Vector bundles

FIBERED PRODUCTS. Given two families $Y_1 \xrightarrow{\pi_1} X$, $Y_2 \xrightarrow{\pi_2} X$ of algebraic manifolds over X (comp. with Example 13.10), then their *fibered product* over X is $Y_1 \underset{X}{\times} Y_2 \overset{\text{def}}{=} \{(y_1, y_2) \in Y_1 \times Y_2 \,|\, \varphi_1(y_1) = \varphi_2(y_2)\}$. In fact, it is an algebraic manifold. If $X = \mathit{Spec}_m\, K$, $Y_i = \mathit{Spec}_m\, A_i$, where K, A_1, A_2 are (finitely generated reduced) k-algebras, then the pull-backs $K \xrightarrow{\pi_i^*} A_i$ equip A_i with K-algebra structure and $Y_1 \underset{X}{\times} Y_2 = \mathit{Spec}_m\, A_1 \underset{K}{\otimes} A_2$, where $A_1 \underset{K}{\otimes} A_2$ is *the tensor product* of K-algebras A_i over K, that is the quotient algebra of $A_1 \otimes A_2$ by an ideal spanned by all differences $(\varkappa a_1) \otimes a_2 - a_1 \otimes (\varkappa a_2)$, where $\varkappa \in K$, $a_i \in A_i$.

Exercice 16.1. Write $A_i \xrightarrow{\alpha_i^*} A_1 \underset{K}{\otimes} A_2$ for two K-algebra homomorphisms sending $a_1 \in A_1$ and $a_2 \in A_2$ to $a_1 \otimes 1$ and $1 \otimes a_2$ respectively. Show that for any K-algebra B and any two homomorphisms of K-algebras $A_i \xrightarrow{g_i^*} B$ there exists a unique homomorphism of K-algebras $A_1 \underset{K}{\otimes} A_2 \xrightarrow{g_1^* \otimes g_2^*} B$ such that $g_i^* = (g_1^* \otimes g_2^*) \circ \alpha_i^*$ for both $i = 1, 2$. Show also that this universality determinate the triple $(\alpha_1^*, \alpha_2^*, A_1 \underset{K}{\otimes} A_2)$ uniquely up to unique isomorphism commuting with α_i^*'s.

HINT. This is completely similar to Exercice 12.8.

So, $Y_1 \underset{X}{\times} Y_2 \subset Y_1 \times Y_2$ is a closed submanifold equipped with two projections $Y_1 \underset{X}{\times} Y_2 \xrightarrow{\alpha_i} Y_i$ and satisfying the following important universal property[1]: for any family $Z \xrightarrow{f} X$ and any two morphisms of X-families $Z \xrightarrow{g_i}$ there exists a unique morphism of X-families $Z \xrightarrow{g_1 \times g_2} Y_1 \underset{X}{\times} Y_2$ such that $g_i = \alpha_i \circ (g_1 \times g_2)$ for both $i = 1, 2$. It is very important that k-algebra $A_1 \underset{K}{\otimes} A_2$ can be *non reduced* even if all three algebras in question are reduced (see Example 16.3 below). In this case $Y_1 \underset{X}{\times} Y_2$ is always considered as *geometric scheme* canonically equipped with the structure algebra $k[Y_1] \underset{k[X]}{\otimes} k[Y_2]$ rather than its underlaying manifold $\mathit{Spec}_m \left(k[Y_1] \underset{k[X]}{\otimes} k[Y_2] \right)_{\text{red}}$.

Example 16.2: base change. Any family $Y \xrightarrow{\pi} X$ can be lifted along any morphism[2] $X' \xrightarrow{f} X$ to the family $Y \underset{X}{\times} X' \xrightarrow{f^*(\pi)} X'$ fitting into commutative diagram

$$\begin{tikzcd} Y \underset{X}{\times} X' \arrow[r] \arrow[d, "f^*(\pi)"'] & Y \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

This procedure is called *a base change*. Algebraically, it is known as *extension of scalars*. For example, given $\mathbb{R}$-algebra (or just a vector space) V, then its complexification is nothing but $\mathbb{C} \underset{\mathbb{R}}{\times} V$.

Example 16.3: scheme restrictions and scheme preimages. Given a closed embedding $Z \overset{\varphi}{\hookrightarrow} X$ and an arbitrary family (i. e. a regular map) $Y \xrightarrow{f} X$, then the base change $Y \underset{X}{\times} Z \xrightarrow{f^*(\varphi)} Z$, of f along φ, is called *a scheme restriction* of the family Y onto the closed submanifold Z and the base change $Z \underset{X}{\times} Y \overset{f^*(\varphi)}{\hookrightarrow} Y$, of f along φ, is called *a scheme preimage* of the closed submanifold $Z \subset X$ under the morphism $Y \xrightarrow{f} X$. If X is affine and Z is given by an ideal $I \subset k[X]$, then geometrically $Z \underset{X}{\times} Y \overset{f^*(\varphi)}{\hookrightarrow} Y$ is a closed embedding of $\mathit{Spec}_m\, ()_{\text{red}}$ into Y, which identifies $f^{-1}(Z)$ with the zero set of the ideal $(f^*)^{-1}(I)$. But the structure algebra $(k[X]/I) \underset{k[X]}{\otimes} k[Y]$ is in general non reduced. For example, a scheme preimage of the cuspidal cubic $Z \subset \mathbb{A}_2$ given by equation $y^2 = x^3$ along the map $\mathbb{A}_1 \xrightarrow{t \mapsto (t, t^2)} \mathbb{A}_2$ (whose image is the parabola $y = x^2$) consists of two points $t = 0$ and $t = 1$ but is equipped with non reduced structure algebra $k[t] \underset{k[x,y]}{\otimes} \big(k[x,y]/(y^2 - x^3)\big) = k[t]/(t^4 - t^3)$ (we have used that x and y act on $k[t]$ as t and t^2).

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[1] note that it is stronger than the set theoretical definition of $Y_1 \underset{X}{\times} Y_2$ given before and obtained by the specialization of the universality in question to $Z = \pi_1^{-1}(x) \times \pi_2^{-1}(x)$ for $x \in X$

[2] mathematically, «family» and «morphism» mean the same; we use different words just to outline the different roles of these maps but it is extremely important that these roles are completely symmetric

Algebraic vector bundle over an algebraic manifold X is an algebraic family of vector spaces over X, i.e. a regular map of algebraic manifolds $E \xrightarrow{\pi} X$ whose fiber $\pi^{-1}(x)$ over any $x \in X$ has a structure of a vector space over k and this structure algebraically depends on x, that is three fiberwise operations: picking the zero: $X \xrightarrow{x \mapsto [0]_x} E$; adding vectors: $E \underset{X}{\times} E \xrightarrow{([u]_x,[v]_x)\mapsto[u+v]_x} E$; and multiplying vectors by constants $I \underset{X}{\times} E \xrightarrow{([\lambda]_x,[v]_x)\mapsto[\lambda v]_x} E$ are the regular morphisms of algebraic manifolds commuting with the projections onto X (we have written above $I = X \times \mathbb{A}_1$ for the trivial line bundle, whose fiber is k, and $[v]_x$ for elements of the fiber $\pi^{-1}(x)$). Two vector bundles $E_1 \xrightarrow{\pi_1} X$, $E_2 \xrightarrow{\pi_2} X$ are called *isomorphic*, if there is an isomorphism of algebraic varieties $E_1 \xrightarrow{\varphi} E_2$ such that $\pi_2 \circ \varphi = \pi_1$ and $\forall\, x \in X$ the restriction $\pi_1^{-1}(x) \xrightarrow{\varphi|_{\pi_1^{-1}(x)}} \pi_2^{-1}(x)$ is linear isomorphism of vector spaces. A vector bundle is called *trivial* of rank d, if it is isomorphic to the direct product $X \times \mathbb{A}_d$ with the standard vector space structure on $\mathbb{A}_d = k^{\oplus d}$, which does not depend on $x \in X$.

A regular map $X \xrightarrow{s} E$ is called *a section*, if $\pi \circ s = \mathrm{Id}_X$, that is $s(x) \in \pi^{-1}(x)\ \forall x$. Each vector bundle has canonical *zero section*, which picks the zero at each fiber. A vector bundle $E \xrightarrow{\pi} X$ is trivial of rank d iff there are d regular sections $X \xrightarrow{s_i} E$ such that $\{s_1(x), \dots, s_d(x)\}$ form a base of $\pi^{-1}(x)\quad \forall\, x \in X$. Indeed, the fiberwise coordinate functions on E w.r.t. these basic vectors give the required isomorphism $E \xrightarrow{\sim} X \times \mathbb{A}_d$.

Locally trivial vector bundle of rank d is a vector bundle $E \xrightarrow{\pi} X$ such that $\forall\, x \in X\ \ \exists$ open $U \ni x$ where the restricted bundle $\pi^{-1}(U) \longrightarrow U$ is trivial of rank d, i.e. has d basic sections $(s_1^{(U)}, s_2^{(U)}, \dots, s_d^{(U)}) : U \longrightarrow \pi^{-1}(U)$. If there are two such trivializations $(s_1^{(U)}, s_2^{(U)}, \dots, s_d^{(U)})$ and $(s_1^{(V)}, s_2^{(V)}, \dots, s_d^{(V)})$ defined, respectively, over some open U, V, then over each $x \in U \cap V$ these two basises are expressed through each other as $(s_1^{(U)}, s_2^{(U)}, \dots, s_d^{(U)}) = (s_1^{(V)}, s_2^{(V)}, \dots, s_d^{(V)}) \cdot \varphi_{VU}$, where $\varphi_{VU} = \varphi_{VU}(x)$ is a non degenerate $d \times d$ - matrix whose entries are regular functions on $U \cap V$ (i-th column of φ_{VU} contains the coordinates of $s_i^{(U)}(x)$ w.r.t. the base $\{s_1^{(V)}(x), \dots, s_d^{(V)}(x)\}$). So, we get the regular maps $U \cap V \xrightarrow{\varphi_{VU}} GL_d(k)$ called *transition functions* between two trivializations. They clearly satisfy the conditions $\varphi_{VU}\varphi_{UW} = \varphi_{VW}$, which hold over any triple intersection $U \cap V \cap W$, and $\varphi_{UV} = \varphi_{VU}^{-1}$, certainly. If we change local basis over each open set by some other, say $(\tilde{s}_1^{(U)}, \tilde{s}_2^{(U)}, \dots, \tilde{s}_d^{(U)}) = (s_1^{(V)}, s_2^{(V)}, \dots, s_d^{(V)}) \cdot \psi_U$ where $\psi_U = \psi_U(x)$ is any non degenerate $d \times d$ matrix whose entries are regular functions on the whole of U, then the transition functions also will be changed by $\tilde{\varphi}_{VU} = \psi_V^{-1}\varphi_{VU}\psi_U$

$GL_d(k)$-valued Čhech's 1-cocycle on X associated with an open covering $X = \cup U_\nu$ is a series of regular maps $U_\alpha \cap U_\beta \xrightarrow{\varphi_{\alpha\beta}} GL_d(k)$ defined for any ordered pair of indexes (α, β) and such that $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$ over $U_\alpha \cap U_\beta$ for all α, β and $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$ over $U_\alpha \cap U_\beta \cap U_\gamma$ for all α, β, γ. Restricting $\varphi_{\alpha\beta}$ onto smaller open sets, we can produce from such the cocycle a cocycle associated with any finer[3] covering inscribed in the initial one. Two cocycles are called *equivalent* or *(co)homologous*, if there exist some common refinement $X = \cup U_\nu$ of their initial open coverings and some regular maps $U_\nu \xrightarrow{\psi_\nu} GL_d(k)$ such that the functions $\varphi_{\alpha\beta}$, $\tilde{\varphi}_{\alpha\beta}$, induced by these cocycles on the refinement, satisfy over each $U_\alpha \cap U_\beta$ the equation $\tilde{\varphi}_{\alpha\beta} = \psi_\alpha^{-1}\varphi_{\alpha\beta}\psi_\beta$. An equivalence class of Čhech 1-cocycles is called *a first Čech cohomology*; the set of these cohomologies is denoted by $H^1(X, GL_d(k))$.

Claim 16.4. *Isomorphism classes of locally trivial vector bundles of rank d are in 1–1 correspondence with the first Čech cohomologies* $\varphi_{\alpha\beta} \in H^1(X, GL_d(k))$.

Proof. Given a Čech cocycle $\varphi_{\alpha\beta}$, construct E as a manifold whose affine charts are $U_\alpha \times \mathbb{A}_d$ glued along $(U_\alpha \cap U_\beta) \times \mathbb{A}_d$ by the rule $U_\alpha \times \mathbb{A}_d \ni (x, v) \longmapsto (x, \varphi(\alpha\beta)(x) \cdot v) \in U_\beta \times \mathbb{A}_d$, where $v \in \mathbb{A}_d$ is a column vector. The cocycle conditions imply that these chard form an atlas and linearity of $\varphi(\alpha\beta)(x)$ for each x implies that the vector space structures of fibers are correctly glued together. Conversely, we have seen in the previous section that for a given vector bundle the transition functions between its trivializations form a Čech cocycle, which is changed by a homologous one under a changing of the trivialization or (what is the same) under a a fiberwise linear isomorphism of the bundle. □

Example 16.5: a tautological vector bundle $S \longrightarrow \mathbb{P}(V)$ is a rank 1 vector subbundle $S \subset \mathbb{P}(V) \times V$ whose fiber over $v \in \mathbb{P}(V)$ is the 1-dimensional subspace of V spanned by v. Over any affine chart $U_\alpha = \{v \in \mathbb{P}(V) \mid \alpha(v) \neq 0\}$, where $\alpha \in V^*$, it can be trivialized by the section $s^{(\alpha)}(v) = (v, v/\alpha(v)) \in \mathbb{P}(V) \times V$, which is a well defined regular function $U_\alpha \xrightarrow{s^{(\alpha)}} S \subset \mathbb{P}(V) \times V$. Since $s^{(\alpha)}(v) = s^{(\beta)}(v) \cdot (\beta(v)/\alpha(v))$ over each $v \in U_\alpha \cap U_\beta$, the transition functions between these trivializations are $\varphi_{\beta\alpha}(v) = \beta(v)/\alpha(v)$, which are well defined regular maps $U_\alpha \cap U_\beta \longrightarrow GL_1(k) = k^*$.

Example 16.6: a tautological vector bundle $S \longrightarrow \mathrm{Gr}(m, V)$ over the Grassmannian, whose points are m-dimensional subspaces $W \subset V$, is a rank m vector subbundle $S \subset \mathrm{Gr}(m, V) \times V$ whose fiber over $W \in \mathrm{Gr}(m, V)$ is the m-dimensional

[3] a covering $X = \cup W_\nu$ is called *finer* than a covering $X = \cup U_\nu$, if $\forall \nu\ \exists \mu\ :\ W_\nu \subset U_\nu$)

subspace $W \subset V$ itself. If we fix a base $\{e_1, e_2, \dots, e_n\}$ in V and for each $I = (i_1 < i_2 < \cdots < i_m) \subset (1, 2, \dots, n)$ consider the standard affine chard $U_I \subset \mathrm{Gr}(m, V)$, which consists of all W projected isomorphically onto the linear span of $\{e_{i1}, e_{i2}, \dots, e_{im}\}$, then S is trivialized over U_I by m sections $s_\nu^{(I)}(W) \subset W$ which form the unique base of W such that the coordinates (written in columns) of the basic vectors form $m \times n$ matrix $M_I(W)$ containing the identity $m \times m$ - submatrix in the rows I. Since for any $W \in U_I \cap U_J$ we have $M_I(W) = M_J(W) \cdot \varphi_{JI}(W)$, where $\varphi_{JI}(W)$ is the inverse matrix for the $m \times m$ submatrix of M_J situated in the rows I, the transition functions between two trivializations $s_\nu^{(I)}(W)$ and $s_\nu^{(J)}(W)$ are given by the maps $W \longmapsto \varphi_{JI}(W) \in GL_m(k)$. Clearly, these are regular maps well defined everywhere in $U_I \cap U_J$.

<u>LINEAR CONSTRUCTIONS WITH VECTOR BUNDLES.</u> Given two locally trivial vector bundles E, F of ranks r, s presented by Čech cocycles $\varphi_{\alpha\beta}$, $\psi_{\alpha\beta}$ over the same open covering $X = \cup U_\nu$, one can form their fiberwise direct sum $E \oplus F$ and tensor product $E \otimes F$ which have ranks $r + s$, rs and are presented by the cocycles $\varphi_{\alpha\beta} \oplus \psi_{\alpha\beta}$ and $\varphi_{\alpha\beta} \otimes \psi_{\alpha\beta}$ (direct sum and tensor product of linear operators). Similarly one can make other tensor constructions, say fiberwise exterior or symmetric powers $\Lambda^m E$, $S^m E$ of a given locally trivial vector bundle E.

<u>PULL BACK.</u> Given a regular map $X \xrightarrow{f} Y$, then any vector bundle $E \twoheadrightarrow Y$ induces a vector bundle $f^*(E) bydef X \underset{Y}{\times} E \twoheadrightarrow X$ over X called *a pull back* of E along f. If E is locally trivial and presented by a Čech cocycle $\varphi_{\alpha\beta}$ over some open covering $Y = \cup U_\nu$, then f^*E is presented by $f^*(\varphi_{\alpha\beta}) = \varphi_{\alpha\beta} \circ f$ over the induced open covering $X = \cup f^{-1}(U_\nu)$.

Exercice 16.7. Let $\mathrm{Gr}(m, V) \xhookrightarrow{p} \mathbb{P}(\Lambda^m V)$ be the Plücker embedding. Check that the pull back $p^* S_{\mathbb{P}}$ of the tautological line bundle on $\mathbb{P}(\Lambda^m V)$ is the maximal exterior power $\Lambda^m S_{\mathrm{Gr}}$ of the tautological line bundle on $\mathrm{Gr}(m, V)$.

<u>PICARD GROUP *Pic*(X)</u>, by the definition, consists of all isomorphism classes of locally trivial rank 1 algebraic vector bundles on X. They form an Abelian group w. r. t. the tensor multiplication, i. e. given two line bundles L, K with Čech cocycles $\varphi_{\alpha\beta}$, $\psi_{\alpha\beta}$, which are just $GL_1(k) = k^*$ - valued functions on $U_\alpha \cap U_\beta$, then their sum in *Pic*(X) equals to the line bundle $E \otimes K$ with the Čech cocycle $\varphi_{\alpha\beta} \cdot \psi_{\alpha\beta}$. The zero element of *Pic*(X) is the trivial line bundle $I = X \times \mathbb{A}_1$. The opposite element for L presented by a Čech cocycle φ_{ij} is the dual bundle $L^* = \mathrm{Hom}(L, I)$ whose Čech cocycle equals $\varphi_{ij}^* = 1/\varphi_{ij}$.

<u>Claim 16.8</u>. *If X is affine and $k[X]$ is factorial, then Pic$(X) = 0$.*

Proof. A trivializing covering $X = \cup U_\alpha$ for a given line bundle L can be always chosen to be such that $U_\alpha = \mathscr{D}(f_\alpha)$ for some finite collection $f_1, f_2, \dots, f_n \in k[X]$. Fix some trivializing sections s_α over each U_α and consider corresponding transition functions $\varphi_{\beta\alpha} = s_\beta / s_\alpha$, which are nowhere vanishing elements of $\mathscr{O}_X(U_\alpha \cap U_\beta) = k[X][1/(f_\alpha f_\beta)]$, that is of the form $f_\alpha^r f_\beta^s$ for some $r, s \in \mathbb{Z}$. Let an irreducible element $q \in k[X]$ appear in $\varphi_{\beta\alpha}$ with a power $m_{\beta\alpha} \in \mathbb{Z}$. If one of these powers is not zero, we can split all f_α's into two nonempty subsets: φ_β's, which are divisible by q, and φ_γ's, which are not. Then for each β the power $m_{\gamma\beta}$ does not depend on γ, because q must disappear in $\varphi_{\gamma_1\gamma_2} = \varphi_{\gamma_1\beta}/\varphi_{\gamma_2\beta}$. Let us write m_β for this power and change all sections s_β by $s'_\beta = q^{m_\beta} \cdot s_\beta$ (this leads to a new basic section, because $Z_q \subset Z_{f_\beta}$). After that q, clearly, disappears in all $\varphi_{\gamma\beta}$ as well as in all $\varphi_{\beta_1\beta_2} = \varphi_{\gamma\beta_2}/\varphi_{\gamma\beta_1}$. Since the set of all q's having non zero $m_{\alpha\beta}$ is finite (it is exhausted by irreducible divisors of f_α's), after a finite series of such the replacements we come to transition functions that have no irreducible factors, i. e. are the constants. □

<u>Corollary 16.9</u>. *Pic*$(\mathbb{A}_n) = 0$. □

<u>Claim 16.10</u>. *Pic$(\mathbb{P}_n) = \mathbb{Z}$ is spanned by the tautological vector bundle S.*

Proof. By Corollary 16.9, any line bundle L can be trivialized over the standard affine chart U_{x_i} by some local nowhere vanishing section s_i. Let us write $t_\nu^{(i)}$, $\nu \neq i$, for the restrictions of linear forms x_ν onto affine hyperplane $x_i = 0$ in $\mathbb{A}_{n+1}$ and use them as affine coordinates on U_{x_i}. The transition function $\varphi_{ij} = s_i/s_j \in k(U_{x_i})$ is a rational function of $t_\nu^{(i)}$ such that its numerator and denominator do not vanish anywhere in U_{x_i} except for $Z_{t_j^{(i)}}$. Hence, $\varphi_{ij} = (t_j^{(i)})^{d_{ij}}$. Since $t_k^{(j)} = x_k/x_j = (x_k/x_i) : (x_j/x_i) = t_k^{(i)}/t_j^{(i)}$, the cocycle conditions $\varphi_{ij} = 1/\varphi_{ji}$ and $\varphi_{jk} = \varphi_{ik}/\varphi_{ij}$ force $d_{ij} = -d_{ji} = d$ with the same d for all i, j. On the other side, for any $d \in \mathbb{Z}$ the functions $\varphi_{ij} = (t_j^{(i)})^d = (x_j/x_i)^d$ form Čech 1-cocycle, i. e. define a line bundle, which we will denote by $\mathscr{O}(-d)$.

Exercice 16.11. Check that $\mathscr{O}(-d) = S^{\otimes d}$.

So, it remains to show that all $\mathscr{O}(d)$ are pairwise non isomorphic. To this aim let us compute their spaces of all regular (possibly, vanishing at some points) global sections. A local section defined everywhere on U_{x_0} has a form $s = f(t^{(0)}{}_1, t^{(0)}{}_2, \dots, t^{(0)}{}_n) \cdot s_0$, where f is any polynomial of n variables. Rewriting s in terms of chart U_{x_i} we get

$$s = f\left(\left(t_1^{(i)}/t_0^{(i)}\right), \dots, \left(t_n^{(i)}/t_0^{(i)}\right) \right) \cdot \left(t_0^{(i)}\right)^d \cdot s_i \,.$$

So, s is extended onto U_{x_i} iff $\deg f \leqslant d$. Hence, for $d < 0$ there are no global sections and for $d \geqslant 0$ there is $\binom{n+d}{d}$-dimensional space of global sections. In particular, all positive $\mathscr{O}(d)$ are mutually different and non isomorphic to negative. Since $\mathscr{O}(-d) = \mathscr{O}(d)^*$, negative $\mathscr{O}(d)$ bundles are pairwise different as well. □

Sheaves of sections. Let $E \xrightarrow{\pi} X$ be a locally trivial vector bundle. Then for any open $U \subset X$ all regular local sections $U \xrightarrow{s} \pi^{-1}(U) \subset E$ form a module over the algebra $\mathscr{O}_X(U)$ of local regular functions on U. This module is denoted by $\Gamma(U, E)$ or $E(U)$. The correspondence $U \longmapsto \Gamma(U, E)$ is called *a sheaf of local sections* of the vector bundle E. Regardless of an evident ambiguity, it is usually denoted by the same letter E. Since the bundle E is locally trivial, the sheaf E is *locally free*, i.e. each point $x \in X$ has an open neighborhood $U \ni X$ such that $\Gamma(U, E)$ is finitely generated free $\mathscr{O}_X(U)$ - module of rank rk E: any collection of trivializing sections for E over U gives a free base for $\Gamma(U, E)$.

Claim 16.12. *Let E be a vector bundle over an affine irreducible variety X and $P_E = \Gamma(E, X)$. Then P_E is finitely generated torsion free*[4] *$k[X]$-module and for any $g \in k[X]$ the module of local sections $\Gamma(\mathscr{D}(g), E)$ coincides with $k[X][g^{-1}] \underset{k[X]}{\otimes} P_E$ (which is the module of fractions*[5] *s/g^m, where $s \in P_E$, $m \in \mathbb{Z}$).*

Proof. Let E be trivialized over some principal open covering $X = \cup \mathscr{D}(f_\nu)$ by local sections $s^{(\nu)}{}_1, s^{(\nu)}{}_2, \dots, s^{(\nu)}{}_r \in \Gamma(\mathscr{D}(f_\nu), E)$. Then the restriction of any section $s \in \Gamma(\mathscr{D}(g), E)$ onto $\mathscr{D}(g) \cap \mathscr{D}(f_\nu) = \mathscr{D}(gf_\nu)$ can be written as

$$s|_{\mathscr{D}(gf_\nu)} = \sum_{i=1}^{r} \frac{h_i}{(gf_\nu)^{m_\nu}} \cdot s_i^{(\nu)}|_{\mathscr{D}(gf_\nu)}$$

So, $\widetilde{s} = g^{\max m_\nu} \cdot s$ is extended onto each $\mathscr{D}(f_\nu)$, that is to a global section of E, and $s = \widetilde{s}/g^m$ as required in the last assertion. To prove the first assertion, write $s_i^{(\nu)}$ as $s_i^{(\nu)} = \widetilde{s}_i^{(\nu)}/f_\nu^{m_{i\nu}}$, where $\widetilde{s}_i^{(\nu)} \in P_E$ are global sections. Then $\widetilde{s}_i^{(\nu)}$ generate P_E over $k[X]$. Indeed, for any $s \in P_E$ and any ν we can write: $s|_{\mathscr{D}(f_\nu} = \dfrac{1}{f_\nu^m} \sum\limits_i g_i^{(\nu)} \cdot \widetilde{s}_i^{(\nu)}$ for some $g_i^{(\nu)} \in k[X]$ and $m \in \mathbb{N}$. Hence, $f_\nu^m \cdot s = \sum\limits_i g_i^{(\nu)} \cdot \widetilde{s}_i^{(\nu)}$ is a $k[X]$-linear combination of $\widetilde{s}_i^{(\nu)}$'s. On the other hand, we can write $1 = \sum\limits_\nu h_\nu f_\nu^m$, because f_ν's have no common zeros. So, $s = \sum_\nu h_\nu f_\nu^m \cdot s = \sum_{\nu i} h_\nu g_i^{(\nu)} \cdot \widetilde{s}_i^{(\nu)}$. Absence of torsion is evident. □

Corollary 16.13. *Under the previous claim conditions, E is trivial iff P_E is free.*

Proof. If $s_1, s_2, \dots, s_r$ form the base of P_E, then, by the claim, their restrictions onto each $\mathscr{D}(f)$ form the base of $\Gamma(\mathscr{D}(f), E)$ over $\mathscr{O}_X(\mathscr{D}(f))$. In particular, r coincides with the number of local trivializing sections, i.e. with the rank of E, and $s_1, s_2, \dots, s_r$ form a base for each fiber, because if some fiber contains a vector laying outside their linear span, then a local section drawn through this vector can not be expressed as $\mathscr{O}_X$-linear combination of s_i's. □

Corollary 16.14. *Each (algebraic, locally trivial) vector bundle over $\mathbb{A}_1$ is trivial.*

Proof. Indeed, P_E is free, because any finitely generated torsion free $k[t]$-module is free[6] □

Exercice 16.15. Show that any nowhere vanishing regular section of a vector bundle over $\mathbb{A}_1$ can included in some system regular sections forming a base in each fiber.

Theorem 16.16. *([7]) Each (algebraic, locally trivial) vector bundle over $\mathbb{P}_1$ splits into a direct sum of line bundles $\mathscr{O}_{\mathbb{P}_1}(d_i)$.*

Proof. Write t for affine coordinate on $\mathbb{A}_1 = \mathbb{P}_1 \setminus \{\infty\}$ and consider two trivializations $e_1^0, e_2^0, \dots, e_r^0$ and $e_1^\infty, e_2^\infty, \dots, e_r^\infty$ of a given vector bundle E defined in $\mathbb{A}_1$ and in $\{\infty\} \cup (\mathbb{A}_1 \setminus \{0\})$. Over $\mathbb{A}_1 \setminus \{0\}$ they are expressed through each other as $(e_1^\infty, e_2^\infty, \dots, e_r^\infty) = (e_1^0, e_2^0, \dots, e_r^0) \cdot \varphi$, where φ is the transition matrix whose entries are rational functions of t without poles in $\mathbb{A}_1 \setminus \{0\}$, that is polynomials in t, t^{-1}. Replacing E by

$$E(m) \overset{\text{def}}{=} E \otimes \mathscr{O}(m) \ ,$$

we multiply all entries of φ by t^m. In particular, we can choose m such that the first column of φ, i.e. e_1^∞ expanded through e_i^0, has no negative powers of t but does not vanish at $t = 0$. This means that e_1^∞ becomes nowhere vanishing global section of E over $\mathbb{P}_1$.

Exercice 16.17. Show that for $m \ll 0$, $E(m)$ will not have the global sections at all.

So, there exists the minimal m such that $e(m)$ admits a nowhere vanishing global section e. Let us now replace E by this $E(m)$. Using induction over the rank, we can suppose that the factor bundle $Q = E/e \cdot \mathscr{O}$ splits as

$$Q = \mathscr{O}(d_2) \oplus \mathscr{O}(d_3) \oplus \ \cdots \ \oplus \mathscr{O}(d_r) \ , \quad \text{where } d_2 \leqslant d_3 \leqslant \ \cdots \ \leqslant d_r$$

[4] A-module M is called *torsion free*, if $am = 0 \Rightarrow a = 0$ or $m = 0$ for $a \in A$, $m \in M$

[5] also called *a localization* of P_E w.r.t. multiplicative set $\{f^k\}$

[6] The same is true for any principal ideal domain. For proof, present the module M in question as F/K, where $F = k[t]^{\oplus n}$ and $K \subset F$ is the kernel of surjection $F \twoheadrightarrow M$ sending basic vectors of F to generators of M. The result follows at once from the *elementary divisors theorem*: each submodule $K \subset F$ is free and there exist some bases $\{e_1, e_2, \dots, e_n\} \subset F$, $\{u_1, u_2, \dots, u_m\} \subset K$ and $f_1, f_2, \dots, f_m \in k[t]$ such that $u_i = f_i \cdot e_i$ for $1 \leqslant i \leqslant m$ (moreover, f_i divides f_j for $i < j$ and the set of those elementary divisors does not depend on a choice of bases). Indeed, theorem forces $F/K = k[t]^{\oplus(n-m)} \oplus T$, where $T = \bigoplus\limits_i (k[t]/(f_i))$ is torsion submodule. The elementary divisors theorem also holds over any principal ideal domain (see Artin's or Van der Warden's » Algebra« textbook; special proof for either $\mathbb{Z}$ or $k[t]$ is extremely fruitful exercise on the Gauss diagonalization and Euclid's division algorithms).

[7] sometimes it is called *Birkhoff - Grothendieck splitting theorem*

By Exercice 16.15, this means that we can take our trivializations $\{e^0_\nu\}$, $\{e^\infty_\nu\}$ with $e^0_1 = e^\infty_1 = e$ and transition matrix

$$(e^\infty_1,\, e^\infty_2,\, \dots,\, e^\infty_r) = \left(e^0_1,\, e^0_2,\, \dots,\, e^0_r\right) \cdot \begin{pmatrix} 1 & f_2 & f_3 & f_4 & \dots & f_r \\ 0 & t^{d_2} & 0 & 0 & \dots & 0 \\ 0 & 0 & t^{d_3} & 0 & \dots & 0 \\ 0 & 0 & 0 & t^{d_4} & \dots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & t^{d_r} \end{pmatrix}$$

where $f_\nu = f_\nu(t, t^{-1})$ are some polynomials in t, t^{-1}. Moreover, adding to all other columns the first column multiplied by appropriate polynomial in t^{-1} (this corresponds to an invertible change of e^∞_ν with $\nu \geqslant 2$), we can make all f_ν to be inside the ideal $(t) \subset k[t]$. This means that all $d_\nu \leqslant 0$. Indeed, if $d_\nu > 0$ for some ν, then e^∞_ν is extended to nowhere vanishing section of $E(-d)$, where $d = GCD(t^{d_\nu}, f_\nu)$ is strictly positive, but this contradicts with our choice of the twist $E \mapsto E(m)$ made before. So. adding to the first row the other rows multiplied by appropriate polynomials in t (this corresponds to an invertible change of e^0_1), we can annihilate all f_ν and get a diagonal transition matrix as required. $\square$