{f} Y$, then any vector bundle $E \twoheadrightarrow Y$ induces a vector bundle $f^*(E) bydef X \underset{Y}{\times} E \twoheadrightarrow X$ over X called *a pull back* of E along f. If E is locally trivial and presented by a Čech cocycle $\varphi_{\alpha\beta}$ over some open covering $Y = \cup U_\nu$, then f^*E is presented by $f^*(\varphi_{\alpha\beta}) = \varphi_{\alpha\beta} \circ f$ over the induced open covering $X = \cup f^{-1}(U_\nu)$.

Exercice 16.7. Let $\mathrm{Gr}(m, V) \overset{p}{\hookrightarrow} \mathbb{P}(\Lambda^m V)$ be the Plücker embedding. Check that the pull back $p^* S_{\mathbb{P}}$ of the tautological line bundle on $\mathbb{P}(\Lambda^m V)$ is the maximal exterior power $\Lambda^m S_{\mathrm{Gr}}$ of the tautological line bundle on $\mathrm{Gr}(m, V)$.

Picard group $Pic(X)$, by the definition, consists of all isomorphism classes of locally trivial rank 1 algebraic vector bundles on X. They form an Abelian group w. r. t. the tensor multiplication, i. e. given two line bundles L, K with Čech cocycles $\varphi_{\alpha\beta}$, $\psi_{\alpha\beta}$, which are just $GL_1(k) = k^*$ - valued functions on $U_\alpha \cap U_\beta$, then their sum in $Pic(X)$ equals to the line bundle $E \otimes K$ with the Čech cocycle $\varphi_{\alpha\beta} \cdot \psi_{\alpha\beta}$. The zero element of $Pic(X)$ is the trivial line bundle $I = X \times \mathbb{A}_1$. The opposite element for L presented by a Čech cocycle φ_{ij} is the dual bundle $L^* = \mathrm{Hom}(L, I)$ whose Čech cocycle equals $\varphi^*_{ij} = 1/\varphi_{ij}$.

Claim 16.8. *If X is affine and $k[X]$ is factorial, then $Pic(X) = 0$.*

Proof. A trivializing covering $X = \cup U_\alpha$ for a given line bundle L can be always chosen to be such that $U_\alpha = \mathscr{D}(f_\alpha)$ for some finite collection $f_1, f_2, \dots, f_n \in k[X]$. Fix some trivializing sections s_α over each U_α and consider corresponding transition functions $\varphi_{\beta\alpha} = s_\beta/s_\alpha$, which are nowhere vanishing elements of $\mathscr{O}_X(U_\alpha \cap U_\beta) = k[X][1/(f_\alpha f_\beta)]$, that is of the form $f^r_\alpha f^s_\beta$ for some $r, s \in \mathbb{Z}$. Let an irreducible element $q \in k[X]$ appear in $\varphi_{\beta\alpha}$ with a power $m_{\beta\alpha} \in \mathbb{Z}$. If one of these powers is not zero, we can split all f_α's into two nonempty subsets: φ_β's, which are divisible by q, and φ_γ's, which are not. Then for each β the power $m_{\gamma\beta}$ does not depend on γ, because q must disappear in $\varphi_{\gamma_1\gamma_2} = \varphi_{\gamma_1\beta}/\varphi_{\gamma_2\beta}$. Let us write m_β for this power and change all sections s_β by $s'_\beta = q^{m_\beta} \cdot s_\beta$ (this leads to a new basic section, because $Z_q \subset Z_{f_\beta}$). After that q, clearly, disappears in all $\varphi_{\gamma\beta}$ as well as in all $\varphi_{\beta_1\beta_2} = \varphi_{\gamma\beta_2}/\varphi_{\gamma\beta_1}$. Since the set of all q's having non zero $m_{\alpha\beta}$ is finite (it is exhausted by irreducible divisors of f_α's), after a finite series of such the replacements we come to transition functions that have no irreducible factors, i. e. are the constants. □

Corollary 16.9. *$Pic(\mathbb{A}_n) = 0$.* □

Claim 16.10. *$Pic(\mathbb{P}_n) = \mathbb{Z}$ is spanned by the tautological vector bundle S.*

Proof. By Corollary 16.9, any line bundle L can be trivialized over the standard affine chart U_{x_i} by some local nowhere vanishing section s_i. Let us write $t^{(i)}_\nu$, $\nu \neq i$, for the restrictions of linear forms x_ν onto affine hyperplane $x_i = 0$ in $\mathbb{A}_{n+1}$ and use them as affine coordinates on U_{x_i}. The transition function $\varphi_{ij} = s_i/s_j \in k(U_{x_i})$ is a rational function of $t^{(i)}_\nu$ such that its numerator and denominator do not vanish anywhere in U_{x_i} except for $Z_{t^{(i)}_j}$. Hence, $\varphi_{ij} = (t^{(i)}_j)^{d_{ij}}$. Since $t^{(j)}_k = x_k/x_j = (x_k/x_i) : (x_j/x_i) = t^{(i)}_k/t^{(i)}_j$, the cocycle conditions $\varphi_{ij} = 1/\varphi_{ji}$ and $\varphi_{jk} = \varphi_{ik}/\varphi_{ij}$ force $d_{ij} = -d_{ji} = d$ with the same d for all i, j. On the other side, for any $d \in \mathbb{Z}$ the functions $\varphi_{ij} = (t^{(i)}_j)^d = (x_j/x_i)^d$ form Čech 1-cocycle, i. e. define a line bundle, which we will denote by $\mathscr{O}(-d)$.

Exercice 16.11. Check that $\mathscr{O}(-d) = S^{\otimes d}$.

So, it remains to show that all $\mathscr{O}(d)$ are pairwise non isomorphic. To this aim let us compute their spaces of all regular (possibly, vanishing at some points) global sections. A local section defined everywhere on U_{x_0} has a form $s = f({t^{(0)}}_1, {t^{(0)}}_2, \dots, {t^{(0)}}_n) \cdot s_0$, where f is any polynomial of n variables. Rewriting s in terms of chart U_{x_i} we get

$$s = f\left(\left(t^{(i)}_1/t^{(i)}_0\right), \dots, \left(t^{(i)}_n/t^{(i)}_0\right) \right) \cdot \left(t^{(i)}_0\right)^d \cdot s_i\,.$$

So, s is extended onto U_{x_i} iff $\deg f \leqslant d$. Hence, for $d < 0$ there are no global sections and for $d \geqslant 0$ there is $\binom{n+d}{d}$-dimensional space of global sections. In particular, all positive $\mathscr{O}(d)$ are mutually different and non isomorphic to negative. Since $\mathscr{O}(-d) = \mathscr{O}(d)^*$, negative $\mathscr{O}(d)$ bundles are pairwise different as well. □

SHEAVES OF SECTIONS. Let $E \xrightarrow{\pi} X$ be a locally trivial vector bundle. Then for any open $U \subset X$ all regular local sections $U \xhookrightarrow{s} \pi^{-1}(U) \subset E$ form a module over the algebra $\mathscr{O}_X(U)$ of local regular functions on U. This module is denoted by $\Gamma(U, E)$ or $E(U)$. The correspondence $U \longmapsto \Gamma(U, E)$ is called *a sheaf of local sections* of the vector bundle E. Regardless of an evident ambiguity, it is usually denoted by the same letter E. Since the bundle E is locally trivial, the sheaf E is *locally free*, i.e. each point $x \in X$ has an open neighborhood $U \ni X$ such that $\Gamma(U, E)$ is finitely generated free $\mathscr{O}_X(U)$ - module of rank $\operatorname{rk} E$: any collection of trivializing sections for E over U gives a free base for $\Gamma(U, E)$.

Claim 16.12. *Let E be a vector bundle over an affine irreducible variety X and $P_E = \Gamma(E, X)$. Then P_E is finitely generated torsion free*[4] *$k[X]$-module and for any $g \in k[X]$ the module of local sections $\Gamma(\mathscr{D}(g), E)$ coincides with $k[X][g^{-1}] \underset{k[X]}{\otimes} P_E$ (which is the module of fractions*[5] *s/g^m, where $s \in P_E$, $m \in \mathbb{Z}$).*

Proof. Let E be trivialized over some principal open covering $X = \cup \mathscr{D}(f_\nu)$ by local sections ${s^{(\nu)}}_1, {s^{(\nu)}}_2, \dots, {s^{(\nu)}}_r \in \Gamma(\mathscr{D}(f_\nu), E)$. Then the restriction of any section $s \in \Gamma(\mathscr{D}(g), E)$ onto $\mathscr{D}(g) \cap \mathscr{D}(f_\nu) = \mathscr{D}(gf_\nu)$ can be written as

$$s|_{\mathscr{D}(gf_\nu)} = \sum_{i=1}^{r} \frac{h_i}{(gf_\nu)^{m_\nu}} \cdot s_i^{(\nu)}|_{\mathscr{D}(gf_\nu)}$$

So, $\widetilde{s} = g^{\max m_\nu} \cdot s$ is extended onto each $\mathscr{D}(f_\nu)$, that is to a global section of E, and $s = \widetilde{s}/g^m$ as required in the last assertion. To prove the first assertion, write $s_i^{(\nu)}$ as $s_i^{(\nu)} = \widetilde{s}_i^{(\nu)}/f_\nu^{m_{i\nu}}$, where $\widetilde{s}_i^{(\nu)} \in P_E$ are global sections. Then $\widetilde{s}_i^{(\nu)}$ generate P_E over $k[X]$. Indeed, for any $s \in P_E$ and any ν we can write: $s|_{\mathscr{D}(f_\nu} = \dfrac{1}{f_\nu^m} \sum\limits_i g_i^{(\nu)} \cdot \widetilde{s}_i^{(\nu)}$ for some $g_i^{(\nu)} \in k[X]$ and $m \in \mathbb{N}$. Hence, $f_\nu^m \cdot s = \sum\limits_i g_i^{(\nu)} \cdot \widetilde{s}_i^{(\nu)}$ is a $k[X]$-linear combination of $\widetilde{s}_i^{(\nu)}$'s. On the other hand, we can write $1 = \sum\limits_\nu h_\nu f_\nu^m$, because f_ν's have no common zeros. So, $s = \sum_\nu h_\nu f_\nu^m \cdot s = \sum_{\nu i} h_\nu g_i^{(\nu)} \cdot \widetilde{s}_i^{(\nu)}$. Absence of torsion is evident. □

Corollary 16.13. *Under the previous claim conditions, E is trivial iff P_E is free.*

Proof. If $s_1, s_2, \dots, s_r$ form the base of P_E, then, by the claim, their restrictions onto each $\mathscr{D}(f)$ form the base of $\Gamma(\mathscr{D}(f), E)$ over $\mathscr{O}_X(\mathscr{D}(f))$. In particular, r coincides with the number of local trivializing sections, i.e. with the rank of E, and $s_1, s_2, \dots, s_r$ form a base for each fiber, because if some fiber contains a vector laying outside their linear span, then a local section drawn through this vector can not be expressed as $\mathscr{O}_X$-linear combination of s_i's. □

Corollary 16.14. *Each (algebraic, locally trivial) vector bundle over $\mathbb{A}_1$ is trivial.*

Proof. Indeed, P_E is free, because any finitely generated torsion free $k[t]$-module is free[6] □

Exercice 16.15. Show that any nowhere vanishing regular section of a vector bundle over $\mathbb{A}_1$ can included in some system regular sections forming a base in each fiber.

Theorem 16.16. *([7]) Each (algebraic, locally trivial) vector bundle over $\mathbb{P}_1$ splits into a direct sum of line bundles $\mathscr{O}_{\mathbb{P}_1}(d_i)$.*

Proof. Write t for affine coordinate on $\mathbb{A}_1 = \mathbb{P}_1 \setminus \{\infty\}$ and consider two trivializations $e_1^0, e_2^0, \dots, e_r^0$ and $e_1^\infty, e_2^\infty, \dots, e_r^\infty$ of a given vector bundle E defined in $\mathbb{A}_1$ and in $\{\infty\} \cup (\mathbb{A}_1 \setminus \{0\})$. Over $\mathbb{A}_1 \setminus \{0\}$ they are expressed through each other as $(e_1^\infty, e_2^\infty, \dots, e_r^\infty) = (e_1^0, e_2^0, \dots, e_r^0) \cdot \varphi$, where φ is the transition matrix whose entries are rational functions of t without poles in $\mathbb{A}_1 \setminus \{0\}$, that is polynomials in t, t^{-1}. Replacing E by

$$E(m) \overset{\text{def}}{=} E \otimes \mathscr{O}(m) \ ,$$

we multiply all entries of φ by t^m. In particular, we can choose m such that the first column of φ, i.e. e_1^∞ expanded through e_i^0, has no negative powers of t but does not vanish at $t = 0$. This means that e_1^∞ becomes nowhere vanishing global section of E over $\mathbb{P}_1$.

Exercice 16.17. Show that for $m \ll 0$, $E(m)$ will not have the global sections at all.

So, there exists the minimal m such that $e(m)$ admits a nowhere vanishing global section e. Let us now replace E by this $E(m)$. Using induction over the rank, we can suppose that the factor bundle $Q = E/e \cdot \mathscr{O}$ splits as

$$Q = \mathscr{O}(d_2) \oplus \mathscr{O}(d_3) \oplus \ \cdots \ \oplus \mathscr{O}(d_r) \ , \quad \text{where } d_2 \leqslant d_3 \leqslant \ \cdots \ \leqslant d_r$$

[4] A-module M is called *torsion free*, if $am = 0 \Rightarrow a = 0$ or $m = 0$ for $a \in A$, $m \in M$

[5] also called *a localization* of P_E w.r.t. multiplicative set $\{f^k\}$

[6] The same is true for any principal ideal domain. For proof, present the module M in question as F/K, where $F = k[t]^{\oplus n}$ and $K \subset F$ is the kernel of surjection $F \twoheadrightarrow M$ sending basic vectors of F to generators of M. The result follows at once from the *elementary divisors theorem*: each submodule $K \subset F$ is free and there exist some bases $\{e_1, e_2, \dots, e_n\} \subset F$, $\{u_1, u_2, \dots, u_m\} \subset K$ and $f_1, f_2, \dots, f_m \in k[t]$ such that $u_i = f_i \cdot e_i$ for $1 \leqslant i \leqslant m$ (moreover, f_i divides f_j for $i < j$ and the set of those elementary divisors does not depend on a choice of bases). Indeed, theorem forces $F/K = k[t]^{\oplus(n-m)} \oplus T$, where $T = \bigoplus\limits_i (k[t]/(f_i))$ is torsion submodule. The elementary divisors theorem also holds over any principal ideal domain (see Artin's or Van der Warden's » Algebra« textbook; special proof for either $\mathbb{Z}$ or $k[t]$ is extremely fruitful exercise on the Gauss diagonalization and Euclid's division algorithms).

[7] sometimes it is called *Birkhoff - Grothendieck splitting theorem*

By Exercice 16.15, this means that we can take our trivializations $\{e_\nu^0\}$, $\{e_\nu^\infty\}$ with $e_1^0 = e_1^\infty = e$ and transition matrix

$$(e_1^\infty,\, e_2^\infty,\, \dots,\, e_r^\infty) = \left(e_1^0,\, e_2^0,\, \dots,\, e_r^0\right) \cdot \begin{pmatrix} 1 & f_2 & f_3 & f_4 & \dots & f_r \\ 0 & t^{d_2} & 0 & 0 & \dots & 0 \\ 0 & 0 & t^{d_3} & 0 & \dots & 0 \\ 0 & 0 & 0 & t^{d_4} & \dots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & t^{d_r} \end{pmatrix}$$

where $f_\nu = f_\nu(t, t^{-1})$ are some polynomials in t, t^{-1}. Moreover, adding to all other columns the first column multiplied by appropriate polynomial in t^{-1} (this corresponds to an invertible change of e_ν^∞ with $\nu \geqslant 2$), we can make all f_ν to be inside the ideal $(t) \subset k[t]$. This means that all $d_\nu \leqslant 0$. Indeed, if $d_\nu > 0$ for some ν, then e_ν^∞ is extended to nowhere vanishing section of $E(-d)$, where $d = GCD(t^{d_\nu}, f_\nu)$ is strictly positive, but this contradicts with our choice of the twist $E \mapsto E(m)$ made before. So. adding to the first row the other rows multiplied by appropriate polynomials in t (this corresponds to an invertible change of e_1^0), we can annihilate all f_ν and get a diagonal transition matrix as required. $\square$

Sheaves.

<u>PRE-SHEAVES.</u> A concept of sheaf is one of the basic ground notions in mathematics just as a concept of set. By the definition, *a pre-sheaf* F of groups (rings, modules, sets, ...) on a topological space X is *a contravariant functor* from a category of open sets (whose morphisms are inclusions) to the category of groups (rings, modules, sets, ...). In other words, F attaches, say a group, $F(U)$, called a group of *local sections*, to each open $U \subset X$ and attaches a group homomorphism $F(U) \xrightarrow{s \mapsto s|_W} F(W)$, called *a restriction* of local sections, to each pair $W \subset U$ in such a way that $\forall\, W \subset U \subset V$ and $\forall\, s \in F(V)$ we have $(s|_U)|_W = s|_W$. By the definition, a morphism of pre-sheaves $E \longrightarrow F$ is a morphism of functors, that is a collection of homomorphisms $E(U) \longrightarrow F(U)$ defined for each U and commuting with the restrictions. Given $x \in X$, *a germ* of a local section of F at x is an equivalence class of pairs (U, s_U), where $U \ni x$ is open, $s_U \in F(U)$, and $(U, s_U) \sim (W, s_W)$, if $s_U|_V = s_W|_V$ for some open V such that $x \in V \subset U \cap W$. The germs of local sections at x form a group (resp. ring, module, set, ...), which is called *a stalk* of F at x and is denoted by F_x.

Exercice 17.1. Show that $F_x = \varinjlim\limits_{U \ni x} F(U)$

Similarly, a stalk F_Z over any closed $Z \subset X$ is defined as $F_Z = \varinjlim\limits_{U \supset Z} F(U)$.

<u>SHEAVES.</u> A pre-sheaf F is called *a sheaf*, if for any family of local sections $s_\nu \in F(U_\nu)$ such that $\forall\, \mu, \nu$ $s_\nu|_{U_\nu \cap U_\mu} = s_\mu|_{U_\nu \cap U_\mu}$ there exists a unique $s \in F(\bigcup\limits_\nu U_\nu)$ with $s|_{U_\nu} = s_\nu$ $\forall \nu$. In particular, for any sheaf F

$$F(U \cup W) = F(U) \times F(W)\,,\ \text{if}\ U \cap W = \varnothing\,, \tag{39}$$

Exercice 17.2. Check that really the structure sheaf of an algebraic variety is a sheaf of rings and the sheaf of local sections of an algebraic vector bundle is a sheaf of modules over the structure sheaf.

For any pre-sheaf F there is a canonical sheaf $\widetilde{F}$ together with a morphism (of pre-sheaves) $F \xrightarrow{\sigma} \widetilde{F}$ such that for any sheaf E and any morphism (of pre-sheaves) $F \xrightarrow{\varphi} E$ there exists a unique morphism $\widetilde{F} \xrightarrow{\widetilde{\varphi}} E$ satisfying $\widetilde{\varphi} \circ \sigma = \varphi$. It is constructed as follows. For each open U consider the direct product of all stalks: $F_U \stackrel{\text{def}}{=} \prod\limits_{x \in U} F_x$ and define $\widetilde{F}(U) \subset F_U$ as a subgroup which consists of all collections of germs $\{s_x\}_{x \in U}$ such that for any $y \in U$ there exist an open neighborhood $W \ni y$ and a local section $t \in F(W)$ whose germ (represented by a pair (W, t)) equals s_x $\forall\, x \in W \cap U$.

Exercice 17.3. Check that this prescription really defines a sheaf, which satisfies the universal property formulated before; moreover this property defines $\widetilde{F}$ up to a unique isomorphism (of pre-sheaves) commuting with σ.

<u>Example 17.4</u>: constant pre-sheaf and constant sheaf. Each, say a group, K trivially gives a pre-sheaf with the same section groups $K(U) = K$ $\forall\, U$ and identical restrictions. This pre-sheaf is not a sheaf, because (39) falls. The associated sheaf $\widetilde{K}$ attaches to each U a group of *locally* constant, that is discrete but continuous, K-valued functions on U, which are constant only over *connected* U. It is called *a constant sheaf* of K.

Exercice 17.5. Compute the real primitive function $\displaystyle\int \frac{dx}{x}$.

HINT. Typical answer: «$\log|x| + C$, where C is a constant» is false: in fact, C is a section of the constant sheaf over $\mathbb{R} \setminus \{0\}$.

<u>Example 17.6</u>: factorization. Consider the unit circle $X = S^1$ and write C^∞, Ω^1 for the sheaves of smooth local functions and smooth local differential 1-forms $f(\vartheta)\, d\vartheta$, where ϑ is natural angle parameter on the circle.

Exercice 17.7. Check that the both really are the sheaf.

The De Rham operator gives a morphism of sheaves $d : C^\infty \xrightarrow{f \mapsto df} \Omega^1$ whose kernel is a constant subsheaf $\mathbb{R} \subset C^\infty$. Its image $U \mapsto B^1(U) = C^\infty/\mathbb{R} \subset \Omega^1$ is a pre-sheaf of smooth local exact 1-forms. But this pre-sheaf is not a sheaf! Indeed, the form $d\vartheta$ is not globally exact, but gives exact forms over any connected $U \subsetneq S^1$ compatible over their intersections; so such local sections can not be glued together!

In other words, naively defined factor sheaf is not a sheaf but only a pre-sheaf. But it would be very nice, if the sheaves of, say Abelian groups, would form an Abelian category, that is with images, cokernels e. t. c. being the sheaves. To this aim the definition of factor-sheaf is modified in order to force the factors to be the sheafs.

Namely, given two sheaves $F \subset E$ of Abelian groups on any topological space, then their factor sheaf E/F *is defined* as a sheaf associated with the pre-sheaf $U \longmapsto F(U)/E(U)$. Thus, a section of E/F over U is a collection of germs (s_α, W_α) such that $s_\alpha \in F(W_\alpha)$, $W_\alpha \subset U$, and $(s_\alpha - s_\beta)|_{W_\alpha \cap W_\beta} \in E(W_\alpha \cap W_\beta)$.

In particular, a local section of the *factor sheaf* $C^\infty/\mathbb{R}$ on S^1 over open $U \subset S^1$ is represented by a covering $U = \cup W_\alpha$ and a collection of exact forms $df_\alpha \in B(W_\alpha)$ such that $f_\alpha - f_\beta = \text{const}$ over $W_\alpha \cap W_\beta$. Now $\delta\vartheta$ becomes a well defined global section of $C^\infty/\mathbb{R}$.

Example 17.8: direct image (push forward) of a sheaf E on X w. r. t. a continuous map $X \xrightarrow{f} Y$ is a sheaf $f_\bullet E$ on Y whose fiber over an open $U \subset Y$ is $E(f^{-1}(U))$. If f is a closed embedding, then the sheaves $f_\bullet E$ on Y are called *sky scrapers* supported on X.

Example 17.9: inverse image (pull back) of a sheaf F on Y w. r. t. a continuous map $X \xrightarrow{f} Y$ is a sheaf $f^\bullet F$ on X whose section over an open $U \subset X$ is a collection of germs $w_x \in F_{f(x)}$ (parameterized by $x \in U$) such that for any $y \in U$ there exist an open $W \ni f(y)$ in Y and a section $s \in F(W)$ whose germ (represented by (s, W)) coincides with w_x for all x from some open neighborhood $V \subset f^{-1}(W) \cap U$, $V \ni y$. In particular, a constant sheaf on X associated with K is a pull back of K under the contraction of X to one point and a stalk of a sheaf at a point is a pull back of this sheaf under the inclusion of the point into the manifold.

Exercice 17.10. Check that $f^\bullet F$ is really a sheaf on X even for a pre-sheaf F on Y (moreover, a sheaf associated with a given pre-sheaf is nothing but a its pull back under the identity map) and prove that for any sheaves E on X and F on Y there exists a natural isomorphism $\mathrm{Hom}_X(f^\bullet F, E) = \mathrm{Hom}_Y(F, f_\bullet E)$.

Example 17.11: localization. Let X be an algebraic manifold (or geometric scheme) with the structure sheaf $\mathscr{O}_X$ and $Z \xhookrightarrow{f} X$ be a closed embedding. Then $f^\bullet \mathscr{O}_X$ is called *a localization* of regular functions on X near Z. For example, if X is affine, $\mathscr{O}_X(X) = A$, and $Z \subset X$ is an irreducible submanifold given by a prime ideal $\mathfrak{p} \subset A$, then $f^\bullet \mathscr{O}_X(Z)$ is a ring of fractions[1] $A_\mathfrak{p} = \{p/q \mid p, q \in A,\ q \notin \mathfrak{p}\}$ (these fractions are added and multiplied by the usual rules and, by the definition, $p_1/q_1 = p_2/q_2$ iff $q_2p_1 - q_1p_2$ is a zero divisor in A). This is a *local ring*, i. e. it contains a unique proper maximal ideal $\mathfrak{m}_\mathfrak{p} = \{p/q \mid p \in \mathfrak{p},\ q \notin \mathfrak{p}\}$ and all elements of $A_\mathfrak{p} \setminus \mathfrak{m}_\mathfrak{p}$ are invertible. In particular, if $Z = x$ is one point with maximal ideal $\mathfrak{m}_x$, then we get the local ring A_x, which coincides with the stalk $\mathscr{O}_{X\,x}$ and consists of germs of local rational functions well defined at x. Its unique maximal ideal $\mathfrak{m}_x \subset A_x$ consists of all rational functions vanishing at x; so $A_x/\mathfrak{m}_x = k$.

$\mathscr{O}$-MODULES. In the framework of algebraic geometry, the sheaves of $\mathscr{O}_X$-modules are those usually considered on an algebraic manifold X. Sections of such a sheaf over an open $U \subset X$ form a module over $\mathscr{O}_X(U)$. A regular map of algebraic manifolds (or geometric schemes) $X \xrightarrow{f} Y$ involves the homomorphism of structure sheaves[2] $\mathscr{O}_Y \xrightarrow{f^*} f_\bullet\mathscr{O}_X$, or equivalently (see the isomorphism from Exercice 17.10), the homomorphism $f^\bullet\mathscr{O}_Y \xrightarrow{f^*} \mathscr{O}_X$. The both send a local regular function on Y to its composition with f and endow $f_\bullet\mathscr{O}_X$ with the structure of $\mathscr{O}_Y$-algebra and $\mathscr{O}_X$ — with the structure of $f^\bullet\mathscr{O}_Y$-algebra. So, given a sheaf E of $\mathscr{O}_X$-modules on X, then, restricting scalars from $f_\bullet\mathscr{O}_X$ to $\mathscr{O}_Y$, we can consider the $f_\bullet\mathscr{O}_X$-module $f_\bullet E$ as an $\mathscr{O}_Y$-module as well. This sheaf of $\mathscr{O}_Y$-modules on Y is denoted by f_*E and is called a *push forward* of the $\mathscr{O}_X$-module E. Dually, given an $\mathscr{O}_Y$-module F on Y, then we can extend scalars from $f^\bullet\mathscr{O}_Y$ to $\mathscr{O}_X$ in the $f^\bullet\mathscr{O}_Y$-module $f^\bullet F$. The resulting $\mathscr{O}_X$-module $\mathscr{O}_X \underset{f^\bullet\mathscr{O}_Y}{\otimes} f^\bullet F$ on X is called a pull back of the $\mathscr{O}_Y$-module F on Y and is denoted by f^*F.

QUASI-COHERENT SHEAVES. A sheaf F of $\mathscr{O}_X$-modules over an affine algebraic manifold X is called *quasi-coherent*, if it is constructed from some $k[X]$-module M_F in the same way as the sheaf of sections of locally trivial vector bundle E was recovered from the module $P_E = \Gamma(X, E)$, of its global sections, in Claim 16.12. More precisely, quasi-coherent sheaf $\widetilde{M}$ associated with $k[X]$-module M takes[3] $\widetilde{M}(U) = \mathscr{O}_X(U) \underset{k[X]}{\otimes} M$ over each open $U \subset X$. In other words, a section of $\widetilde{M}$ over U is a collection of fractions m_x/f_x parameterized by $x \in U$ such that $\forall\, y \in U$ there exist an open neighborhood $V \subset U$, $V \ni y$ and a pair $m \in M$, $f \in \mathscr{O}_X(V)$ satisfying two properties: $f(y) \neq 0$ and $m_x/f_x = m/f \quad \forall\, x \in V \cap \mathscr{D}(f)$. A quasi coherent sheaf $\widetilde{M}$ is called *coherent*, if the module M is finitely generated over $k[X]$. In general case, an $\mathscr{O}_X$-module $\mathscr{M}$ over an arbitrary geometric scheme X is called quasicoherent (resp. coherent), if there exists an affine open covering (resp. a finite affine open covering)$X = \cup X_i$ such that $\mathscr{M}(X_i)$ is a quasicoherent (resp. coherent) $\mathscr{O}_{X_i}$-module for each i. The quasi coherence is preserved by pull-backs and push forwards, moreover, the pull-back of a coherent module is also coherent. In a contrast, the direct image of a coherent module is often non finitely generated. For example, taking a push forward under the contraction of X to a point p, we actually restrict the scalars from $k[X]$ to $k = \mathscr{O}_p$; after that, say $\mathscr{O}_X$, becomes infinite dimensional; an other typical example gives the inclusion $X = \mathbb{A}_1 \setminus \{0\} \xhookrightarrow{i} \mathbb{A}_1$ where $i_*\mathscr{O}_X = \widetilde{k[t, t^{-1}]}$, which is not finitely generated over $\widetilde{k[t]} = \mathscr{O}_{\mathbb{A}_1}$.

[1] also called a *localization* of A at $\mathfrak{p}$

[2] it is uniquely defined by f in the case of manifolds and is a part of definition of a regular map in the case of geometric schemes

[3] the tensor product $* \underset{k[X]}{\otimes} *$ used here is the tensor product of $k[X]$-modules defined precisely as in Exercice 16.1: if K is a k-algebra and M_1, M_2 — two K-modules, then $M_1 \underset{K}{\otimes} M_2$ is the quotient of $M_1 \otimes M_2$ by a subspace spanned by all $(\varkappa m_1) \otimes m_2 - m_1 \otimes (\varkappa m_2)$, where $\varkappa \in K$, $m_i \in M_i$

Example 17.12: sheaves of ideals and closed subschemes. $\mathscr{O}_X$-submodule $\mathscr{I} \subset \mathscr{O}_X$ is called *a sheaf of ideals*. It is coherent, because its sections $\mathscr{I}(U)$ over affine $U \subset X$ form (maybe non reducible) ideal $I \subset k[U]$. The zero set $Z_I \subset U$ is a closed algebraic subset of U equipped with the structure of geometric scheme given by the k-algebra $k[U]/I$. These local closed subschemes are glued together into one closed geometric subscheme Z equipped with the regular map $Z \overset{f}{\hookrightarrow} X$ whose pull back takes $\mathscr{O}_X$-module E to $\mathscr{O}_Z = \mathscr{O}_X/\mathscr{I}$ module $\left(\mathscr{O}_X/\mathscr{I}\right) \underset{\mathscr{O}_X}{\otimes} E$. Sines multiplication by $\mathscr{I}$ acts trivially on this module, all its sections over any open $W \subset X \setminus Z$, where $\mathscr{I}(W) = \mathscr{O}_X(W)$, should vanish. So, Z can be also described as the support of the factor sheaf $F = \mathscr{O}_X/\mathscr{I}$, i.e. $Z = \{x \in X \mid F_x \neq 0\}$. By this reason, a pull back map along closed embedding $Z \hookrightarrow X$ is usually called a *restriction* of an $\mathscr{O}_X$-modules onto Z. In particular, if $Z = x$ is one point and $\mathscr{I} = \mathfrak{m}_x$ is its maximal ideal, then the restriction of an $\mathscr{O}_X$-module E onto x, i.e. $\left(\mathscr{O}_X/\mathfrak{m}_x\right) \underset{\mathscr{O}_X}{\otimes} E$, is nothing but $E_x/\mathfrak{m}_x \cdot E_x$, where E_x is a stalk of E at x. The ideal sheaf $\mathscr{I}$ of a closed subscheme $Z \subset X$ is locally free iff Z can be given locally by one equation; in particular, an ideal sheaf of any nonempty subscheme of codimension $\geqslant 2$ is not locally free.

Sheaves of algebras and affine morphisms. Given a sheaf $\mathscr{A}$ of $\mathscr{O}_X$-algebras which is quasi coherent as a sheaf of $\mathscr{O}_X$-modules, then over any affine $U \subset X$ we have a finitely generated $k[U]$-algebra A and two maps: $k[U] \xrightarrow{\alpha^*} A$ and $k[U][x_1, x_2, \dots, x_n] \twoheadrightarrow A$ which give a map $Spec_m\, A \xrightarrow{\alpha} U$ and a closed embedding $Spec_m\, A \hookrightarrow U \times \mathbb{A}_n$. The maps α are glued together into a global map $Y \xrightarrow{\widetilde{a}} X$, where Y is a geometric scheme with a structure sheaf $\mathscr{A}$ locally (over affine $U \subset X$) isomorphic to a closed subscheme in $U \times \mathbb{A}_n$ (in general, n depends on U). Morphisms of the type $Y \longrightarrow X$ just described are called *affine morphisms*. The Example 17.12 can be considered as a particular case of this construction where $\mathscr{A} = \mathscr{O}_X/\mathscr{I}$. An other important particular case is given in the next example.

Example 17.13: vector bundles. Given a coherent $\mathscr{O}_X$-module F, a sheaf of its symmetric algebras[4]

$$\mathscr{S}(F) = \bigoplus_{d \geqslant 0} S^d(F) \tag{40}$$

satisfies the above conditions, i.e. is a quasi coherent $\mathscr{O}_X$-algebra. The corresponding affine morphism is denoted as $\mathbb{V}(F^*) \xrightarrow{\pi} X$ and called a *vector bundle dual to F*. By the construction, the fiber $\pi^{-1}(p)$ over a given $p \in X$ is the maximal spectrum of the k-algebra $\left(\mathscr{O}_X/\mathfrak{m}_x\right) \underset{\mathscr{O}_X}{\otimes} \mathscr{S}(F) = \bigoplus_{d \geqslant 0} S^d(F_x/\mathfrak{m}_x \cdot F_x)$ (comp. with the Example 17.12), that is a vector space $(F_x/\mathfrak{m}_x \cdot F_x)^*$ over $k = \mathscr{O}_{X\,x}/(\mathfrak{m}_x \cdot \mathscr{O}_{X\,x})$ (recall that $\mathbb{A}(V) = Spec_m\,(\bigoplus_{d \geqslant 0} S^d V^*)$). In other words, $\mathbb{V}(F^*) \xrightarrow{\pi} X$ is a vector bundle (possibly not locally trivial) and sections of F are naturally identified with functions $\mathbb{V}(F^*) \longrightarrow k$ which are linear along fibers of π.

Exercice 17.14. Show that vector bundle $\mathbb{V}(F^*)$ is locally trivial iff F is locally free $\mathscr{O}_X$-module.
Hint. See Corollary 16.13.

Sheaves of graded algebras, projective morphisms, and cones. Given a sheaf $\mathscr{A} = \bigoplus_{m \geqslant 0} \mathscr{A}_m$ of *graded* $\mathscr{O}_X$-algebras such that: (1) $\mathscr{A}_m = \mathscr{O}_X$; (2) $\mathscr{A}_1$ is coherent $\mathscr{O}_X$-module; (3) $\mathscr{A}$ is generated by $\mathscr{A}_1$ as $\mathscr{O}_X$-algebra, then $\mathscr{A}$ is quasicoherent and over any affine $U \subset X$ we have a finitely generated graded $k[U]$-algebra $A = \oplus A_k = k[U][x_0, x_1, \dots, x_n]/I$, where $x_0, x_1, \dots, x_n \in A_1$ are some generators of $k[U]$-module A_1 (in general, n may depend on U) and $I \subset k[U][x_0, x_1, \dots, x_n]$ is a graded ideal. So, we have a closed subscheme $Y_U \subset U \times \mathbb{P}_n$ given by I and equipped with the closed morphism $Y_U \xrightarrow{p_U} U$ induced by the projection $U \times \mathbb{P}_n \longrightarrow U$. The maps p_U are glued together into a global map $Y \xrightarrow{p} X$. Such a map p is called a *projective morphism* and the geometric scheme Y is denoted $\mathbb{P}\mathrm{roj}\,(\mathscr{A})$ and called a *projective spectrum* of graded $\mathscr{O}_X$-algebra $\mathscr{A}$. Note that epimorphism of $\mathscr{O}_X$-algebras $\mathscr{S}(\mathscr{A}_1) \xrightarrow{\varphi^*} \mathscr{A}$ (induced by the multiplication) gives a closed embedding $Spec_m \mathscr{A} \overset{\varphi}{\hookrightarrow} \mathbb{V}(\mathscr{A}_1^*)$ and $\mathbb{P}\mathrm{roj}\,(\mathscr{A}) \subset \mathbb{P}\mathbb{V}(\mathscr{A}_1^*)$ is nothing but the fiberwise projectivization of $\operatorname{im}\varphi$ (see Example 17.15 below).

Example 17.15: projective bundles. As in Example 17.13, the symmetric algebra (40) of any coherent $\mathscr{O}_X$-module F satisfies the above conditions (1) – (3), that is produces the projective morphism $\mathbb{P}\mathrm{roj}\,(\mathscr{S}(F)) \xrightarrow{p} X$. Certainly, the manifold $\mathbb{P}\mathrm{roj}\,(\mathscr{S}(F))$ is nothing but the projective bundle obtained from the vector bundle $\mathbb{V}(F^*)$ (see Example 17.13) by the fiberwise projectivization. It usually denoted by $\mathbb{P}(F^*)$ or $\mathbb{P}(\mathbb{V}(F^*))$, because the fiber $p^{-1}(x)$ at $x \in X$ is the projective space $\mathbb{P}\left((F_x/\mathfrak{m}_x \cdot F_x)^*\right)$ over $k = \mathscr{O}_{X\,x}/(\mathfrak{m}_x \cdot \mathscr{O}_{X\,x})$ and sections of F can be considered as fiberwise linear homogeneous forms on $\mathbb{P}(F^*)$. Such the forms form a locally free sheaf $\mathscr{O}_{\mathbb{P}(F^*)}$-modules of rank 1; this sheaf (as well as the corresponding line bundle over $\mathbb{P}(\mathbb{V}(F^*))$) is denoted by $\mathscr{O}_{\mathbb{P}(F^*)}(1)$.

[4] the symmetric algebra of an $\mathscr{O}_X$-module is defined precisely in the same way as the symmetric algebra of a vector space (see Claim 4.13 and discussion around it) with only difference: all the tensor products are taken over $\mathscr{O}_X$ rather than over k

In general case, the restriction of the line bundle $\mathscr{O}_{\mathbb{P}\mathbb{V}(\mathscr{A}_1^*)}(1)$ from $\mathbb{P}\mathbb{V}(\mathscr{A}_1^*)$ onto $\mathbb{P}\mathrm{roj}(\mathscr{A})$ is denoted $\mathscr{O}_{\mathbb{P}\mathrm{roj}(\mathscr{A})}(1)$ as well.

Non projectivized affine morphism $Spec_m\mathscr{A} \xhookrightarrow{\pi} X$ is called a *dual cone* of the graded $\mathscr{O}_X$-algebra $\mathscr{A}$ and denoted $C(\mathscr{A})$. There is canonical *zero section* $X \xhookrightarrow{\sigma_0} C(\mathscr{A})$ given by the augumentation homomorphism $\mathscr{A} \xrightarrow{\sigma_0^*} \mathscr{O}_X$, which is identical on $\mathscr{A}_0$ and takes to zero all $\mathscr{A}_i$ with $i > 0$.

Projective spaces

1◇1. Let S^dV^* be the space of all degree d homogeneous polynomials on n-dimensional linear space V. Find $\dim S^dV^*$.

1◇2 (Veronese map). Under the previous problem conditions, let $V^* \xrightarrow{v_d} S^dV^*$ take a linear form ℓ to its d-th power ℓ^d. Does the image of v_d lay in a hyperplane or its linear span exhaust the whole of S^dV^*?

1◇3. Let V be n-dimensional vector space over a finite field consisting of q elements. How many **a)** basises **b)** k-dimensional subspaces are there in V? **c)** How many points are there in $\mathbb{P}(V)$?

1◇4*. Let $G_n^k(q)$ be the number of k-dimensional vector subspaces in n-dimensional vector space over a finite field of q elements. Compute $\lim\limits_{q\to 1} G_n^k(q)$.

1◇5. Consider the projective closures of affine curves
a) $y = x^2$ **b)** $y = x^3$ **c)** $y^2 + (x-1)^2 = 1$ **d)** $y^2 = x^2(x+1)$
Write down their homogeneous equations and their affine equations in two other standard affine charts on $\mathbb{P}_2$. Try to draw all these affine curves.

1◇6. Let the real Euclidian plane $\mathbb{R}^2$ be included in $\mathbb{CP}_2$ as the real part of the standard affine chart U_0. Find two points of $\mathbb{CP}_2$ such that any Euclidean circle will contain them after comlexification and projective closuring.

1◇7 (Pythagorean triples). Consider $\mathbb{P}_2$ with homogeneous coordinates $(t_0 : t_1 : t_2)$. Let $\ell \subset \mathbb{P}_2$ be the line $t_2 = 0$, $Q \subset \mathbb{P}_2$ be the conic $t_0^2 + t_1^2 = t_2^2$, and $O = (1 : 0 : 1) \in Q$. For each $P = (p : q : 0) \in \ell$ find coordinates of the intersection point $Q \cap (OP)$ different from O and show that the projection from O maps Q bijectively onto ℓ. Find some polynomials $a(p,q)$, $b(p,q)$, $c(p,q)$ whose values on $\mathbb{Z}\times\mathbb{Z}$ give, up to a common factor, all integer Pythagorian triples $a^2 + b^2 = c^2$ (and only such the triples).

1◇8 (projecting twisted cubic). Let $\mathbb{P}_1 = \mathbb{P}(V^*)$ be the space of linear forms (considered up to constant factor) in two variables (t_0, t_1), $\mathbb{P}_3 = \mathbb{P}(S^3V^*)$ be the space of cubic forms in (t_0, t_1), and $\mathbb{P}_1 \xhookrightarrow{v_3} \mathbb{P}_3$ be the cubic Veronese map, like in prb. 1◇2. Its image is called *the twisted cubic* C_3. Calculate the projection of C_3:

a) from the point t_0^3 to the plane spanned by $3\,t_0^2t_1$, $3\,t_0t_1^2$, and t_1^3
b) from the point $3\,t_0^2t_1$ to the plane spanned by t_0^3, $3\,t_0t_1^2$, and t_1^3
c) from the point $t_0^3 + t_1^3$ to the plane spanned by t_0^3, $3\,t_0^2t_1$, and $3\,t_0t_1^2$

More precisely, write explicit parametric representation for the projection in appropriate coordinates, then find its affine and homogeneous equation. Do that for several affine charts on the projection target plane. In each case, find degree of the the curve and try to draw it. Has it selfintersections and/or cusps?

Quadrics and conics

2◇1. Consider the quadratic form $q(A) = \det A$ on the space of square 2×2-matrices $Mat_2(k)$. Describe its polarization, i. e. what is the number $\widetilde{q}(A,B)$ in terms of two matrices[1] A and B.

2◇2 (continuation of prb. 1◇6). Under the conditions of prb. 1◇6, show that any quadric on $\mathbb{CP}_2$, which pass through two points you have found in prb. 1◇6 and has at least 3 points inside the initial real Euclidian plane, looks there as a circle.

2◇3 (Euclidean polarities). Consider a circle in the real Euclidean affine plane. How to draw[2]:
a) the polar of a given point (especially, when the point is inside the circle)
b) the pole of a given line (especially, when the line does not intersect the circle)
Describe geometrically a polarity w. r. t. an 'imaginary circle' $x^2 + y^2 = -1$.

2◇4. Show that all conics passing through the points $a = (1:0:0)$, $b = (0:1:0)$, $c = (0:0:1)$, $d = (1:1:1)$ form a line in the space of all conics. Write an explicit equation[3] for these conic family and find all singular conics inside it.

2◇5 (1 – 1 correspondence on a conic). Let $Q \subset \mathbb{P}_2$ be non singular conic considered together with some rational parameterization $\mathbb{P}_1 \xrightarrow{\sim} Q$. Show that for any bijection $Q \xrightarrow{\gamma} Q$ induced by a linear automorphism of $\mathbb{P}_1$ there exist two points $p_1, p_2 \in Q$ and a line $\ell \subset \mathbb{P}_2$ such that $x \xmapsto{\gamma} y$ iff $\pi_\ell^{p_1} x = \pi_\ell^{p_2} y$. Were are the fixed points of this map? Is it possible, using only the ruler, to find (some) p_1, p_2, ℓ for γ given by its action on 3 points $a, b, c, \in Q$?

2◇6*. Using only the ruler, draw a triangle inscribed in a given non singular conic Q and such that his sides a, b, c pass through 3 given points A, B, C. How many solutions may have this problem?

Hint. Start 'naive' drawing from any $p \in Q$ and denote by $\gamma(p)$ your return point after passing trough A, B, C. Is $p \longmapsto \gamma(p)$ a projective isomorphism of kind described in prb. 2◇5?

2◇7*. Formulate and solve projectively dual problem to the previous one.

2◇8*. Describe a general algorithm for reducing a trigonometric equation $f(\sin(x), \cos(x)) = 0$, where f is an arbitrary quadratic polynomial in two variables, to a simple equation $\cos(x) = \alpha$, where α contains at most cubic irrationalities.

Hint. The problem is how to compute explicitly 4 intersection points of 2 quadrics $f(x,y) = 0$ and $x^2 + y^2 = 1$; but the same intersection points can be produced by any two quadrics from the same pencil. A good idea is to intersect two singular conics of this pencil.

2◇9. How many lines cross each of 4 given pairwise skew lines in:
a) $\mathbb{CP}_3$ **b)** $\mathbb{RP}_3$ **c*)** $\mathbb{C}^3$ **d*)** $\mathbb{R}^3$
Find all possible answers and indicate those are stable under small perturbations.

2◇10. Consider the Segre embedding[4] $\mathbb{P}(U_-^*) \times \mathbb{P}(U_+) \xrightarrow{\sim} Q_S \subset \mathbb{P}(\mathrm{Hom}(U_-, U_+))$ and let a point $p \in \mathbb{P}(\mathrm{Hom}(U_-, U_+)) \setminus Q_S$ correspond to an isomorphism $\mathbb{P}(U_-) \xrightarrow{\varphi} \mathbb{P}(U_+)$. Consider any $u \in \mathbb{P}(U_-)$ and let $\xi = \mathrm{Ann}\,(u) \in \mathbb{P}(U_-^*)$ ([5]). Draw a plane $\pi_{p,\ell}$ through the point p and the line $\ell = s(\xi \times \mathbb{P}_1) \subset Q_S$. Prove that $\pi_{p,\ell} \cap Q_S$ is a reducible conic of the form $\ell \cup \ell'$, where the second component ℓ' has the form $s(\mathbb{P}_1 \times t)$ for some $t \in \mathbb{P}(U_+)$, which coincides with $\varphi(u)$.

[1] for example, the standard Euclidean norm $\|\,(a_{ij})\,\| \overset{\text{def}}{=} \sum a_{ij}^2$ is polarized to $(A,B) = \mathrm{tr}\,\left(A \cdot {}^tB\right)$; polarization of $\det(A)$ should look similarly with something else instead of ${}^tB$ ...

[2] using ruler and compasses

[3] it should be a quadratic form whose coefficients depend linearly on two homogeneous parameters

[4] see lecture 4 and "Tensor Guide"

[5] i. e. if $\xi = (\xi_0 : \xi_1)$, then $u = (\xi_1 : -\xi_0)$

Plücker – Segre – Veronese interaction

3◇1. Consider two lines $\ell_1, \ell_2 \subset \mathbb{P}_3$ and denote by $\ell_1^\times, \ell_2^\times \subset \mathbb{P}_3^\times$ two pencils of planes passing through these lines. Take any 3 non collinear points a, b, c such that no two of them are coplanar with either ℓ_1 or ℓ_2. Write $\ell_1^\times \xrightarrow{\gamma_{abc}} \ell_2^\times$ for a linear projective isomorphism that sends 3 planes passing through a, b, c in $\ell_1^\times$ to the corresponding planes in $\ell_2^\times$.

Describe the incidence graph $\Gamma_{abc} \overset{\text{def}}{=} \bigcup\limits_{\pi \in \ell_1^\times} (\pi \cap \gamma_{abc}(\pi))$ ruled by the intersection lines of γ_{abc}-incident planes, if: **a)** $\ell_1 \cap \ell_2 = \varnothing$ **b)** $\ell_1 \cap \ell_2 \neq \varnothing$

3◇2. Let $A \in \mathrm{Hom}(U,V) \simeq U^* \otimes V$, $B \in \mathrm{Hom}(V,W) \simeq V^* \otimes W$ be two linear maps decomposed as $A = \sum \alpha_\nu \otimes a_\nu$, $B = \sum \beta_\mu \otimes b_\mu$ with $\alpha_\nu \in U^*$, $a_\nu \in V$, $\beta_\mu \in V^*$, $b_\mu \in W$. Decompose similarly their product $B \circ A \in \mathrm{Hom}(U,W) \simeq U^* \otimes W$.

3◇3. Are the following decompositions valid for any vector space V over a field of zero characteristic: **a)** $V^{\otimes 2} \simeq S^2V \oplus \Lambda^2V$ **b)** $V^{\otimes 3} \simeq S^3V \oplus \Lambda^3V$ (If yes, give a proof, if no, give an explicit example of a tensor, which can not be decomposed in this way.)

3◇4 (spinor decomposition). Let $V = \mathrm{Hom}(U_-, U_+)$, where $\dim U_\pm = 2$. Show that $V = U_-^* \otimes U_+$ and $V^{\otimes 2} = \underbrace{\Big(\big(S^2U_-^* \otimes S^2U_+\big) \oplus \big(\Lambda^2U_-^* \otimes \Lambda^2U_+\big)\Big)}_{S^2V} \bigoplus \underbrace{\Big(\big(S^2U_-^* \otimes \Lambda^2U_+\big) \oplus \big(\Lambda^2U_-^* \otimes S^2U_+\big)\Big)}_{\Lambda^2V}$.

3◇5. Let $G \subset \mathbb{P}_3 = \mathbb{P}(V)$ be a non singular quadric given by a quadratic form g whose polarization is $\widetilde{g}$. Show that bilinear form $\Lambda^2\widetilde{g}$ on Λ^2V, which acts on decomposable bivectors as

$$\Lambda^2\widetilde{g}(\, v_1 \wedge v_2 \,,\, w_1 \wedge w_2 \,) \overset{\text{def}}{=} \det \begin{pmatrix} \widetilde{g}(v_1,w_1) & \widetilde{g}(v_1,w_2) \\ \widetilde{g}(v_2,w_1) & \widetilde{g}(v_2,w_2) \end{pmatrix} ,$$

is symmetric and non degenerate, and write its explicit Gram matrix in a convenient base (say, coming from an orthonormal base for g in V). Show that the intersection of the corresponding quadric $\Lambda^2G \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2V)$ with the Plücker quadric consists of all tangent lines to $G \subset \mathbb{P}_3$.

3◇6. Under the previous problems notations, let $\mathrm{Gr}(2,V)$ be the Grassmannian variety, of lines in $\mathbb{P}_3 = \mathbb{P}(V)$. Show that the Plücker embedding $\mathrm{Gr}(2,V) \hookrightarrow \mathbb{P}(\Lambda^2V)$ sends two line families living on the Segre quadric $G \subset \mathbb{P}(V) = \mathbb{P}(\mathrm{Hom}(U_-,U_+))$ to a pair of non singular plane conics that are cut out the Plücker quadric $P \subset \mathbb{P}(\Lambda^2V)$ by two complementary planes $\Lambda_- = \mathbb{P}\left(S^2U_-^* \otimes \Lambda^2U_+\right)$ and $\Lambda_+ = \mathbb{P}\left(\Lambda^2U_-^* \otimes S^2U_+\right)$ laying in $\mathbb{P}(\Lambda^2\mathrm{Hom}(U_-,U_+))$ via prb. 3◇4. Moreover, the both conics are embedded into these planes via Veronese, that is, we have the following commutative diagram (Plücker is dotted, because it maps lines into points):

$$\begin{tikzcd}
\mathbb{P}(U_+) \arrow[rrr, hook, "\text{Veronese}"] & & & \mathbb{P}(S^2U_+) \simeq \Lambda_+ \arrow[d, hook] \\
\mathbb{P}_1^+ \times \mathbb{P}_1^- \arrow[u, "\pi_+"] \arrow[d, "\pi_-"'] \arrow[r, "\text{Segre}", "\sim"'] & G \subset \mathbb{P}\mathrm{Hom}(U_-,U_+) \arrow[rr, dashed, "\text{Plücker}"] & & P \subset \mathbb{P}\begin{pmatrix}\Lambda^2U_-^* \otimes S^2U_+ \\ \oplus \\ S^2U_-^* \otimes \Lambda^2U_+\end{pmatrix} \\
\mathbb{P}(U_-^*) \arrow[rrr, hook, "\text{Veronese}"'] & & & \mathbb{P}(S^2U_-^*) \simeq \Lambda_- \arrow[u, hook]
\end{tikzcd}$$

3◇7. Is there a 2×4 - matrix whose 2×2 - minors are: **a)** $\{2,3,4,5,6,7\}$ **b)** $\{3,4,5,6,7,8\}$ (If no, explain why, if yes, give an explicit example.)

Some (multi) linear algebra

4◇1. Is it true that any rank 1 matrix of size $m \times n$ can be written as a product of some $m \times 1$ and $1 \times n$ matrices?

4◇2. Show that for any two vector spaces U, W there exists a canonical isomorphism

$$U^* \otimes W \xrightarrow{\sim} \mathrm{Hom}(U, W)\,,$$

which sends $\xi \otimes w$ to rank 1 operator $u \longmapsto \xi(u) \cdot w$.

4◇3. Let $\{e_1, e_2, \dots, e_d\} \subset V$ and $\{x_1, x_2, \dots, x_d\} \subset V^*$ be dual bases. Show that the tensor $\sum\limits_{\nu} x_\nu \otimes e_\nu \in V^* \otimes V$ doesn't depend on a choice of the dual bases.

4◇4. Check for any vector space V a series of canonical isomorphisms:

$$\mathrm{Hom}(V, V) \simeq V^* \otimes V \xrightarrow{\tau} (V \otimes V^*)^* \simeq \mathrm{Hom}(V, V)^*$$

where τ takes $\xi \otimes v$ to a linear form that sends $v' \otimes \xi'$ to the full contraction $\xi(v')\xi'(v)$. The resulting correlation $\mathrm{Hom}(V, V) \xrightarrow{\sim} \mathrm{Hom}(V, V)^*$ corresponds to some bilinear form $t(A, B) \overset{\text{def}}{=} \tau A(B)$ on $\mathrm{Hom}(V, V)$. Is this form symmetric? How it looks in terms of matrices? What is the corresponding quadratic form?

4◇5. Let $A = (a_{ij})$ be $n \times n$ - matrix whose entries are considered as independent variables. Fix a collection of m matrix elements $a_{i_\nu j_\nu}$, where $1 \leqslant \alpha \leqslant m$. Calculate the polarization

$$\frac{\partial^m \det A}{\partial a_{i_1 j_1}\, \partial a_{i_2 j_2} \,\cdots\, \partial a_{i_m j_m}}$$

a) for $m = 1$; **b)** for $m = 2$; **c*)** for any m.

Hint. Use Laplace relations: write A_m for $\binom{n}{m} \times \binom{n}{m}$ matrix whose entries are $m \times m$-minors of A and $\widehat{A}_m$ for the matrix of their algebraic complements; then $\det A = \binom{n}{m}^{-1} \mathrm{tr}\left(A_m \cdot {}^t\widehat{A}_m\right)$.

4◇6*. Prove the following Taylor expansion for the polynomial function $\det(A)$ on the space of $n \times n$ - matrices:

$$\det(\lambda A + \mu B) = \sum_{p+q=n} \binom{n}{p}^{-1} \cdot \lambda^p \mu^q \cdot \sum_{\substack{IJ:\\ \#I=\#J=p}} (-1)^{|I|+|J|} a_{IJ} b_{\widehat{I}\widehat{J}} \tag{41}$$

where (a_{IJ}) is $p \times p$-minor of $A = (a_{ij})$ situated in rows $I = (i_1, i_2, \dots, i_p)$ and columns $J = (j_1, j_2, \dots, j_p)$ and $\left(b_{\widehat{I}\widehat{J}}\right)$ is $q \times q$-minor of $B = (b_{ij})$ situated in the complementary rows $\widehat{I} = \{1, 2, \dots, n\} \setminus I$ and the complementary columns $\widehat{J} = \{1, 2, \dots, n\} \setminus J$ (in the notations from the hint to the previous problem, the last right sum in (41) actually equals $\mathrm{tr}\left(A_p \cdot {}^t\widehat{B}_q\right)$).

4◇7. Let $e = \{e_1, e_2, \dots, e_d\} \subset V$ be a base. Denote by $\sigma_\mu(e) \subset \mathrm{Gr}(n, V)$ a Schubert cycle defined as the closure of all U whose matrix with respect to the base e has a strong step form predicted by the Young diagram μ. Show that if two Schubert cycles $\sigma_\lambda(e)$ and $\sigma_\mu(e')$ have complementary non zero codimensions $|\lambda| + |\mu| = n(d - n)$ and two bases e and e' are general enough, then $\sigma_\lambda(e) \cap \sigma_\mu(e') = \varnothing$ for all λ, μ except for the case when the diagrams λ, μ are complementary, i.e. $\mu_{n+1-i} = d - n + 1 - \lambda_i$ (if we put them together appropriately, then we get the $n \times (d-n)$-rectangle). Show also that for complementary diagrams the intersection consists, typically, of one point.

Hint. To start with, take $e'_i = e_{d+1-i}$

Hypersurfaces, tangents, and singularities

5◇1. In $\mathbb{P}_n$, consider 2-dimensional plane π and two codimension 2 subspaces L_1, L_2 such that $p_1 = L_1 \cap \pi$ and $p_2 = L_2 \cap \pi$ are two distinct points on π. Take $a, b, c \in \pi$ such that no 3 of p_1, p_2, a, b, c are collinear. Write $\ell_1 = L_1^\times \subset \mathbb{P}_n^\times$, $\ell_2 = L_2^\times \subset \mathbb{P}_n^\times$ for two pencils of hyperplanes passing through L_1, L_2 respectively, and $\gamma_{abc} \colon \ell_1 \xrightarrow{\sim} \ell_2$ — for the usual projective linear isomorphism defined by a, b, c. Show that its incidence graph $\bigcup\limits_{H \in \ell_1} (H \cap \gamma_{abc}(H)) \subset \mathbb{P}_n$ is a quadric, find the rank and describe the singular points of this quadric in the both cases: $\dim(L_1 \cap L_2)$ equals either $(n-3)$ or $(n-4)$.

5◇2. Let $S \subset \mathbb{P}_3$ be a surface ruled by all lines tangent to the twisted cubic $C_3 \subset \mathbb{P}_3$. Write down an explicit equation for S, find its degree and all singular points.

5◇3. Let $S \subset \mathbb{P}_5 = \mathbb{P}(S^2V^*)$ be the space of singular conics on $\mathbb{P}_2 = \mathbb{P}(V)$. Show that singular points of S correspond to double lines in $\mathbb{P}(V)$ and $\mathit{Sing}(S)$ coincides with an image of the Veronese embedding $\mathbb{P}(V^*) \xhookrightarrow{v_2} \mathbb{P}_5$. For non singular $q \in S$, which corresponds to splitted conic $\ell_1 \cup \ell_2 \subset \mathbb{P}(V)$, prove that the tangent space T_qS, for S at q, consits of all conics passing through $\ell_1 \cap \ell_2$.

5◇4* (generalisation of prb. 5◇3). Let $S \subset \mathbb{P}(S^2V^*)$ be the space of singular quadrics on $\mathbb{P}_n = \mathbb{P}(V)$. We write $q \in S$ for a point of S and $Q_q \in \mathbb{P}(V)$ for the corresponding singular quadric.

a) Show that q is non singular point of S iff Q_q is a simple cone (i.e. has just one vertex point).

b) Show that the tangent space $T_qS \subset \mathbb{P}(S^2V^*)$ at non singular $q \in S$ consists of all quadrics passing through the vertex of Q_q.

5◇5 (plane cubics).

a) How many singular points may have a plane cubic curve and what could be their multiplicities?

b) Classify all reducible cubics up to a projective linear isomorphism.

c) Show that irreducible singular cubics are rational and (up to a projective linear isomorphism) are exhausted by $y^2 = x^3$ (nodal cubic) and $y^2 = x^2(x+1)$ (cuspidal cubic).

HINT. rationality may be proved via projection from a singular point; to get nice coordinate representations put the singular point at the origin and draw the axes in such a way that either $y = 0$ would be the cuspidal tangent line or $y = \pm x$ would be the pair of nodal tangent lines.

d) How many tangent lines come to a smooth cubic curve from a generic point on $\mathbb{P}_2$?

e) How many inflection points are there on a smooth cubic?

f*) Show that any non singular cubic may be presented in appropriate affine coordinates by equation $y^2 = x^3 + px + q$.

HINT. see: *C. H. Clemens. A scrapbook of complex curve theory.* Plenum Press. But try to simplify (or to modify) the arguments by your own geometric and/or multilinear arguments

g*) Show that 3 non-inflection tangents which are drown from an inflection point on a smooth cubic meet this cubic in 3 collinear points.

HINT. look at Clemens'es book (*loc. cit.*) and make his arguments to be solid by adding your own details

5◇6. For the plane curves **a)** $(x_0 + x_1 + x_2)^3 = 27\, x_0 x_1 x_2$ **b)** $(x^2 - y + 1)^2 = y^2(x^2+1)$ find all[1] singular points, compute their multiplicities, look how many branches come to each singularity and what are their geometric tangents.

HINT. to analyze local geometry, blow up the singularity: take affine coordinates (x, y) whose origin is the singularity and substitute $x = \alpha\, t$, $y = \beta\, t$ in the curve equation $f(x, y) = 0$; now $(\alpha : \beta)$ parameterizes the line pencil through the origin and $f(\alpha, \beta;\, t) = 0$, as an equation in t, gives the intersection of line $(\alpha\, \beta)$ with the curve; tangent lines are those where the multiplicity of zero root jumps. (Answer for (b): 3 ordinary cusps at … find where!)

[1] including possible singularities at the infinity in (b)

Polynomial ideals

6◇1. Give an example of proper non-principal ideal in **a)** $\mathbb{C}[x,y]$ **b)** $\mathbb{Z}[x]$.

6◇2. Let a polynomial f vanish along a hypersurface given in $\mathbb{C}^n$ by a polynomial equation $g=0$. Prove that each irreducible factor of g divides f.

6◇3. Prove that any algebraic set in $\mathbb{C}^2$ is a finite union of points and curves (recall that a curve is a zero set of one polynomial).

6◇4. Let $J=(xy,yz,zx)\subset\mathbb{C}[x,y,z]$. Describe $V(J)\subset\mathbb{A}^3$ and $I(V(J))\subset\mathbb{C}[x,y,z]$. Is it possible to define the same variety by 2 polynomial equations?

6◇5. Find $f\in I(V(J))\setminus J$ for $J=(x^2+y^2-1,y-1)\subset\mathbb{C}[x,y]$.

6◇6. Describe $V(J)\subset\mathbb{A}^3$ and $I(V(J))\subset\mathbb{C}[x,y,z]$ for:
a) $J=(xy,(x-y)z)$ **b)** $J=(xy+yz+zx,x^2+y^2+z^2)$

6◇7. Which of the following three facts about ideals in $k[x_1,x_2,\dots,x_n]$ (k is an arbitrary field) are true? (Prove the true ones and give counter-examples for the other.)
a) $\sqrt{IJ}=\sqrt{I\cap J}$
b) $\sqrt{IJ}=\sqrt{I}\sqrt{J}$
c) $(I=\sqrt{I}\ \&\ J=\sqrt{J})\quad\Rightarrow\quad IJ=\sqrt{IJ}$

6◇8. Let $B\supset A$ be an extension of commutative rings such that B is finitely generated as A-module. Prove that $\mathfrak{m}B\neq B$ for any maximal ideal $\mathfrak{m}\subset A$.

6◇9. Which of the following three rings are Noetherian?
a) $\left\{ f(z)=\dfrac{p(z)}{q(z)}\in\mathbb{C}(z)\ \middle|\ q(z)\neq 0 \text{ for } |z|\leqslant 1\right\}$;
b) power series $f(z)\in\mathbb{C}[[z]]$ converging everywhere on $\mathbb{C}$;
c) $\left\{ f(x,y)\in\mathbb{C}[x,y]\ \middle|\ \dfrac{\partial^{i+j}f}{\partial x^i\partial y^j}=0\quad\forall\, 0\leqslant i+j\leqslant n\right\}$, where $n\in\mathbb{N}$ is fixed.

6◇10*. Show that any finitely generated[1] field is finite (as a set).

6◇11*. Show that the ideal generated in $\mathbb{C}[x_0,x_1,x_2,x_3]$ by all homogeneous polynomials that vanish along the twisted cubic in $\mathbb{CP}_3$
a) is generated by 3 quadratic polynomials
b) can't be generated by 2 polynomials

[1] in absolute sense, i. e. as $\mathbb{Z}$-algebra w. r. t. the action $m\cdot a\stackrel{\text{def}}{=}\underbrace{a+a+\ \cdots\ +a}_{m\text{ times}}$

Plane curves

7◇1. Describe all complex plane projective quintics that have singularities of multiplicity 4 at two given distinct points $a, b \in \mathbb{P}_2$.

HINT. such a quintic have to contain (multiple) line (a, b). Answer: 3-dimensional projective space of equations of the form ... find it!

7◇2. Let $C \subset \mathbb{P}_2$ be a curve with affine equation $x^2y + x\,y^2 = x^4 + y^4$.

a) What kind of singularity has C at the origin?

b) Has C any other singular points (for example, at infinity)?

c) Find the local intersection number at the origin of C with a curve having a simple cusp with tangent line $x = y$.

7◇3. Using Plücker relations, compute the class and the number of inflection points for a smooth plane cubic curve.

7◇4. Let $C \subset \mathbb{P}_2$ be degree d curve, $q \notin C$ lie neither on an inflection tangency nor on a geometric tangency through a singular point of C, and $C_q^{(d-1)}$ be the $(d-1)$-th degree polar of q w. r. t. C. Show that $(C, C_q^{(d-1)})_p$ equals: **a)** 1, if p is smooth; **b)** 3, if p is an ordinary cusp; **c)** $m(m-1)$, if p is an ordinary m-typle node.

HINT. In (a), (b) p is smooth on $C_q^{(d-1)}$ as well and $T_pC_q^{(d-1)} \neq T_pC$ in (a) but $T_pC_q^{(d-1)}$ coincides with the cuspidal tangent in (b). In (c) p is an $(m-1)$-typle point on $C_q^{(d-1)}$ but each geometrical tangency of C at p is transversal to $C_q^{(d-1)}$ and hence intersects it with multiplicity $(m-1)$.

7◇5. Given an irreducible curve $C \subset \mathbb{P}_2$ (maybe singular), show that the closure of a set formed in $\mathbb{P}_2^\times$ by the tangent lines through the smooth points of C is an algebraic curve.

7◇6. Using Plücker relations, list all complex plane quartics with the simplest singularities (ordinary double nodes and cusps) w. r. t. how many cusps, nodes, double tangents and inflexion points may they have. Which of them have to be reducible?

7◇7. Show that smooth plane quartic curve either has a tangent line intersecting the curve just ones with multiplicity 4 or has 28 bitangent lines (touching the curve in two distinct points).

7◇8. Find all lines on the singular projective surface with affine equation $xyz = 1$.

HINT. Show that there are no lines in the initial affine chart

Algebraic manifolds

8◇1 (Zariski topology). Let $X = Spec_m A$ be affine algebraic set. Check that the sets

$$V(I) = \{ x \in X \mid f(x) = 0 \quad \forall f \in I \}$$

produced by all ideals $I \subset A$ satisfy the closed sets axioms of the topology.

8◇2. Prove that any open covering of affine algebraic variety contains a finite sub-covering.

8◇3. Give an example if affine algebraic set X and open $U \subset X$ such that $\mathcal{O}_X(U)$ is not finitely generated as k-algebra.

8◇4. Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be affine algebraic sets.
- **a)** Show that $X \times Y$ is affine algebraic subset in $\mathbb{A}^{n+m}$.
- **b)** Give $X \times Y \subset \mathbb{A}^{n+m}$ by explicit equations (assuming that the equations for X, Y are known).
- **c)** Show that $X \times Y$ is irreducible as soon both X, Y are.

8◇5. Prove that the maximal spectrum of a finite dimensional[1] k-algebra is a finite set and deduce from this that any finite morphism has only finite (or empty) fibers.

8◇6. Give an example of regular morphism of affine algebraic sets $X \xrightarrow{\varphi} Y$ such that all fibers of φ are finite (or empty) but φ is not a finite morphism.

8◇7. Prove that a projection of affine hypersurface $V(f) \subset \mathbb{A}^n$ from any point $p \notin V(f)$ onto any hyperplane $H \not\ni p$ is dominant.

8◇8* (Noether's normalization lemma). Show that any affine hypersurface $V(f) \subset \mathbb{A}^n$ admits a finite surjection onto some hyperplane $\mathbb{A}^{n-1} \subset \mathbb{A}^n$.

8◇9. Prove that $\dim(X \times Y) = \dim X + \dim Y$

8◇10. Let $X \xrightarrow{\varphi} Y$ be a regular morphism of algebraic manifolds. Show that isolated[2] points of fibers $\varphi^{-1}(y)$ draw an open subset of X when y runs through Y.
Hint. Use Chevalley's theorem on semi-continuity (lecture 13).

8◇11. Show that an image of a regular dominant morphism contains an open dense subset.

8◇12* (constructivity theorem of Chevalley). Prove that an image of any regular morphism of algebraic varieties is *constructive*, i.e. can be constructed from a finite number of open and closed subsets by a finite number of unions, intersections, and taking complements.

8◇13 (quadratic transformation). Show that the prescription $(t_0 : t_1 : t_2) \longmapsto (t_0^{-1} : t_1^{-1} : t_2^{-1})$ is extended to a rational map $\mathbb{P}_2 \overset{q}{\dashrightarrow} \mathbb{P}_2$ defined everywhere except for 3 points; find these points; clarify how does q act on a triangle (triple of lines) with the vertexes at these 3 points; find $\operatorname{im} q$.

8◇14 (graph of rational map). Let $X \overset{\psi}{\dashrightarrow} Y$ be a rational map defined on open dense $U \subset X$. By the definition, its *graph* $\Gamma_\psi \subset X \times Y$ is the Zariski closure of $\{(x, \psi(x)) \in X \times Y \mid x \in U\}$.
- **a)** Show that a graph of the natural rational map $\mathbb{A}^{n+1} \dashrightarrow \mathbb{P}_n$, which sends $P \in \mathbb{A}^{n+1}$ to $(OP) \in \mathbb{P}_n$, is isomorphic to the blow up of the origin.
- **b*)** Try to describe a graph of the quadratic transformation from prb. 8◇13, in particular, describe the fibers of its projections onto the both, source and target, $\mathbb{P}_2$'s.

[1] as a vector space over k

[2] a point $p \in M$ is called *isolated* point of a subset $M \subset X$ in a topological space X, if it has an open neighborhood $U \ni p$ such that $U \cap M = \{p\}$

27 lines

9◇1 (Schläflische Doppelsechs). The 'double six line configuration' is constructed as follows. Let

$$[0],\,[1],\,\dots,\,[5]\subset\mathbb{P}_3$$

be six lines such that $[1],\dots,[5]$ are mutually skew, $[0]$ intersects all of them, and each of $[1],\dots,[5]$ does not either touch or lay on the quadric drown through any 3 other. Show that: **a)** $\forall\, i=1,\dots,5$ $\exists$ unique line $[i']\neq[0]$ such that $[i']\cap[j]\neq\varnothing\ \forall\, j\neq i$;

b) $[i']\cap[i]=[i']\cap[j']=\varnothing$ for all $i=1,\dots,5$ and for all $j\neq i$;

c) each of $[1'],\dots,[5']$ does not either touch or lay on the quadric drown through any 3 other;

d) there exists a unique line $[0']$ that intersects each of $[1'],\dots,[5']$

Hint. Let $[0'_1]\neq[1]$ and $[0'_2]\neq[2]$ be the lines, which intersect all $[1'],\dots,[5']$ except for $[1']$ and $[2']$ respectively; show that they have the same intersection points p_3,p_4,p_5 with $[3'],[4'],[5']$, which may be recovered geometrically using only the lines $[3]$, $[4]$, $[5]$, $[3']$, $[4']$, $[5']$, and $[0]$.

9◇2. Show that each double six line configuration lies on a smooth cubic surface and explain how to find the other 15 lines laying on it.

9◇3. Can a smooth cubic surface $S\subset\mathbb{P}_3$ have a plane section that splits into a smooth conic and its tangent line?

9◇4 (projecting a smooth cubic). Let $S\subset\mathbb{P}_3$ be a smooth cubic surface, $p\in S$ be outside the lines laying on S, $\pi\not\ni p$ be any plane, and $Q=\{q\in\pi\mid(pq)\text{ touches }S\text{ outside }p\}$ be the apparent contour of S visible from p and projected from p onto π. Show that:

a) each plane section passing through p and any line $\ell\subset S$ contains precisely 2 distinct tangent lines coming from p onto S;

Hint. Look at the (smooth!) residue conic $\big(S\cap(p\ell)\big)\setminus\ell$.

b) $Q\subset\pi$ is a smooth quartic;

Hint. Look at the discriminant of $S|_{(pq)}\setminus\{p\}$.

c) Q has precisely 28 distinct double tangents, which are exhausted by $T_pS\cap\pi$ and projections (from p onto π) of lines laying on S;

Hint. Use the Plücker relation to compute the number of bitangents.

d) deduce from the previous assertions a new proof of the existence of precisely 27 lines on a smooth cubic surface are projections of 27 lines laying on S.

9◇5*. Show that any smooth cubic $S\subset\mathbb{P}_3$ can be given in appropriate coordinate system by equation $\varphi_1\varphi_2\varphi_3+\psi_1\psi_2\psi_3=0$, where all φ_i, ψ_j are linear homogeneous forms.

Hint. Use a line $\ell\subset S$ and 5 planes passing through it and intersecting S in a triple of distinct lines

9◇6. Let $\ell_1,\ell_2\subset S$ be two skew lines on a smooth cubic surface $S\subset\mathbb{P}_3$. Show that the prescription:

$$p\longmapsto(\ell\cap\ell_1,\ell\cap\ell_2)$$

where $p\in S\setminus(\ell_1\cup\ell_2)$ and ℓ is a unique line through p meeting the both lines ℓ_i, can be extended to a regular morphism $S\xrightarrow{\varrho}\mathbb{P}_1\times\mathbb{P}_1=\ell_1\times\ell_2$. Show also that:

a) ϱ contracts 5 lines on S to some points on $\mathbb{P}_1\times\mathbb{P}_1$;

b) ϱ is rational isomorphism[1], i.e. there is a rational map $U\xrightarrow{\varrho^{-1}}S$ defined on some open dense $U\subset\mathbb{P}_1\times\mathbb{P}_1$ such that $\varrho\circ\varrho^{-1}=\mathrm{Id}_U$ and $\varrho^{-1}\circ\varrho=\mathrm{Id}_W$ for some open dense $W\subset S$.

9◇7. Let $p\in S$ be a singular point of a (singular) cubic surface in $\mathbb{P}_3$. Show that there is at least one (but in general 6) lines laying on S and passing through p.

[1] this means, in particular, that S is rational, i.e. admits a rational parameterization

Program for written examination

Projective geometry

PROJECTIVE SPACES. Homogeneous coordinates, affine charts, projective closure of affine manifold, projective duality, projection from a subspace onto a complementary subspace, projective linear isomorphisms, all rational bijections $\mathbb{P}_1 \leftrightsquigarrow \mathbb{P}_1$ are induced by linear automorphisms.

PROJECTIVE QUADRICS. Quadratic forms on V = bilinear symmetric forms on $V \times V$ = corelations $V \longrightarrow V^*$; singular points and tangent space; intersections with lines, linear subspaces laying on a quadric; polar map induced by non singular quadric; examples: Veronese – Segre – Plücker; how many lines intersect given 4 skew lines in 3D space?

PLANE CONICS. Rational parameterization, representation of projective linear transformations of lines by conics and projective linear automorphisms of conics; Pascal's Hexagon and Poncelet's Porism (for $n = 3$), plane drawings by the ruler, space of conics, pencils of conics, how many conics touch 5 given lines in a linearly generic position?

GRASSMANNIANS. Affine charts, cell decomposition, Plücker embedding and Plücker coordinates, linear span of skew symmetric tensor and Plücker relations; working example: $\mathrm{Gr}(2,4)$, Plücker quadric and line configurations in 3D space, line pencils and line nets, how do two line rulings of the Segre quadric look at $\mathrm{Gr}(2,4) \subset \mathbb{P}_5$?

PROJECTIVE HYPERSURFACES. Tangent space and singular points, space of hypersurfaces, resultant of 2 homogeneous binary forms, general resultant systems, projection $\mathbb{P}_n \times \mathbb{A}_n \twoheadrightarrow \mathbb{A}_n$ is closed, lines on surfaces $S \subset \mathbb{P}_3$, 27 lines on a smooth cubic surface, double six line configuration.

Multilinear and commutative algebra

TENSOR PRODUCTS. Tensor products of vector spaces, modules and algebras, Segre and Veronese varieties of decomposable tensors, isomorphism $\mathrm{Hom}(U,V) = U^* \otimes V$, contractions, linear span of a tensor.

SYMMETRIC ALGEBRA. Linear span of the Veronese curve and Arongold's principle, rational curves are projections of Veronese curve $\Rightarrow$ smooth plane cubic is irrational, polarization of polynomials, linear span of symmetric tensor, partial derivatives are the contractions, equations for Veronese varieties.

EXTERIOR ALGEBRA. Linear base change in Grassmanian polynomial, Laplase relations, reduction of Grassmannian quadretic form, contractions of Grassmanian polynomials and linear span of skew symmetric tensor.

HILBERT THEOREMS. Noetherian rings, finitely generated algebras over a field are Noetherian, Nullstellensatz, reduced finitely generated algebras over algebraically closed field k = affine algebraic varieties over k, affine algebraic-geometric dictionary, Zariski topology.

FINITE MORPHISMS. Integer elements and integer ring extensions, Gauss lemma, finitely generated algebra over a field admits division only if it is finite dimensional algebraic extension of this field, finite morphisms of affine algebraic varieties are closed and send proper closed subsets to proper closed subsets, minimal polynomial over a normal ring is integer.

Algebraic manifolds

CONCEPT OF MANIFOLD. Structure sheaf and its sections over the principal open sets, general concept of the manifold, separability, product and fibered product of algebraic manifolds, reduction to the diagonal: graph of morphism, morphisms from projective varieties are closed (in part. global regular functions on projective varieties are exhausted by constants).

NORMALIZATION. Blow up construction, projection of projective variety is finite; affine normalization, normalization in algebraic families.

DIMENSION. Combinatoric definition, dimension of affine space, dimension of the product of two manifolds, dimension of a hypersurface, dimensions of fibers of a regular morphism, estimations for dimensions of intersections.

VECTOR BUNDLES. Locally trivial vector bundles and locally free $\mathcal{O}$-modules; line bundles and Picard group, A is factorial $\Rightarrow$ $Pic(Spec_m A) = 0$, working examples: $Pic(\mathbb{P}_n) = \mathbb{Z} = Pic(\mathrm{Gr}(k,n))$; finitely generated $k[x]$-modules: all vector bundles over $\mathbb{A}_1$ are trivial and over $\mathbb{P}_1$ split as $\oplus\,\mathcal{O}(d_\nu)$.

Elementary Geometry

1◇1. Find a condition on 5 lines in $\mathbb{P}_2$ necessary and sufficient for the existence of a unique non-singular conic touching all these lines.

1◇2. Consider the complex plane quartic[1]

$$(x_0^2 + x_1^2)^2 + 3\,x_0^2 x_1 x_2 + x_1^3 x_2 = 0 \tag{42}$$

a) Find all its singular points over $\mathbb{C}$.
b) Describe their local structure over $\mathbb{C}$ (i. e. geometrical tangents and their intersection multiplicities).
c) Find a rational parameterization for this quartic.

Hint. Use a projection from the singular point onto a line.

d) Show that any irreducible plain quartic with a singularity of multiplicity 3 is rational.

1◇3.
a) Show that all plane quintics[2] that have an ordinary cusp (of multiplicity 2) at $(0:0:1)$ with the cuspidal tangency $t_1 = 0$ form a projective subspace in the space of all plane quintics.
b) Find the dimension of this subspace.
c) Compute the local intersection multiplicity at $(0:0:1)$ between such a quintic and the quartic (42) from prb. 1◇2.

1◇4. ([3]) Is there a 2×4 - matrix whose 2×2 - minors are:
a) $\{2, 3, 4, 5, 6, 7\}$
b) $\{3, 4, 5, 6, 7, 8\}$
(If no, explain why, if yes, give an explicit example.)

Hint. Use some congruence reasons instead of fingering all possible permutations...

1◇5. Prove the Arongold principle: the space of homogeneous degree d polynomials (in several variables) over a field of zero characteristic is linerly spanned by d-th powers of linear forms[4].

1◇6. Write $S \subset \mathbb{P}(S^2 V^*)$ for the hypersurface of singular quadrics in $\mathbb{P}(V)$. Show that a simple quadratic cone[5] with vertex at $p \in \mathbb{P}(V)$ is a non-singular point of S and the tangent hyperplane to S at this point consists of all quadrics passing through p.

1◇7. Let A be a finitely generated k-algebra. Show that if A is finite dimensional as a vector space over k, then $Spec_m\, A$ is a finite set.

[1] i. e. a plain curve of degree 4
[2] i. e. plane curves of degree 5
[3] this problem was the 7-th problem of the 3-th home task; if you already have shown me your solutions, then you do not need to do this again, certainly
[4] this implies that any linear assertion about polynomials, e. g. the Taylor formula, is true as soon it holds for d-th powers of all linear forms
[5] i. e. a quadric which has precisely one singular point (called a vertex)

Advanced Geometry

2◇1. Prove that any hypersurface in $\mathbb{A}_n$ admits a finite surjective morphism onto $\mathbb{A}_{n-1}$.
HINT. Use appropriate projection.

2◇2. Write $\mathbb{P}_N = \mathbb{P}(S^4V^*)$ for the space of quartic hypersurfaces in $\mathbb{P}_3 = \mathbb{P}(V)$ $(\dim V = 4)$. Show that all quartics containing a line form a hypersurface in $\mathbb{P}_N$ (in other words, there is a polynomial Φ in the coefficients of variable quartic form $F(x_0 : x_1 : x_2 : x_3)$ such that $\Phi(F) = 0$ iff the quartic $F = 0$ contains a line).

2◇3. How many triple intersection points[1] have 27 lines on a smooth cubic surface?
HINT. In the local museum of math. things there is a plaster model of a smooth cubic on which all 27 lines are precisely drown. More seriously, this is the most difficult problem in this test; so, don't worry if you will fare from a complete analysis — any progress will be marked highly. What about the Fermat cubic, say?

2◇4. Show that any nowhere vanishing regular section of the trivial rank r vector bundle over $\mathbb{A}_1$ can included in some system of r regular sections that form a base in each fiber[2].

2◇5. Consider the standard covering of the Grassmannian $\mathrm{Gr}(m,n)$, of m-dimensional subspaces in k^n, by affine charts U_I consisting of $W \subset k^n$ which are isomorphically projected onto m-dimensional subspace spanned by i_1-th, i_2-th, ..., i_m-th basic vectors of k^n along all the other $(n-m)$ basic vectors[3]. Let us present a point $W \in U_I$ by $n \times m$ - matrix $M_I(W)$, whose columns are the coordinates of vectors forming a unique base of W such that $k \times k$ - submatrix situated in i_1-th, i_2-th, ..., i_m-th rows is the identity matrix. We consider other $(n-m)\cdot m$ matrix elements (staying outside I-rows) as affine coordinates of W in the chart U_I. Let $S \twoheadrightarrow \mathrm{Gr}(m,n)$ be the tautological vector subbundle of $k^n \times \mathrm{Gr}(m,n)$ whose fiber over a point $W \in \mathrm{Gr}(m,n)$ is the subspace $W \subset k^n$.

a) Construct some trivializing basic sections $s_1^{(I)}, s_2^{(I)}, \dots, s_m^{(I)}$ for S over each U_I and describe corresponding transition matrices $\Phi_{IJ} = \Phi_{IJ}(W)$, which satisfy

$$(s_1^{(I)}, s_2^{(I)}, \dots, s_m^{(I)}) \cdot \Phi_{IJ} = (s_1^{(J)}, s_2^{(J)}, \dots, s_m^{(J)})$$

everywhere in $U_I \cap U_J \subset U_I$.

HINT. Write M_{IJ} for the $m \times m$ - submatrix of M_I situated in J-rows; then M_J is easily expressed through M_I and M_{IJ}.

b) Do the same for the line bundle $D = \Lambda^m S$ and for each its tensor power $D^{\otimes d}$.

c) Prove that any line bundle L over $\mathrm{Gr}(m,n)$ is isomorphic to some $D^{\otimes d}$.

HINT. Write D_{IJ} for determinant $\det M_{IJ}$ of the matrix introduced in the previous hint. The transition function φ_{IJ}, of L, is a rational function in matrix elements of M_I regular and non-vanishing everywhere on U_I except for the zero set of D_{IJ}.

[1] that is, the points where some 3 out of 27 lines are intersecting simultaneously
[2] more honorary (and not obligatory, certainly!) problem is to do the same over $\mathbb{A}_n$
[3] as usually, $I = (i_1, i_2, \dots, i_m)$ runs through all increasing collections of m elements of $\{1, 2, \dots, n\}$