Abstract nonsense appendix

<u>CATEGORIES.</u> Let us use the notion «a category» without giving its formal definition precisely[1]. Informally, *a category* $\mathscr{C}$ consists of *objects*, which form a class[2] denoted by $\mathrm{Ob}\,\mathscr{C}$, and for each pair of objects $X, Y \in \mathrm{Ob}\,\mathscr{C}$ there is a set of *morphisms* $\mathrm{Hom}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y)$, which are distinct for distinct pairs X, Y (usually they are visualized as arrows $X \xrightarrow{\varphi} Y$). This data satisfy the following crucial properties:

- for any ordered triple of objects $X, Y, Z \in \mathrm{Ob}\,\mathscr{C}$ there is *a composition map*
$$\mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \xrightarrow{(\varphi,\psi)\mapsto\varphi\circ\psi} \mathrm{Hom}(X,Z)\ ,$$
which is associative, i.e. $(\chi\circ\varphi)\circ\psi = \chi\circ(\varphi\circ\psi)$ (as soon as they are defined);
- for any $X \in \mathrm{Ob}\,\mathscr{C}$ there is a unique *identity morphism* $\mathrm{Id}_X \in \mathrm{Hom}(X,X)$ that satisfies $\varphi\circ\mathrm{Id}_X = \varphi$ and $\mathrm{Id}_X\circ\psi = \psi$ for any morphisms $X \xrightarrow{\varphi} Y$, $Y \xrightarrow{\psi} X$ and any $Y \in \mathrm{Ob}\,\mathscr{C}$.

For example, the following categories should be well known: topological spaces and continuous maps as the morphisms; finitely generated k-algebras with unity and algebra homomorphisms preserving unity; affine algebraic varieties with regular maps.

More exotic examples: each partially ordered set can be considered as a category where $\mathrm{Hom}(X,Y)$ either contains just one arrow (if $X \leqslant Y$) or is empty (if X and Y are non comparable); each monoid[3] can be considered as a category with just one object X and $\mathrm{Hom}(X,X)$ equal to the monoid itself.

Two objects $X, Y \in \mathrm{Ob}\,\mathscr{C}$ are called *isomorphic*, if there are two arrows[4] $X \underset{\psi}{\overset{\varphi}{\rightleftarrows}} Y$ such that $\varphi\circ\psi = \mathrm{Id}_Y$, $\psi\circ\varphi = \mathrm{Id}_X$.

Given a category $\mathscr{C}$, one can always construct *an opposite category* $\mathscr{C}^{\circ}$ with the same objects: $\mathrm{Ob}\,\mathscr{C}^{\circ} = \mathrm{Ob}\,\mathscr{C}$ and inverted arrows: $\mathrm{Hom}_{\mathscr{C}^{\circ}}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(Y,X)$. The duality $\mathscr{C} \leftrightarrow \mathscr{C}^{\circ}$ is called *reversing of arrows*.

We have seen that the category of finitely generated k-algebras looks like the opposite category to the category of affine algebraic varieties over the same field k. To make this statement more precise we need a tool «for comparing» the categories.

<u>FUNCTORS</u> are «homomorphisms of categories». More precisely, *a covariant functor* $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is a map $\mathrm{Ob}\,\mathscr{C} \xrightarrow{X\mapsto F(X)} \mathrm{Ob}\,\mathscr{D}$ together with a collection of maps $\mathrm{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{\varphi\mapsto F(\varphi)} \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$ defined for each pair $X, Y \in \mathrm{Ob}\,\mathscr{C}$ and preserving the compositions, i.e. satisfying $F(\varphi\circ\psi) = F(\varphi)\circ F(\psi)$ as soon as $\varphi\circ\psi$ is defined (in particular, always $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$).

Dually, *a contravariant functor* $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is a covariant functor $\mathscr{C}^{\circ} \xrightarrow{F} \mathscr{D}$. In other words, it is an «antihomomorphism of categories», that is takes $\mathrm{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{\varphi\mapsto F(\varphi)} \mathrm{Hom}_{\mathscr{D}}(F(Y),F(X))$ for each pair $X, Y \in \mathrm{Ob}\,\mathscr{C}$ and satisfies $F(\varphi\circ\psi) = F(\psi)\circ F(\varphi)$.

For example, the dualization, which takes each vector space V over k to its dual V^* and each linear map $V \xrightarrow{\varphi} W$ to the dual map $W^* \xrightarrow{\varphi^*} V^*$, is a contravariant functor from the category of vector spaces and linear maps to itself. The double dualization gives then an example of a covariant functor.

For any $\mathscr{C}$ we always have *the identity functor* $\mathscr{C} \xrightarrow{\mathrm{Id}_{\mathscr{C}}} \mathscr{C}$, which acts identically on the objects and the arrows. An other trivial series of examples is given by *forgetful functors* acting from categories of sets equipped with an extra structure[5] and morphisms, which are maps of sets preserving this extra structure, to the category $\mathscr{S}et$, of all sets (without any structure). Such a functor also acts identically on objects and arrows — it just forgets the extra structure. Less trivial is

[1] like the notion «a set» is utilized usually during studying the calculus — and our reasons are quite similar: a deep settling down into logical casuistry is inevitable when speaking, say, about «the category of all sets»...

[2] say, class of sets, when category of sets is considered

[3] that is semigroup with unity

[4] called *inverse isomorphisms*

[5] it may be geometric: say, a topology, a differentiable manifold structure e. t. c. ; or algebraic: say, a structure of a group, ring, e. t. c.

Example 15.1: Hom-functors. Each $X \in \mathrm{Ob}\mathscr{C}$ produces two functors from $\mathscr{C}$ to category of sets. A covariant functor $\mathscr{C} \xrightarrow{h^X} \mathscr{S}et$ takes an object Y to $h^X(Y) \overset{\text{def}}{=} \mathrm{Hom}(X,Y)$ and an arrow $Y_1 \xrightarrow{\varphi} Y_2$ to the composition map $h^X(\varphi)$: $h^X(Y_1) = \mathrm{Hom}(X,Y_1) \xrightarrow{\psi\mapsto\varphi\circ\psi} \mathrm{Hom}(X,Y_2) = h^X(Y_2)$. A contravariant functor $\mathscr{C} \xrightarrow{h_X} \mathscr{S}et$ takes an object Y to $h_X(Y) \overset{\text{def}}{=} \mathrm{Hom}(Y,X)$ and an arrow $Y_1 \xrightarrow{\varphi} Y_2$ to the composition map

$$h_X(\varphi) \,:\, h_X(Y_2) = \mathrm{Hom}(Y_2,X) \xrightarrow{\psi\mapsto\psi\circ\varphi} \mathrm{Hom}(Y_1,X) = h_X(Y_1)\,.$$

Exercice 15.2. Show that in the category $\mathscr{M}od(K)$, of modules over commutative ring K with K-linear morphisms, the functor h^X takes any exact triple of modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ to an exact triple

$$0 \longrightarrow \mathrm{Hom}(X,A) \longrightarrow \mathrm{Hom}(X,B) \longrightarrow \mathrm{Hom}(X,C)$$

whose rightmost arrow is non surjective in general. Formulate and prove the similar property of the contravariant functor h_X.

Natural transformations. Given two (covariant) functors $F,G \,:\, \mathscr{C} \longrightarrow \mathscr{D}$, *a morphism of functors*[6] $F \overset{f}{\longmapsto} G$ is a collection of arrows $F(X) \xrightarrow{f_X} G(X) \in \mathrm{Hom}_{\mathscr{D}}(F(X),G(X))$ (parameterized by $X \in \mathrm{Ob}\mathscr{C}$) such that for any morphism $X \xrightarrow{\varphi} Y$ in $\mathscr{C}$ we have the following commutative square[7] of morphisms in $\mathscr{D}$:

$$\begin{tikzcd} F(X) \arrow[r, "F(\varphi)"] \arrow[d, "f_X"'] & F(Y) \arrow[d, "f_Y"] \\ G(Y) \arrow[r, "G(\varphi)"] & G(Y) \end{tikzcd} \tag{36}$$

For example, the canonical embeddings $V \overset{i_V}{\hookrightarrow} V^{**}$ of vector spaces into their double dual[8] present a natural transformation from the identical functor on the category of vector spaces to the double dualization functor.

Clearly, two natural transformations can be composed and the identity maps give an identity transformation from any functor to itself. So we have

Claim 15.3. *For any two categories $\mathscr{C},\mathscr{D}$ all covariant functors $\mathscr{C} \xrightarrow{F} \mathscr{D}$ form a category $\mathscr{F}un\,(\mathscr{C},\mathscr{D})$ whose morphisms are natural transformations of functors.* □

Equivalence of categories. A functor $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is called *an equivalence* of categories, if there is a functor $\mathscr{D} \xrightarrow{G} \mathscr{C}$ (called *quasi-inverse* to F) such that the composition GF is isomorphic to $\mathrm{Id}_{\mathscr{C}}$ in category $\mathscr{F}un\,(\mathscr{C},\mathscr{C})$ and the composition FG is isomorphic to $\mathrm{Id}_{\mathscr{D}}$ in category $\mathscr{F}un\,(\mathscr{D},\mathscr{D})$. Note that our requirement «be isomorphic» to the identical functor is much weaker than an other possible request «coincide with» the identical functor.

For example, consider the category k^n that has a unique object — n-dimensional coordinate vector space over k and all linear maps $k^n \longrightarrow k^n$ as arrows. There is a natural functor $k^n \xrightarrow{I} \mathscr{V}ec_n(k)$ sending this category to the category of *all* n-dimensional vector space over k. This functor is an equivalence of categories: to construct (some) quasi-inverse $\mathscr{V}ec_n(k) \xrightarrow{G} k^n$ just fix a base in each vector space V (this identifies V with the standard coordinate vector space) and present all linear maps by their matrices (i. e. by linear transformations of the coordinate vector space). This example has the straightforward generalization:

Claim 15.4. *A functor $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is an equivalence of categories iff it is full*[9]*, faithful*[10]*, and each $Y \in \mathrm{Ob}\mathscr{D}$ is isomorphic to $F(X)$ for some $X \in \mathrm{Ob}\mathscr{D}$ (depending on Y).*

Proof. For any $Y \in \mathrm{Ob}\mathscr{D}$ fix some isomorphism $i_Y : Y \xrightarrow{\sim} F(X)$ existing by our assertion and write $G(Y)$ for such X (it depends on Y). For any arrow $Y_1 \xrightarrow{\varphi} Y_2$ define $G(Y_1) \xrightarrow{G(\varphi)} G(Y_2)$ as an arrow that corresponds to the arrow $i_{Y_2}\circ\varphi\circ i_{Y_1}^{-1} \,:\, F(G(Y_1)) \longrightarrow F(G(Y_2))$ under the assumed bijection[11]

$$\mathrm{Hom}_{\mathscr{C}}(G(Y_1),G(Y_2)) \xrightarrow{F} \mathrm{Hom}_{\mathscr{D}}(FG(Y_1),FG(Y_2))\,.$$

[6] also called *a natural transformation* of functors

[7] a diagram of morphisms in a category is called *commutative* if the compositions of arrows taken along different passes joining any two vertexes coincide

[8] sending a vector $v \in V$ to the corresponding evaluation functional $V^* \xrightarrow{\mathrm{ev}_v} k$

[9] F is called *full*, if all maps $\mathrm{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{\varphi\mapsto F(\varphi)} \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$ are surjective

[10] F is called *faithful*, if all maps $\mathrm{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{\varphi\mapsto F(\varphi)} \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$ are injective

[11] «full and faithful» in our assumptions means precisely this bijectivity

Now it remains to make an easy exercise below. $\square$

Exercice 15.5. Verify that $\mathscr{D} \xrightarrow{G} \mathscr{C}$ just constructed: a) is a functor; b) is quasi-inverse to F.

<u>Representable functors.</u> A contravariant functor $\mathscr{C} \xrightarrow{F} \mathscr{S}et$ (or a covariant functor $\mathscr{C}^{\circ} \xrightarrow{F} \mathscr{S}et$), is called *representable*, if it is isomorphic in the category $\mathscr{F}un\,(\mathscr{C}^{\circ}, \mathscr{S}et)$ to the functor $h_X : Y \longmapsto \mathrm{Hom}(Y, X)$ associated with some $X \in \mathrm{Ob}\,\mathscr{C}$ as it was described in Example 15.1. Such an object X is called *a representing object* for F. In fact, it is rather «the object» than «an object», because for any two isomorphisms $h_{X_1} \xleftarrow{f_1} F \xrightarrow{f_2} h_{X_2}$ in the category $\mathscr{F}un\,(\mathscr{C}^{\circ}, \mathscr{S}et)$ there exists *a unique isomorphism* $X_1 \xrightarrow{\sigma} X_2$ in the original category $\mathscr{C}$ such that for any $Y \in \mathrm{Ob}\,\mathscr{C}$ the morphism $h_{X_1}(Y) \xrightarrow{(f_2 f_1^{-1})_Y} h_{X_2}(Y)$ (prescribed by the natural transformation $f_2 f_1^{-1}$) coincides with the composition map $\mathrm{Hom}(Y, X_1) \xrightarrow{\psi \mapsto \sigma\circ\psi} \mathrm{Hom}(Y, X_2)$. This follows from slightly more abstract

<u>Claim 15.6</u>. *A full faithful covariant functor $\varrho : \mathscr{C} \longrightarrow \mathscr{F}un\,(\mathscr{C}^{\circ}, \mathscr{D})$ is defined by the prescriptions: for any object X $\varrho(X) = h_X$; for any arrow $X_1 \xrightarrow{\varphi} X_2$ $\varrho(\varphi)$ is a natural transformation $h_{X_1} \longmapsto h_{X_2}$ whose action $\varrho(\varphi)_Y$ at $Y \in \mathrm{Ob}\,\mathscr{C}$ is given by the map $h_{X_1}(Y) = \mathrm{Hom}(Y, X_1) \xrightarrow{\psi \mapsto \varphi\circ\psi} \mathrm{Hom}(Y, X_2) = h_{X_2}(Y)$.*

Proof. Write shortly $\mathscr{C}^{\times}$ for the category $\mathscr{F}un\,(\mathscr{C}^{\circ}, \mathscr{S}et)$. The following two steps are almost obvious:

Exercice 15.7. Check that $\varrho(\varphi)$ defined above really gives a natural transformation of functors $h_{X_1} \longmapsto h_{X_2}$ for any arrow $X_1 \xrightarrow{\varphi} X_2$.

Exercice 15.8. Check that $\varrho(\varphi_1\circ\varphi_2) = \varrho(\varphi_1)\circ\varrho(\varphi_2)$, i. e. we have really constructed a functor $\mathscr{C} \xrightarrow{\varrho} \mathscr{C}^{\times}$.

It remains to check that the map $\mathrm{Hom}_{\mathscr{C}}(X_1, X_2) \xrightarrow{\varrho} \mathrm{Hom}_{\mathscr{C}^{\times}}(h_{X_1}, h_{X_2})$ is bijective for all $X_1, X_2 \in \mathrm{Ob}\,\mathscr{C}$. To construct the inverse map $\mathrm{Hom}_{\mathscr{C}^{\times}}(h_{X_1}, h_{X_2}) \xrightarrow{\lambda} \mathrm{Hom}_{\mathscr{C}}(X_1, X_2)$, consider an arbitrary natural transformation $f : h_{X_1} \mapsto h_{X_2}$, which is an element of $\mathrm{Hom}_{\mathscr{C}^{\times}}(h_{X_1}, h_{X_2})$. Its action at X_1 gives a map

$$h_{X_1}(X_1) = \mathrm{Hom}_{\mathscr{C}}(X_1, X_1) \xrightarrow{f_{X_1}} \mathrm{Hom}_{\mathscr{C}}(X_1, X_2) = h_{X_2}(X_1)\,.$$

We put $\lambda(f) \stackrel{\text{def}}{=} f_{X_1}(\mathrm{Id}_{X_1}) \in \mathrm{Hom}_{\mathscr{C}}(X_1, X_2)$, which is an arrow $X_1 \xrightarrow{\lambda(f)} X_2$ in $\mathscr{C}$. Then, clearly, $\lambda(\varrho(\varphi)) = \varphi$ for any morphism $X_1 \xrightarrow{\varphi} X_2$, because of $\varrho(\varphi)_{X_1}(\mathrm{Id}_{X_1}) = \varphi\circ\mathrm{Id}_{X_1} = \varphi$. Vice versa, we have $\varrho(\lambda(f)) = f$ for any $f : h_{X_1} \mapsto h_{X_2}$, because each arrow $Y \xrightarrow{\psi} X_1$ produces a commutative diagram of type (36):

$$\begin{tikzcd}
h_{X_1}(X_1) \arrow[r, "h_{X_1}(\psi)"] \arrow[d, "f_{X_1}"'] & h_{X_1}(Y) \arrow[d, "f_Y"] \\
h_{X_2}(X_1) \arrow[r, "h_{X_2}(\psi)"] & h_{X_2}(Y)
\end{tikzcd}$$

which says that right vertical map f_Y takes any $\psi \in h_{X_1}(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X_1)$ (considered as $h_{X_1}(\psi)$-image of $\mathrm{Id}_{X_1} \in h_{X_1}(X_1)$) to the $h_{X_2}(\psi)$-image of $f_{X_1}(\mathrm{Id}_{X_1}) = \lambda(f)$, i. e. to $\lambda(f)\circ\psi$, and this is precisely the action of $\varrho(\lambda(f))$ on ψ. $\square$

Exercice 15.9. A covariant functor $\mathscr{C} \xrightarrow{F} \mathscr{S}et$ is called *corepresentable*, if it is isomorphic in the category $\mathscr{F}un\,(\mathscr{C}, \mathscr{S}et)$ to the functor $h^X : Y \longmapsto \mathrm{Hom}(X, Y)$ associated with some $X \in \mathrm{Ob}\,\mathscr{C}$ (called *corepresentig object* for F). Show that a full faithful contravariant functor $\varrho^{\circ} : \mathscr{C} \longrightarrow \mathscr{F}un\,(\mathscr{C}, \mathscr{D})$ is defined by the prescriptions: for any object X $\varrho^{\circ}(X) = h^X$; for any arrow $X_1 \xrightarrow{\varphi} X_2$ $\varrho^{\circ}(\varphi)$ is a natural transformation $h^{X_2} \longmapsto h^{X_1}$ whose action $\varrho^{\circ}(\varphi)_Y$ at $Y \in \mathrm{Ob}\,\mathscr{C}$ is given by the map

$$h^{X_2}(Y) = \mathrm{Hom}(X_2, Y) \xrightarrow{\psi \mapsto \psi\circ\varphi} \mathrm{Hom}(X_1, Y) = h^{X_1}(Y)\,.$$

Hint. Just reverse all arrows in the above discussion.

<u>Defining objects by «universal properties».</u> The Claim 15.6 gives a strategy for reproducing several set-theoretical constructions (such as a direct product of sets e. t. c.) in a context of an arbitrary category $\mathscr{C}$. By the definition, a result of some «natural operation» with objects X_i is an object X such that for any Y a set $\mathrm{Hom}(Y, X)$ coincides with the result of the original set-theoretical version of this operation applied to the sets $\mathrm{Hom}(Y, X_i)$. More precisely, X should represent a functor, which takes Y to the result of the set-theoretical operation with $\mathrm{Hom}(Y, X_i)$'s. Such a definition is implicit and does not guarantee an existence of X — the functor in question can be not representable. But if a representing object exists, then it automatically satisfies some «universal properties» and is unique up to unique isomorphism preserving these properties.

<u>Example 15.10</u>: a product $A \times B$, of $A, B \in \mathrm{Ob}\,\mathscr{C}$, is an object representing a functor $Y \mapsto \mathrm{Hom}(Y, A) \times \mathrm{Hom}(Y, B)$ from $\mathscr{C}^{\circ}$ to $\mathscr{S}et$ (as soon as this functor is representable). In more details, for any Y we should have an isomorphism

$\beta_Y : \mathrm{Hom}(Y, A \times B) \xrightarrow{\sim} \mathrm{Hom}(Y, A) \times \mathrm{Hom}(Y, B)$ functorial w.r.t. arrows $Y_1 \longrightarrow Y_2$. Following the ideas from the proof of Claim 15.6, we can put here $Y = A \times B$ and write

$$\beta_{A\times B}(\mathrm{Id}_{A\times B}) \in \mathrm{Hom}(A \times B, A) \times \mathrm{Hom}(A \times B, B)$$

as (π_A, π_B) for appropriate arrows $A \xleftarrow{\pi_A} A \times B \xrightarrow{\pi_B} B$.

Exercice 15.11. Show that the triple $A \xleftarrow{\pi_A} A \times B \xrightarrow{\pi_B} B$ satisfies the following universal property: for any two morphisms $A \xleftarrow{\varphi} Y \xrightarrow{\psi} B$ there exists a unique morphism $Y \xrightarrow{\varphi\times\psi} A \times B$ such that $\varphi = \pi_A \circ (\varphi \times \psi)$ and $\psi = \pi_B \circ (\varphi \times \psi)$.

Exercice 15.12. Show that for any other triple $A \xleftarrow{\pi'_A} C \xrightarrow{\pi'_B} B$ satisfying the above universal property there exists a unique isomorphism $\gamma : C \xrightarrow{\sim} A \times B$ such that $\pi_A \circ \gamma = \pi'_A$ and $\pi_B \circ \gamma = \pi'_B$

Example 15.13: a coproduct $A \otimes B$, of $A, B \in \mathrm{Ob}\,\mathscr{C}$, is an object corepresenting a functor $Y \mapsto \mathrm{Hom}(A, Y) \times \mathrm{Hom}(B, Y)$ from $\mathscr{C}$ to $\mathscr{S}et$. Reversing arrows in the previous example, we can characterize it by the following universal property: there are two morphisms $A \xrightarrow{i_A} A \otimes B \xleftarrow{i_B} B$ such that for any two arrows $A \xrightarrow{\varphi} Y \xleftarrow{\psi} B$ in $\mathscr{C}$ there exists a unique morphism $A \otimes B \xrightarrow{\varphi\otimes\psi} Y$ such that $\varphi = (\varphi \otimes \psi) \circ i_A$ and $\psi = (\varphi \otimes \psi) \circ i_B$.

Exercice 15.14. Show that such the triple $A \xrightarrow{i_A} A \otimes B \xleftarrow{i_B} B$ (if exists) is unique up unique isomorphism commuting with i-arrows.

Exercice 15.15. Show that if one of two coproducts $A \otimes (B \otimes C)$, $(A \otimes B) \otimes C$ exists, then the other one exists as well and is isomorphic to the first. Prove the same statement also for the products.

Limits. The above definitions are just very particular examples of the following more abstract construction. Fix some category $\mathscr{N}$ (called *category of indexes* — one can treat $\mathscr{N}$ as a kind of «ordered set», where order is given by arrows, but what follows will be valid for *an arbitrary* $\mathscr{N}$ as well).

A functor $\mathscr{N} \xrightarrow{X} \mathscr{C}$ is nothing but a family of objects $X_\nu \in \mathrm{Ob}\,\mathscr{C}$ indexed by $i \in \mathrm{Ob}\,\mathscr{N}$ and morphisms $X_\nu \xrightarrow{\varphi_{\nu\mu}} X_\mu$ indexed by all arrows $i \longrightarrow j$ in $\mathscr{N}$. For example, if $\mathscr{N}$ really is a partially ordered set satisfying the extra condition $\forall\, i, j \;\exists\, k : k > i,\; k > j$, then a functor $\mathscr{N} \xrightarrow{X} \mathscr{C}$ is what is usually called *a direct spectrum* or *a direct system of morphisms*[12] in $\mathscr{C}$.

Further, the category $\mathscr{C}$, we are working in, can be tautologically sent into category $\mathscr{F}un\,(\mathscr{N}, \mathscr{C})$ by a functor $X \mapsto \overline{X}$ that attaches a constant family each object $X \in \mathrm{Ob}\,\mathscr{C}$ to the $\overline{X}$ (whose $\overline{X}_\nu \equiv X$, $\varphi_{\nu\mu} \equiv \mathrm{Id}_X$ $\forall\nu$) and each arrow $X \xrightarrow{\psi} Y$ to the corresponding morphism of constant families[13].

Now, given an arbitrary family X_ν, $\varphi_{\nu\mu}$, that is a functor $\mathscr{N} \xrightarrow{X} \mathscr{C}$, its *projective limit* is an object $\varprojlim X_\nu \in \mathscr{C}$ representing a contravariant functor $Y \longmapsto \mathrm{Hom}_{\mathscr{F}un\,(\mathscr{N},\mathscr{C})}(\overline{Y}, X)$ from $\mathscr{C}$ to $\mathscr{S}et$. Dually, *an inductive limit* $\varinjlim X_\nu$ corepresents a covariant functor $Y \longmapsto \mathrm{Hom}_{\mathscr{F}un\,(\mathscr{N},\mathscr{C})}(X, \overline{Y})$. So, by the definition, we have an isomorphism $\mathrm{Hom}_{\mathscr{C}}(Y, \varprojlim X_\nu) = \mathrm{Hom}_{\mathscr{F}un\,(\mathscr{N},\mathscr{C})}(\overline{Y}, X)$ functorial in Y. Applying it to $Y = \varprojlim X_\nu$, we get a natural transformation $\overline{\varprojlim X_\nu} \xrightarrow{\pi} X$ corresponding to $\mathrm{Id}_{\varprojlim X_\nu} \in \mathrm{Hom}_{\mathscr{C}}(\varprojlim X_\nu, \varprojlim X_\nu)$. This is a family of morphisms $\varprojlim X_\nu \xrightarrow{\pi_\nu} X_\nu$ such that $\pi_\mu = \varphi_{\nu\mu}\pi_\nu$ for all arrows $\varphi_{\nu\mu}$ in the family $\{X_\nu\}$. It satisfies the following universal property: for any object $Y \in \mathrm{Ob}\,\mathscr{C}$ equipped with a family of arrows[14] $Y \xrightarrow{\psi_\nu} X_\nu$ such that $\psi_\mu = \varphi_{\nu\mu}\psi_\nu$ there exists a unique morphism $Y \xrightarrow{\alpha} \varprojlim X_\nu$ such that $\psi_\nu = \pi_\nu \circ \alpha$ $\forall\nu$.

Exercice 15.16. Show that projective limit is uniquely characterized by this universal property (up to unique isomorphism commuting with π_ν's).

Exercice 15.17. Show that inductive limit $\varinjlim X_\nu$ is equipped with canonical maps $X_\nu \xrightarrow{i_\nu} \varinjlim X_\nu$ and satisfies the following universal property: given an object $Y \in \mathrm{Ob}\,\mathscr{C}$ with a family of arrows $X_\nu \xrightarrow{\psi_\nu} Y$ such that $\psi_\mu = \psi_\nu\varphi_{\mu\nu}$ (which give a natural transformation $X \overset{\psi}{\longmapsto} \overline{Y}$ in $\mathscr{F}un\,(\mathscr{N}, \mathscr{C})$), then there exists a unique morphism $\varinjlim X_\nu \xrightarrow{\alpha} Y$ such that $\psi_\nu = \alpha \circ \pi_\nu$ $\forall\nu$.

Exercice 15.18. Let $\mathscr{N}$ be an arbitrary partially ordered set (considered as a category). Show that any family of $\mathscr{N}$-indexed sets $\mathscr{N} \xrightarrow{X} \mathscr{S}et$ has $\varinjlim X$.

[12] dually, a functor $\mathscr{N}^{\circ} \xrightarrow{X} \mathscr{C}$ is called in this case *a inverse spectrum* or *a inverse system of morphisms*

[13] clearly $\overline{X} \overset{\psi}{\mapsto} \overline{Y}$ is a natural transformation of functors

[14] that is for any natural transformation $\overline{Y} \overset{\psi}{\longmapsto} X$ in $\mathscr{F}un\,(\mathscr{N}, \mathscr{C})$

Hint. A right queue of X is a sequence of elements $x_\nu \in X_\nu$ indexed by some $S \subset \operatorname{Ob}\mathcal{N}$ such that all $\mu > \nu$ belong to S as soon as $\nu \in S$ and $\varphi_{\mu\nu}(x_\mu) = x_\nu\ \forall\, \mu, \nu \in S$. Two right queues $\{x_\alpha\}$, $\{y_\beta\}$ are called equivalent, if $\forall\, x_\alpha,\ y_\beta\ \exists\, \gamma > \alpha, \beta$: $\varphi_{\alpha\gamma}(x_\alpha) = \varphi_{\beta\gamma}(y_\beta)$. Check that a set of all equivalence classes of right queues satisfies the universal properties defining $\varinjlim X$.

Exercice 15.19. Let $\mathcal{N} = \mathbb{N}$ be the set of all positive integers with the standard order. Find $\varprojlim A_n$ and $\varinjlim A_n$ of Abelian groups $A_n = \mathbb{Z}/p^n\mathbb{Z}$ w.r.t. an inverse system of canonical factorizations $\psi_{nm} : \mathbb{Z}/p^n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z}$ ($\forall\ m < n$) and w.r.t. a direct system of standard inclusions $\varphi_{mn} : \mathbb{Z}/p^m\mathbb{Z} \xhookrightarrow{[1]\mapsto[p^{n-m}]} \mathbb{Z}/p^n\mathbb{Z}$ (again $\forall\ m < n$).

Hint. $\varprojlim A_n = \mathbb{Z}_p$ is the set of all p-adic integers and $\varinjlim A_n \subset \mathbb{Q}/\mathbb{Z}$ consists of (mod $\mathbb{Z}$)-classes of fractions z/p^ℓ whose denominator is a power of p (so called p-rational numbers).

Exercice 15.20. Let $\mathcal{N} = \mathbb{N}$ as above but with the partial ordering prescribed by the divisibility. Find $\varprojlim A_n$ and $\varinjlim A_n$ of $A_n = \mathbb{Z}/n\mathbb{Z}$ w.r.t. an inverse system of factorizations $\psi_{nm} : \mathbb{Z}/n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}$ ($\forall\, m|n$) and w.r.t. a direct system of inclusions $\varphi_{mn} : \mathbb{Z}/m\mathbb{Z} \xhookrightarrow{[1]\mapsto[n/m]} \mathbb{Z}/n\mathbb{Z}$ (again $\forall\, m|n$).

Hint. $\varinjlim A_n = \mathbb{Q}/\mathbb{Z}$ and $\varprojlim A_n = \prod\limits_p \mathbb{Z}_p$ is the product of all rings of p-adic integer numbers (so called *completion* of $\mathbb{Z}$).

Example 15.21: fibered products (also called Cartesian squares, or coamalgams) are defined in an arbitrary category $\mathscr{C}$ as projective limits w.r.t. the category of indexes $\mathcal{N} = \{\bullet \longrightarrow \bullet \longleftarrow \bullet\}$ (3 objects and 2 non-identical arrows). Any functor $\mathcal{N} \longrightarrow \mathscr{C}$ is a diagram $X \xrightarrow{\xi} B \xleftarrow{\eta} Y$ in $\mathscr{C}$. Its projective limit is denoted by $X \underset{B}{\times} Y$ and called *a fibered product of X, Y over B*. It comes with the following commutative square (called *a Cartesian square*)

$$\begin{tikzcd} & X\underset{B}{\times}Y \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[dr, "\xi"'] & & Y \arrow[dl, "\eta"] \\ & B & \end{tikzcd} \tag{37}$$

which is universal in the following sense: for any other commutative square

$$\begin{tikzcd} & Z \arrow[dl, "\varphi'"'] \arrow[dr, "\psi'"] & \\ X \arrow[dr, "\xi"'] & & Y \arrow[dl, "\eta"] \\ & B & \end{tikzcd}$$

there exists a unique morphism $Z \xrightarrow{\varphi' \times \psi'} X \underset{B}{\times} Y$: $\varphi' = \varphi\circ(\varphi' \times \psi')$, $\psi' = \psi\circ(\varphi' \times \psi')$. Upper part of diagram (37) is uniquely (up to unique isomorphism commuting with φ, ψ) defined by this universality.

Example 15.22: amalgams (also called co-Cartesian squares, or coproducts) are inductive limits w.r.t. the index category $\mathcal{N}^{\circ} = \{\bullet \longleftarrow \bullet \longrightarrow \bullet\}$. Their expanded definition is obtained from the previous one by reversing the arrows: *an amalgam* of a diagram $X \xleftarrow{\xi} B \xrightarrow{\eta} Y$ is an universal (*co-Cartesian*) commutative square

$$\begin{tikzcd} & X\underset{B}{\otimes}Y & \\ X \arrow[ur, "\varphi"] & & Y \arrow[ul, "\psi"'] \\ & B \arrow[ul, "\xi"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

such that for any other commutative square

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "\varphi'"] & & Y \arrow[ul, "\psi'"'] \\ & B \arrow[ul, "\xi"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

there exists a unique morphism $X \underset{B}{\otimes} Y \xrightarrow{\varphi'\otimes\psi'} Z$ satisfying $\varphi' = (\varphi'\otimes\psi')\circ\varphi$, $\psi' = (\varphi'\otimes\psi')\circ\psi$.

Abelian categories. Categories appearing in commutative algebra and geometry typically have extra structures on their morphisms $\mathrm{Hom}(X,Y)$: usually we can add morphisms, form their kernels, images e. t. c. A category $\mathscr{C}$ is called *additive*, if it satisfies the following properties:

- bifunctor $X, Y \longmapsto \mathrm{Hom}(X,Y)$ takes its values in the category of Abelian groups $\mathscr{A}b$ instead of $\mathscr{S}et$, i.e. $\mathrm{Hom}(X,Y)$ is an Abelian group $\forall\, X,Y\in \mathrm{Ob}\,\mathscr{C}$ and the composition
$$\mathrm{Hom}(Y,Z)\times \mathrm{Hom}(X,Y) \xrightarrow{(\varphi,\psi)\mapsto\varphi\circ\psi} \mathrm{Hom}(X,Z)\ ,$$
is bilinear: $(\varphi_1+\varphi_2)\circ(\psi_1+\psi_2) = f_1\circ\psi_1 + f_1\circ\psi_2 + f_2\circ\psi_1 + f_2\circ\psi_2$;
- there is *a zero object* $0\in\mathrm{Ob}\,\mathscr{C}$ such that $\mathrm{Hom}(0,0)=0$ is the zero group;

  Exercice 15.23. Deduce from the previous property that $\mathrm{Hom}(X,0)=\mathrm{Hom}(0,X)=0\ \forall\, X\in\mathrm{Ob}\,\mathscr{C}$ and 0 is defined by this property up to unique isomorphism (namely, the zero morphism $0\xrightarrow{0}0'$).
- for any pair of objects A, B there exist a diagram[15]:
$$A \underset{\pi_A}{\overset{i_A}{\rightleftarrows}} A\oplus B \underset{\pi_B}{\overset{i_B}{\leftrightarrows}} B \tag{38}$$
such that $\pi_b\circ i_A = 0$, $\pi_A\circ i_B = 0$, $\pi_A\circ i_A = \mathrm{Id}_A$, $\pi_B\circ i_B = \mathrm{Id}_B$ and $i_A\circ\pi_A + i_B\circ\pi_B = \mathrm{Id}_{A\oplus B}$.

  Exercice 15.24. Show that $A\oplus B$ is defined by this property up to unique isomorphism commuting with i's and π's.

One can emulate all natural constructions known for Abelian groups in a context of an arbitrary additive category $\mathscr{C}$. For example, define *a kernel* of an arrow $A\xrightarrow{\varphi}B$ in $\mathscr{C}$ as an object representing a functor
$$C \longmapsto \ker\left(\mathrm{Hom}(C,A)\xrightarrow{\gamma\mapsto\varphi\circ\gamma}\mathrm{Hom}(C,B)\right)$$
from $\mathscr{C}$ to $\mathscr{A}b$. If exists, the representing object $\ker(\varphi)$ comes with canonical map[16] $\ker(\varphi)\xrightarrow{\varkappa}A$ satisfying $\varphi\circ\varkappa=0$ and the following universality: for any arrow $C\xrightarrow{\gamma}A$ such that $\varphi\circ\gamma=0$ there exists a unique morphism $C\xrightarrow{\psi}\ker(\varphi)$ such that $\varkappa\circ\psi=\gamma$. This property fixes the kernel up to unique isomorphism commuting with $\varkappa$. Reversing arrows, we define *a cokernel* of $A\xrightarrow{\varphi}B$ as a universal morphism $B\xrightarrow{\chi}\mathrm{coker}\,(\varphi)$ such that $\chi\circ\varphi=0$ and for any arrow $B\xrightarrow{\gamma}C$ such that $\gamma\circ\varphi=0$ there exists a unique morphism $\mathrm{coker}\,(\varphi)\xrightarrow{\psi}C$ such that $\psi\circ\chi=\gamma$. Again, $\mathrm{coker}\,(\varphi)$ is uniquely defined by this property (up to unique isomorphism commuting with χ).

Exercice 15.25. Show that in the direct sum diagram (38) the arrow $A\xrightarrow{i_A}A\oplus B$ gives the kernel of the arrow $A\oplus B\xrightarrow{\pi_B}B$ and the arrow $B\xrightarrow{i_B}A\oplus B$ gives the kernel of the arrow $A\oplus B\xrightarrow{\pi_A}A$.

From the main theorem about homomorphisms of groups we expect two ways in which an image of arrow $A\xrightarrow{\varphi}B$ can be defined. Namely, $\mathrm{im}\,\varphi$ should be isomorphic to both: the kernel of $B\xrightarrow{\chi}\mathrm{coker}\,(\varphi)$ and the cokernel of $\ker(\varphi)\xrightarrow{\varkappa}A$. An additive category $\mathscr{C}$ is called *Abelian*, if it satisfies

- each arrow $A\xrightarrow{\varphi}B$ has kernel $\ker(\varphi)$, cokernel $\mathrm{coker}\,(\varphi)$ and is decomposed as
$$\begin{tikzcd} & \mathrm{im}\,(\varphi) \arrow[dr, "\beta"] & \\ A \arrow[ur, "\alpha"] \arrow[rr, "\varphi"] & & B \end{tikzcd}$$
where $\mathrm{im}\,\varphi \simeq \mathrm{coker}\left(\ker(\varphi)\xrightarrow{\varkappa}A\right)\simeq\ker\left(B\xrightarrow{\chi}\mathrm{coker}\,(\varphi)\right)$.

[15]its middle term $A\oplus B$ is called *a direct sum* of A, B and all the diagram is called *a splitted exact triple*

[16]image of $\mathrm{Id}_{\ker\varphi}$ under the canonical identification of $\mathrm{Hom}(\ker\varphi,\ker\varphi)$ with $\ker\left(\mathrm{Hom}(\ker\varphi,A)\xrightarrow{\gamma\mapsto\varphi\circ\gamma}\mathrm{Hom}(\ker\varphi,B)\right)$

A morphism φ in Abelian category is called *surjective* (or *an epimorphism*), if $\operatorname{coker}\varphi = 0$. If $\ker\varphi = 0$, then φ is called *injective* (or *an monomorphism*).

Exercice 15.26. Show that in Abelian category:

a) $\ker(\varphi) \xrightarrow{\varkappa} A$ is injective and $B \xrightarrow{\chi} \operatorname{coker}(\varphi)$ is surjective for any arrow $A \xrightarrow{\varphi} B$;

b) φ is an isomorphism iff it is simultaneously surjective and injective.

Exercice 15.27. Check that in any additive category all squares

$$\begin{tikzcd} A \oplus B \arrow[r, shift left, "\pi_B"] \arrow[d, shift left, "\pi_A"] & B \arrow[l, shift left, "i_B"] \arrow[d, shift left, "0"] \\ A \arrow[u, shift left, "i_A"] \arrow[r, shift left, "0"] & 0 \arrow[l, shift left, "0"] \arrow[u, shift left, "0"] \end{tikzcd}$$

(coming from (38)) are simultaneously Cartesian and co-Cartesian.

Claim 15.28. *In any Abelian category there exist all fibered products and amalgams.*

Proof. To complete an arbitrary triple $X \xrightarrow{\xi} B \xleftarrow{\eta} Y$ to Cartesian square, write $K \xrightarrow{\varkappa} X \oplus Y$ for the kernel of morphism $\delta = \xi \circ \pi_X - \eta \circ \pi_Y \,:\, X \oplus Y \longrightarrow B$. Then a square

$$\begin{tikzcd} & K \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[dr, "\xi"'] & & Y \arrow[dl, "\eta"] \\ & B & \end{tikzcd}$$

where $\varphi = \pi_X \circ \varkappa$, $\psi = \pi_Y \circ \varkappa$ is commutative (because $\xi\varphi - \eta\psi = \delta\varkappa = 0$) and universal (because for any other triple $X \xleftarrow{\varphi'} Z \xrightarrow{\psi'} Y$ such that $\xi\varphi' = \eta\psi'$ only the canonical map[17] $\zeta = \varphi' \oplus \psi' : Z \longrightarrow X \oplus Y$ satisfies $\pi_X\zeta = \varphi'$, $\pi_Y\zeta = \psi'$ and can be lifted to an arrow $Z \xrightarrow{\zeta'} K$, since of $\delta\zeta = \xi\varphi' - \eta\psi' = 0$). $\square$

Exercice 15.29. Show that a diagram $X \xleftarrow{\xi} B \xrightarrow{\eta} Y$ is completed to co-Cartesian square by a cokernel $X \oplus Y \xrightarrow{\chi} Q$ of a morphism $\delta = i_X \circ \xi - i_Y \circ \eta \,:\, B \longrightarrow X \oplus Y$.

Exercice 15.30. Show that for any fibered product (37) in Abelian category:

a) ξ is surjective $\Rightarrow$ ψ is surjective;

b) $K \xrightarrow{\varkappa} X \underset{B}{\times} Y$ is the kernel of $\varphi \Rightarrow K \xrightarrow{\psi \circ \varkappa} Y$ is the kernel of η.

Langauge of elements. Let $\mathscr{C}$ be an Abelian category. Define *an element* a of $A \in \operatorname{Ob}\mathscr{C}$ as a class of equivalent arrows $X \xrightarrow{\alpha} A$ w.r.t. relation:

$$\left(X_1 \xrightarrow{\alpha_1} A\right) \sim \left(X_2 \xrightarrow{\alpha_2} A\right) \iff \exists \text{ surjections } X_1 \overset{\pi_1}{\twoheadleftarrow} Z \overset{\pi_2}{\twoheadrightarrow} X_2 \,:\, \alpha_1\pi_1 = \alpha_2\pi_2 \,.$$

Exercice 15.31. Show that this relation is transitive.

Hint. Use Claim 15.28 and Exercice 15.30.

Write $|A|$ for a set of all elements of $A \in \operatorname{Ob}\mathscr{C}$. Element $0 \in |A|$ represented by the zero morphism $0 \xrightarrow{0} A$ is called *the zero element*. If an element a is represented by $X \xrightarrow{\alpha} A$, then its *opposite* $-a$ is an element $X \xrightarrow{-\alpha} A$. Given two elements a, b represented by $X \xrightarrow{\alpha} A$, $Y \xrightarrow{\beta} A$, define their *sum* $a + b$ by a morphism $X \oplus Y \xrightarrow{\alpha\pi_X + \beta\pi_Y} A$.

Exercice 15.32. Show that all these definitions are correct and provide $|A|$ with a structure of an Abelian group.

Exercice 15.33*. Verify that in category $\mathscr{A}b$, of Abelian groups, $|A|$ is isomorphic to A as an Abelian group.

For a morphism $A \xrightarrow{\varphi} B$ in Abelian category $\mathscr{C}$ define a map $|A| \xrightarrow{\varphi} |B|$ by taking an element a presented by $X \xrightarrow{h} A$ to an element $\varphi(a)$ presented by the composition $X \xrightarrow{\varphi\circ h} B$.

Exercice 15.34. Verify that this definition is correct and $|A| \xrightarrow{\varphi} |B|$ is a homomorphism of Abelian groups.

Exercice 15.35. Verify the following rules of diagram searching:

a) $A \xrightarrow{\varphi} B$ is mono $\iff (\forall\, a \in |A| \;\; \varphi(a) = 0 \Rightarrow a = 0) \iff (\forall\, a_1, a_2 \in |A| \;\; \varphi(a_1) = \varphi(a_2) \Rightarrow a_1 = a_2)$;

[17] predicted by Exercice 15.27

b) $A \xrightarrow{\varphi} B$ is epi $\iff (\forall\, b \in |B|\ \exists\, a \in |A| : \varphi(a) = b)$; c) $\varphi = 0 \iff \varphi(a) = 0\ \forall\, a$;

d) $\operatorname{im}(A \xrightarrow{\varphi} B) = \ker(B \xrightarrow{\psi} C \iff (\psi\varphi = 0\ \&\ \forall\, b \in |B| : \psi(b) = 0\ \exists\, a \in |A| : \varphi(a) = \beta)$

Exercice 15.36*. Try to prove the following

Theorem 15.37. $\mathscr{C} \xhookrightarrow{A \mapsto |A|} \mathscr{A}b$ *is a faithful functor from an arbitrary Abelian category* $\mathscr{C}$ *into category of Abelian groups.* □

This section shows that one can work with objects of *an arbitrary* Abelian category[18] as if they are Abelian groups. In particular, we can use diagram searching and other «elementary business», say in the category of Abelian sheaves on a manifold.

[18]which is an abstract category just satisfying four axioms listed in the previous section