Sheaves.

PRE-SHEAVES. A concept of sheaf is one of the basic ground notions in mathematics just as a concept of set. By the definition, *a pre-sheaf* F of groups (rings, modules, sets, ...) on a topological space X is *a contravariant functor* from a category of open sets (whose morphisms are inclusions) to the category of groups (rings, modules, sets, ...). In other words, F attaches, say a group, $F(U)$, called a group of *local sections*, to each open $U \subset X$ and attaches a group homomorphism $F(U) \xrightarrow{s \mapsto s|_W} F(W)$, called *a restriction* of local sections, to each pair $W \subset U$ in such a way that $\forall\, W \subset U \subset V$ and $\forall\, s \in F(V)$ we have $(s|_U)|_W = s|_W$. By the definition, a morphism of pre-sheaves $E \longrightarrow F$ is a morphism of functors, that is a collection of homomorphisms $E(U) \longrightarrow F(U)$ defined for each U and commuting with the restrictions. Given $x \in X$, *a germ* of a local section of F at x is an equivalence class of pairs (U, s_U), where $U \ni x$ is open, $s_U \in F(U)$, and $(U, s_U) \sim (W, s_W)$, if $s_U|_V = s_W|_V$ for some open V such that $x \in V \subset U \cap W$. The germs of local sections at x form a group (resp. ring, module, set, ...), which is called *a stalk* of F at x and is denoted by F_x.

Exercice 17.1. Show that $F_x = \varinjlim_{U \ni x} F(U)$

Similarly, a stalk F_Z over any closed $Z \subset X$ is defined as $F_Z = \varinjlim_{U \supset Z} F(U)$.

SHEAVES. A pre-sheaf F is called *a sheaf*, if for any family of local sections $s_\nu \in F(U_\nu)$ such that $\forall\, \mu, \nu$ $s_\nu|_{U_\nu \cap U_\mu} = s_\mu|_{U_\nu \cap U_\mu}$ there exists a unique $s \in F(\bigcup_\nu U_\nu)$ with $s|_{U_\nu} = s_\nu$ $\forall \nu$. In particular, for any sheaf F

$$F(U \cup W) = F(U) \times F(W)\,, \text{ if } U \cap W = \varnothing\,, \tag{39}$$

Exercice 17.2. Check that really the structure sheaf of an algebraic variety is a sheaf of rings and the sheaf of local sections of an algebraic vector bundle is a sheaf of modules over the structure sheaf.

For any pre-sheaf F there is a canonical sheaf $\widetilde{F}$ together with a morphism (of pre-sheaves) $F \xrightarrow{\sigma} \widetilde{F}$ such that for any sheaf E and any morphism (of pre-sheaves) $F \xrightarrow{\varphi} E$ there exists a unique morphism $\widetilde{F} \xrightarrow{\widetilde{\varphi}} E$ satisfying $\widetilde{\varphi} \circ \sigma = \varphi$. It is constructed as follows. For each open U consider the direct product of all stalks: $F_U \stackrel{\text{def}}{=} \prod_{x \in U} F_x$ and define $\widetilde{F}(U) \subset F_U$ as a subgroup which consists of all collections of germs $\{s_x\}_{x \in U}$ such that for any $y \in U$ there exist an open neighborhood $W \ni y$ and a local section $t \in F(W)$ whose germ (represented by a pair (W, t)) equals s_x $\forall\, x \in W \cap U$.

Exercice 17.3. Check that this prescription really defines a sheaf, which satisfies the universal property formulated before; moreover this property defines $\widetilde{F}$ up to a unique isomorphism (of pre-sheaves) commuting with σ.

Example 17.4: constant pre-sheaf and constant sheaf. Each, say a group, K trivially gives a pre-sheaf with the same section groups $K(U) = K$ $\forall\, U$ and identical restrictions. This pre-sheaf is not a sheaf, because (39) falls. The associated sheaf $\widetilde{K}$ attaches to each U a group of *locally* constant, that is discrete but continuous, K-valued functions on U, which are constant only over *connected* U. It is called *a constant sheaf* of K.

Exercice 17.5. Compute the real primitive function $\displaystyle\int \frac{dx}{x}$.

HINT. Typical answer: «$\log|x| + C$, where C is a constant» is false: in fact, C is a section of the constant sheaf over $\mathbb{R} \setminus \{0\}$.

Example 17.6: factorization. Consider the unit circle $X = S^1$ and write C^∞, Ω^1 for the sheaves of smooth local functions and smooth local differential 1-forms $f(\vartheta)\, d\vartheta$, where ϑ is natural angle parameter on the circle.

Exercice 17.7. Check that the both really are the sheaf.

The De Rham operator gives a morphism of sheaves $d\,:\, C^\infty \xrightarrow{f \mapsto df} \Omega^1$ whose kernel is a constant subsheaf $\mathbb{R} \subset C^\infty$. Its image $U \mapsto B^1(U) = C^\infty/\mathbb{R} \subset \Omega^1$ is a pre-sheaf of smooth local exact 1-forms. But this pre-sheaf is not a sheaf! Indeed, the form $d\vartheta$ is not globally exact, but gives exact forms over any connected $U \subsetneq S^1$ compatible over their intersections; so such local sections can not be glued together!

In other words, naively defined factor sheaf is not a sheaf but only a pre-sheaf. But it would be very nice, if the sheaves of, say Abelian groups, would form an Abelian category, that is with images, cokernels e. t. c. being the sheaves. To this aim the definition of factor-sheaf is modified in order to force the factors to be the sheafs.

Namely, given two sheaves $F \subset E$ of Abelian groups on any topological space, then their factor sheaf E/F *is defined* as a sheaf associated with the pre-sheaf $U \longmapsto F(U)/E(U)$. Thus, a section of E/F over U is a collection of germs (s_α, W_α) such that $s_\alpha \in F(W_\alpha)$, $W_\alpha \subset U$, and $(s_\alpha - s_\beta)|_{W_\alpha \cap W_\beta} \in E(W_\alpha \cap W_\beta)$.

In particular, a local section of the *factor sheaf* $C^\infty/\mathbb{R}$ on S^1 over open $U \subset S^1$ is represented by a covering $U = \cup W_\alpha$ and a collection of exact forms $df_\alpha \in B(W_\alpha)$ such that $f_\alpha - f_\beta = \text{const}$ over $W_\alpha \cap W_\beta$. Now $\delta\vartheta$ becomes a well defined global section of $C^\infty/\mathbb{R}$.

Example 17.8: direct image (push forward) of a sheaf E on X w. r. t. a continuous map $X \xrightarrow{f} Y$ is a sheaf $f_\bullet E$ on Y whose fiber over an open $U \subset Y$ is $E(f^{-1}(U))$. If f is a closed embedding, then the sheaves $f_\bullet E$ on Y are called *sky scrapers* supported on X.

Example 17.9: inverse image (pull back) of a sheaf F on Y w. r. t. a continuous map $X \xrightarrow{f} Y$ is a sheaf $f^\bullet F$ on X whose section over an open $U \subset X$ is a collection of germs $w_x \in F_{f(x)}$ (parameterized by $x \in U$) such that for any $y \in U$ there exist an open $W \ni f(y)$ in Y and a section $s \in F(W)$ whose germ (represented by (s, W)) coincides with w_x for all x from some open neighborhood $V \subset f^{-1}(W) \cap U$, $V \ni y$. In particular, a constant sheaf on X associated with K is a pull back of K under the contraction of X to one point and a stalk of a sheaf at a point is a pull back of this sheaf under the inclusion of the point into the manifold.

Exercice 17.10. Check that $f^\bullet F$ is really a sheaf on X even for a pre-sheaf F on Y (moreover, a sheaf associated with a given pre-sheaf is nothing but a its pull back under the identity map) and prove that for any sheaves E on X and F on Y there exists a natural isomorphism $\mathrm{Hom}_X(f^\bullet F, E) = \mathrm{Hom}_Y(F, f_\bullet E)$.

Example 17.11: localization. Let X be an algebraic manifold (or geometric scheme) with the structure sheaf $\mathscr{O}_X$ and $Z \xhookrightarrow{f} X$ be a closed embedding. Then $f^\bullet \mathscr{O}_X$ is called *a localization* of regular functions on X near Z. For example, if X is affine, $\mathscr{O}_X(X) = A$, and $Z \subset X$ is an irreducible submanifold given by a prime ideal $\mathfrak{p} \subset A$, then $f^\bullet \mathscr{O}_X(Z)$ is a ring of fractions[1] $A_\mathfrak{p} = \{p/q \mid p, q \in A,\ q \notin \mathfrak{p}\}$ (these fractions are added and multiplied by the usual rules and, by the definition, $p_1/q_1 = p_2/q_2$ iff $q_2p_1 - q_1p_2$ is a zero divisor in A). This is a *local ring*, i. e. it contains a unique proper maximal ideal $\mathfrak{m}_\mathfrak{p} = \{p/q \mid p \in \mathfrak{p},\ q \notin \mathfrak{p}\}$ and all elements of $A_\mathfrak{p} \setminus \mathfrak{m}_\mathfrak{p}$ are invertible. In particular, if $Z = x$ is one point with maximal ideal $\mathfrak{m}_x$, then we get the local ring A_x, which coincides with the stalk $\mathscr{O}_{X\,x}$ and consists of germs of local rational functions well defined at x. Its unique maximal ideal $\mathfrak{m}_x \subset A_x$ consists of all rational functions vanishing at x; so $A_x/\mathfrak{m}_x = k$.

$\mathscr{O}$-MODULES. In the framework of algebraic geometry, the sheaves of $\mathscr{O}_X$-modules are those usually considered on an algebraic manifold X. Sections of such a sheaf over an open $U \subset X$ form a module over $\mathscr{O}_X(U)$. A regular map of algebraic manifolds (or geometric schemes) $X \xrightarrow{f} Y$ involves the homomorphism of structure sheaves[2] $\mathscr{O}_Y \xrightarrow{f^*} f_\bullet \mathscr{O}_X$, or equivalently (see the isomorphism from Exercice 17.10), the homomorphism $f^\bullet \mathscr{O}_Y \xrightarrow{f^*} \mathscr{O}_X$. The both send a local regular function on Y to its composition with f and endow $f_\bullet \mathscr{O}_X$ with the structure of $\mathscr{O}_Y$-algebra and $\mathscr{O}_X$ — with the structure of $f^\bullet \mathscr{O}_Y$-algebra. So, given a sheaf E of $\mathscr{O}_X$-modules on X, then, restricting scalars from $f_\bullet \mathscr{O}_X$ to $\mathscr{O}_Y$, we can consider the $f_\bullet \mathscr{O}_X$-module $f_\bullet E$ as an $\mathscr{O}_Y$-module as well. This sheaf of $\mathscr{O}_Y$-modules on Y is denoted by $f_* E$ and is called a *push forward* of the $\mathscr{O}_X$-module E. Dually, given an $\mathscr{O}_Y$-module F on Y, then we can extend scalars from $f^\bullet \mathscr{O}_Y$ to $\mathscr{O}_X$ in the $f^\bullet \mathscr{O}_Y$-module $f^\bullet F$. The resulting $\mathscr{O}_X$-module $\mathscr{O}_X \underset{f^\bullet \mathscr{O}_Y}{\otimes} f^\bullet F$ on X is called a pull back of the $\mathscr{O}_Y$-module F on Y and is denoted by $f^* F$.

QUASI-COHERENT SHEAVES. A sheaf F of $\mathscr{O}_X$-modules over an affine algebraic manifold X is called *quasi-coherent*, if it is constructed from some $k[X]$-module M_F in the same way as the sheaf of sections of locally trivial vector bundle E was recovered from the module $P_E = \Gamma(X, E)$, of its global sections, in Claim 16.12. More precisely, quasi-coherent sheaf $\widetilde{M}$ associated with $k[X]$-module M takes[3] $\widetilde{M}(U) = \mathscr{O}_X(U) \underset{k[X]}{\otimes} M$ over each open $U \subset X$. In other words, a section of $\widetilde{M}$ over U is a collection of fractions m_x/f_x parameterized by $x \in U$ such that $\forall\, y \in U$ there exist an open neighborhood $V \subset U$, $V \ni y$ and a pair $m \in M$, $f \in \mathscr{O}_X(V)$ satisfying two properties: $f(y) \neq 0$ and $m_x/f_x = m/f \quad \forall\, x \in V \cap \mathscr{D}(f)$. A quasi coherent sheaf $\widetilde{M}$ is called *coherent*, if the module M is finitely generated over $k[X]$. In general case, an $\mathscr{O}_X$-module $\mathscr{M}$ over an arbitrary geometric scheme X is called quasicoherent (resp. coherent), if there exists an affine open covering (resp. a finite affine open covering)$X = \cup X_i$ such that $\mathscr{M}(X_i)$ is a quasicoherent (resp. coherent) $\mathscr{O}_{X_i}$-module for each i. The quasi coherence is preserved by pull-backs and push forwards, moreover, the pull-back of a coherent module is also coherent. In a contrast, the direct image of a coherent module is often non finitely generated. For example, taking a push forward under the contraction of X to a point p, we actually restrict the scalars from $k[X]$ to $k = \mathscr{O}_p$; after that, say $\mathscr{O}_X$, becomes infinite dimensional; an other typical example gives the inclusion $X = \mathbb{A}_1 \setminus \{0\} \xhookrightarrow{i} \mathbb{A}_1$ where $i_* \mathscr{O}_X = \widetilde{k[t, t^{-1}]}$, which is not finitely generated over $\widetilde{k[t]} = \mathscr{O}_{\mathbb{A}_1}$.

[1] also called a *localization* of A at $\mathfrak{p}$

[2] it is uniquely defined by f in the case of manifolds and is a part of definition of a regular map in the case of geometric schemes

[3] the tensor product $* \underset{k[X]}{\otimes} *$ used here is the tensor product of $k[X]$-modules defined precisely as in Exercice 16.1: if K is a k-algebra and M_1, M_2 — two K-modules, then $M_1 \underset{K}{\otimes} M_2$ is the quotient of $M_1 \otimes M_2$ by a subspace spanned by all $(\varkappa m_1) \otimes m_2 - m_1 \otimes (\varkappa m_2)$, where $\varkappa \in K$, $m_i \in M_i$

Example 17.12: sheaves of ideals and closed subschemes. $\mathcal{O}_X$-submodule $\mathcal{I} \subset \mathcal{O}_X$ is called *a sheaf of ideals*. It is coherent, because its sections $\mathcal{I}(U)$ over affine $U \subset X$ form (maybe non reducible) ideal $I \subset k[U]$. The zero set $Z_I \subset U$ is a closed algebraic subset of U equipped with the structure of geometric scheme given by the k-algebra $k[U]/I$. These local closed subschemes are glued together into one closed geometric subscheme Z equipped with the regular map $Z \overset{f}{\hookrightarrow} X$ whose pull back takes $\mathcal{O}_X$-module E to $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$ module $(\mathcal{O}_X/\mathcal{I}) \underset{\mathcal{O}_X}{\otimes} E$. Sines multiplication by $\mathcal{I}$ acts trivially on this module, all its sections over any open $W \subset X \setminus Z$, where $\mathcal{I}(W) = \mathcal{O}_X(W)$, should vanish. So, Z can be also described as the support of the factor sheaf $F = \mathcal{O}_X/\mathcal{I}$, i.e. $Z = \{x \in X \mid F_x \neq 0\}$. By this reason, a pull back map along closed embedding $Z \hookrightarrow X$ is usually called a *restriction* of an $\mathcal{O}_X$-modules onto Z. In particular, if $Z = x$ is one point and $\mathcal{I} = \mathfrak{m}_x$ is its maximal ideal, then the restriction of an $\mathcal{O}_X$-module E onto x, i.e. $(\mathcal{O}_X/\mathfrak{m}_x) \underset{\mathcal{O}_X}{\otimes} E$, is nothing but $E_x/\mathfrak{m}_x \cdot E_x$, where E_x is a stalk of E at x. The ideal sheaf $\mathcal{I}$ of a closed subscheme $Z \subset X$ is locally free iff Z can be given locally by one equation; in particular, an ideal sheaf of any nonempty subscheme of codimension $\geqslant 2$ is not locally free.

Sheaves of algebras and affine morphisms. Given a sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras which is quasi coherent as a sheaf of $\mathcal{O}_X$-modules, then over any affine $U \subset X$ we have a finitely generated $k[U]$-algebra A and two maps: $k[U] \xrightarrow{\alpha^*} A$ and $k[U][x_1, x_2, \dots, x_n] \twoheadrightarrow A$ which give a map $Spec_m\, A \xrightarrow{\alpha} U$ and a closed embedding $Spec_m\, A \hookrightarrow U \times \mathbb{A}_n$. The maps α are glued together into a global map $Y \xrightarrow{\widetilde{\alpha}} X$, where Y is a geometric scheme with a structure sheaf $\mathcal{A}$ locally (over affine $U \subset X$) isomorphic to a closed subscheme in $U \times \mathbb{A}_n$ (in general, n depends on U). Morphisms of the type $Y \longrightarrow X$ just described are called *affine morphisms*. The Example 17.12 can be considered as a particular case of this construction where $\mathcal{A} = \mathcal{O}_X/\mathcal{I}$. An other important particular case is given in the next example.

Example 17.13: vector bundles. Given a coherent $\mathcal{O}_X$-module F, a sheaf of its symmetric algebras[4]

$$\mathcal{S}(F) = \bigoplus_{d \geqslant 0} S^d(F) \tag{40}$$

satisfies the above conditions, i.e. is a quasi coherent $\mathcal{O}_X$-algebra. The corresponding affine morphism is denoted as $\mathbb{V}(F^*) \xrightarrow{\pi} X$ and called a *vector bundle dual to F*. By the construction, the fiber $\pi^{-1}(p)$ over a given $p \in X$ is the maximal spectrum of the k-algebra $(\mathcal{O}_X/\mathfrak{m}_x) \underset{\mathcal{O}_X}{\otimes} \mathcal{S}(F) = \bigoplus_{d \geqslant 0} S^d(F_x/\mathfrak{m}_x \cdot F_x)$ (comp. with the Example 17.12), that is a vector space $(F_x/\mathfrak{m}_x \cdot F_x)^*$ over $k = \mathcal{O}_{X\,x}/(\mathfrak{m}_x \cdot \mathcal{O}_{X\,x})$ (recall that $\mathbb{A}(V) = Spec_m\,(\bigoplus_{d \geqslant 0} S^d V^*)$). In other words, $\mathbb{V}(F^*) \xrightarrow{\pi} X$ is a vector bundle (possibly not locally trivial) and sections of F are naturally identified with functions $\mathbb{V}(F^*) \longrightarrow k$ which are linear along fibers of π.

Exercice 17.14. Show that vector bundle $\mathbb{V}(F^*)$ is locally trivial iff F is locally free $\mathcal{O}_X$-module.
Hint. See Corollary 16.13.

Sheaves of graded algebras, projective morphisms, and cones. Given a sheaf $\mathcal{A} = \bigoplus_{m \geqslant 0} \mathcal{A}_m$ of *graded* $\mathcal{O}_X$-algebras such that: (1) $\mathcal{A}_m = \mathcal{O}_X$; (2) $\mathcal{A}_1$ is coherent $\mathcal{O}_X$-module; (3) $\mathcal{A}$ is generated by $\mathcal{A}_1$ as $\mathcal{O}_X$-algebra, then $\mathcal{A}$ is quasicoherent and over any affine $U \subset X$ we have a finitely generated graded $k[U]$-algebra $A = \oplus A_k = k[U][x_0, x_1, \dots, x_n]/I$, where $x_0, x_1, \dots, x_n \in A_1$ are some generators of $k[U]$-module A_1 (in general, n may depend on U) and $I \subset k[U][x_0, x_1, \dots, x_n]$ is a graded ideal. So, we have a closed subscheme $Y_U \subset U \times \mathbb{P}_n$ given by I and equipped with the closed morphism $Y_U \xrightarrow{p_U} U$ induced by the projection $U \times \mathbb{P}_n \twoheadrightarrow U$. The maps p_U are glued together into a global map $Y \xrightarrow{p} X$. Such a map p is called a *projective morphism* and the geometric scheme Y is denoted $\mathbb{P}\mathrm{roj}\,(\mathcal{A})$ and called a *projective spectrum* of graded $\mathcal{O}_X$-algebra $\mathcal{A}$. Note that epimorphism of $\mathcal{O}_X$-algebras $\mathcal{S}(\mathcal{A}_1) \xrightarrow{\varphi^*} \mathcal{A}$ (induced by the multiplication) gives a closed embedding $Spec_m \mathcal{A} \overset{\varphi}{\hookrightarrow} \mathbb{V}(\mathcal{A}_1^*)$ and $\mathbb{P}\mathrm{roj}\,(\mathcal{A}) \subset \mathbb{P}\mathbb{V}(\mathcal{A}_1^*)$ is nothing but the fiberwise projectivization of $\operatorname{im}\varphi$ (see Example 17.15 below).

Example 17.15: projective bundles. As in Example 17.13, the symmetric algebra (40) of any coherent $\mathcal{O}_X$-module F satisfies the above conditions (1) – (3), that is produces the projective morphism $\mathbb{P}\mathrm{roj}\,(\mathcal{S}(F)) \xrightarrow{p} X$. Certainly, the manifold $\mathbb{P}\mathrm{roj}\,(\mathcal{S}(F))$ is nothing but the projective bundle obtained from the vector bundle $\mathbb{V}(F^*)$ (see Example 17.13) by the fiberwise projectivization. It usually denoted by $\mathbb{P}(F^*)$ or $\mathbb{P}(\mathbb{V}(F^*))$, because the fiber $p^{-1}(x)$ at $x \in X$ is the projective space $\mathbb{P}\left((F_x/\mathfrak{m}_x \cdot F_x)^*\right)$ over $k = \mathcal{O}_{X\,x}/(\mathfrak{m}_x \cdot \mathcal{O}_{X\,x})$ and sections of F can be considered as fiberwise linear homogeneous forms on $\mathbb{P}(F^*)$. Such the forms form a locally free sheaf $\mathcal{O}_{\mathbb{P}(F^*)}$-modules of rank 1; this sheaf (as well as the corresponding line bundle over $\mathbb{P}(\mathbb{V}(F^*))$) is denoted by $\mathcal{O}_{\mathbb{P}(F^*)}(1)$.

[4] the symmetric algebra of an $\mathcal{O}_X$-module is defined precisely in the same way as the symmetric algebra of a vector space (see Claim 4.13 and discussion around it) with only difference: all the tensor products are taken over $\mathcal{O}_X$ rather than over k

In general case, the restriction of the line bundle $\mathscr{O}_{\mathbb{PV}(\mathscr{A}_1^*)}(1)$ from $\mathbb{PV}(\mathscr{A}_1^*)$ onto $\mathbb{P}\mathrm{roj}\,(\mathscr{A})$ is denoted $\mathscr{O}_{\mathbb{P}\mathrm{roj}\,(\mathscr{A})}(1)$ as well.

Non projectivized affine morphism $Spec_m \mathscr{A} \xhookrightarrow{\pi} X$ is called a *dual cone* of the graded $\mathscr{O}_X$-algebra $\mathscr{A}$ and denoted $C(\mathscr{A})$. There is canonical *zero section* $X \xhookrightarrow{\sigma_0} C(\mathscr{A})$ given by the augumentation homomorphism $\mathscr{A} \xrightarrow{\sigma_0^*} \mathscr{O}_X$, which is identical on $\mathscr{A}_0$ and takes to zero all $\mathscr{A}_i$ with $i > 0$.