Working example: lines on surfaces

VARIETY OF LINES ON SURFACES OF GIVEN DEGREE. We are going to analyze the set of lines laying on a surface $S \subset \mathbb{P}_3$ of a given degree d.

Exercice 14.1. Carry out the complete analysis for $d = 2$.

To this aim consider the space $\mathbb{P}_N = \mathbb{P}(S^dV^*)$, of surfaces of degree d in $\mathbb{P}_3 = \mathbb{P}(V)$, and identify the set of all lines in $\mathbb{P}(V)$ with the Plücker quadric $Q_P \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2V)$. Let $\Gamma \overset{\text{def}}{=} \{ (S, \ell) \in \mathbb{P}_N \times Q_P \mid \ell \subset S \} \subset \mathbb{P}_N \times Q_P$ be the incidence graph.

Claim 14.2. *Γ is closed submanifold of $\mathbb{P}_N \times Q_P$.*

Proof. 2-dimensional subspace spanned by $u, w \in V$ coincides with the image of the contraction map $V^* \longrightarrow V$, which sends $\xi \in V^*$ to $\langle \xi , u \wedge w \rangle$. So, the line $\ell = (uw)$ lies on a surface S given by $F = 0$ iff $F(\langle \xi , u \wedge w \rangle) \cong 0$ identically in $\xi \in V^*$. In coordinates, let e_ν form a base of V, ξ_ν be the coordinates of ξ w.r.t. the dual base of V^*, and $u \wedge w = \sum\limits_{\mu\neq\nu} p_{\mu\nu}\, e_\mu \wedge e_\nu$, where $p_{\mu\nu} = -p_{\nu\mu}$ are the associated Plücker coordinates on $\mathbb{P}_5 \supset Q$. Then $\langle \xi , u \wedge w \rangle = \sum\limits_i \xi_i \cdot (\sum_\nu p_{i\nu} e_\nu)$. Substitute this into F, expand the result through the monomials in ξ, and write down that all coefficients of this expansion vanish — this gives a system of polynomial equations on the coefficients of F and p_{ij} describing $\Gamma \subset \mathbb{P}_N \times Q_P \subset \mathbb{P}_N \times \mathbb{P}_5$. □

Claim 14.3. *Projection $\Gamma \xrightarrow{\pi_2} Q_P$ is surjective; all its fibers are projective spaces of dimension $d(d + 1)(d + 5)/6 - 1$.*

Proof. Let a line $\ell \subset \mathbb{P}(V)$ be given by $x_0 = x_1 = 0$. Then $S \supset \ell$ iff S has an equation $0 = x_2 \cdot F_2(x) + x_3 \cdot F_3(x)$, where $F_2, F_3 \in S^{d-1}V^*$ are arbitrary homogeneous polynomials. These equations form a vector space W, which coincides with the image of the linear operator $S^{d-1}V^* \oplus S^{d-1}V^* \xrightarrow{(f,g)\mapsto x_2f+x_3g} S^dV^*$ whose kernel consists of all (f, g) such that $x_2f = -x_3g$ that is possible iff $f = x_3h$ and $g = -x_2h$ for some $h \in S^{d-2}V^*$. Hence, the kernel is isomorphic to $S^{d-2}V^*$ and $\dim W = 2 \dim(S^{d-1}V^*) - \dim(S^{d-1}V^*) = \Big(2\, d(d + 1)(d + 2) - (d - 1)d(d + 1)\Big)/6 = d(d + 1)(d + 5)/6$. □

Corollary 14.4. *Γ is an irreducible projective manifold of dimension $d(d + 1)(d + 5)/6 + 3$.*

Proof. This follows at once from Corollary 13.32 and Theorem 13.28. □

Claim 14.5. *A generic[1] surface $S_d \subset \mathbb{P}_3$ of degree $d \geqslant 4$ does not contain lines.*

Proof. By Corollary 13.13, the image of the projection $\Gamma \xrightarrow{\pi_1} \mathbb{P}_N$, that is the set of all surfaces containing some lines, is closed irreducible submanifold of $\mathbb{P}_N = \mathbb{P}(S^dV^*)$. By Theorem 13.28, its dimension equals $\dim\Gamma$ minus the minimal dimension of non-empty fibres of π_1. In particular, the image is proper, as soon as $\dim\Gamma < N$, i. e. if $d(d+1)(d+5)/6+3 < (d + 1)(d + 2)(d + 3)/6$, which holds for all $d \geqslant 4$. □

Claim 14.6. *Each cubic surface $S_3 \subset \mathbb{P}_3$ contains lines; generically, this line set is finite.*

Proof. Taking in the previous proof $d = 3$, we get $\dim\Gamma = N = 19$ and π_1 is surjective, if it has a non-empty 0-dimensional fiber. Let us find, say, all the lines on a cubic C with affine equation $xyz = 1$. This affine piece does not contain the lines at all, because $x = x_0 + \alpha\, t$, $y = y_0 + \beta\, t$, $z = z_0 + \gamma\, t$ lies on C iff $\alpha\beta\gamma = 0$, $\alpha\beta z_0 + \beta\gamma x_0 + \gamma\alpha y_0 = 0$, and $\alpha y_0z_0 + \beta x_0z_0 + \gamma x_0y_0 = 0$, but $x_0y_0z_0 = 1$, which leads to contradiction when we go from the left to the right: for example, $\alpha = 0 \Rightarrow \beta = 0$ or $\gamma = 0 \Rightarrow \beta = \gamma = 0$. To describe C at infinity, put $x = x_1/x_0$, $y = x_2/x_0$, $z = x_3/x_0$ and rewrite its equation as $x_1x_2x_3 = x_0^3$. Thus, $C \cap \{x_0 = 0\}$ consists of 3 lines: $x_i = x_0 = 0$, $i = 1, 2, 3$. □

Exercice 14.7*. Find all lines on the (smooth) Fermat cubic C_F, given by $\sum x_i^3 = 0$.

HINT. C_F is preserved by the permutations of the coordinates; up to such a permutation, a pair of linear equations for $\ell \subset C_F$ is reduced, by Gauss, to $x_0 = \alpha x_2 + \beta x_3$, $x_1 = \gamma x_2 + \delta x_3$; now substitute, show that $\alpha\beta\gamma\delta = 0$, and find them...

LINES ON A SMOOTH CUBIC. Now, let $S \subset \mathbb{P}_3$ be a smooth cubic surface with an equation $F(x) = 0$.

Claim 14.8. *Reducible plane section of S splits either into a line and a smooth conic or into a triple of distinct lines.*

Proof. We have to show that a plane section $\pi\cap S$ can not contain a double line component. If there is a double line $\ell \subset \pi\cap S$, take the coordinates where π is given by $x_2 = 0$ and ℓ by $x_2 = x_3 = 0$. Then $F(x) = x_2Q(x) + x_3^2L(x) = 0$ for some linear L and quadratic Q. Let a be an intersection point of ℓ with the quadric $Q(x) = 0$. Then $x_2(a) = x_3(a) = Q(a) = 0$ and all partial derivatives $\partial F/\partial x_i$ vanish at a, i. e. S is singular at a. □

Corollary 14.9. *A point of S can belong at most to 3 lines laying on S and these lines should be coplanar.*

Proof. Indeed, all lines passing through $p \in S$ and laying on S belong to $S \cap T_pS$. □

Claim 14.10. *Given $\ell \subset S$, there are precisely 5 distinct planes $\pi_1, \pi_2, \dots, \pi_5$ containing ℓ and intersecting S in a triple of lines; moreover, if $\pi_i \cap S = \ell \cup \ell_i \cup \ell_i'$, then $\ell_i \cap \ell_j = \ell_i \cap \ell_j' = \ell_i' \cap \ell_j' = \varnothing\ \forall\, i \neq j$ (in particular, S contains some skew lines) and any line on S skew to ℓ must intersect for each i precisely one of ℓ_i, ℓ_i'.*

[1] i. e. any from some dense Zariski open subset in the space of all hypersurfaces

Proof. Fix a base $\{e_0, e_1, e_2, e_3\}$ for V such that the line $\ell = (e_0e_1)$ (given by equations $x_2 = x_3 = 0$) lays on S. Then the equation $F(x) = 0$, defining S, can be written in this base as:

$$L_{00}(x_2,x_3)\cdot x_0^2 + 2\,L_{01}(x_2,x_3)\cdot x_0x_1 + L_{11}(x_2,x_3)\cdot x_1^2 + 2\,Q_0(x_2,x_3)\cdot x_0 + 2\,Q_1(x_2,x_3)\cdot x_1 + R(x_2,x_3) = 0 \qquad (35)$$

where $L_{ij}, Q_\nu, R \in k[x_2, x_3]$ are homogeneous of degrees $1, 2, 3$ respectively. Let us parameterize the plain pencil through ℓ by the points $e_\vartheta = \vartheta_2e_2 + \vartheta_3e_3 \in (e_2e_3)$ and write $(t_0 : t_1 : t_2)$ for homogeneous coordinates in the plane $\pi_\vartheta = (e_0e_1e_\vartheta)$ w.r.t. these 3 base points. The equation for the plane conic $(\pi_\vartheta \cap S) \setminus \ell$ is obtained from (35) by the substitution $x = (t_0 : t_1 : \vartheta_2t_3 : \vartheta_3t_3)$ and cancelling the common factor t_3. The resulting conic has the Gram matrix

$$G = \begin{pmatrix} L_{00}(\vartheta) & L_{01}(\vartheta) & Q_0(\vartheta) \\ L_{01}(\vartheta) & L_{11}(\vartheta) & Q_1(\vartheta) \\ Q_0(\vartheta) & Q_1(\vartheta) & R(\vartheta) \end{pmatrix}$$

whose determinant $D(\vartheta_2,\vartheta_3) = L_{00}(\vartheta)L_{11}(\vartheta)R(\vartheta) + 2\,L_{01}(\vartheta)Q_0(\vartheta)Q_1(\vartheta) - L_{11}(\vartheta)Q_0^2(\vartheta) - L_{00}(\vartheta)Q_1^2(\vartheta) - L_{01}(\vartheta)^2R(\vartheta) \in k[\vartheta_2,\vartheta_3]$ is homogeneous of degree 5 in $\vartheta = (\vartheta_2 : \vartheta_3)$, i.e. has 5 roots counted with multiplicities. We have to show that all these roots are simple. Each root corresponds to the splitting of the conic into a pair of lines, whose intersection point lies either on ℓ or outside ℓ.

In the first case, we can fix a base in such a way that two lines in question are $\ell' = (e_0e_2)$ and $\ell'' = \big(e_0\,(e_1+e_2)\big)$ given by equations $x_3 = x_1 = 0$ and $x_3 = (x_1 - x_2) = 0$; this corresponds to the root $\vartheta = (1:0)$ and its multiplicity equals the highest power of ϑ_3 which divides $D(\vartheta_2,\vartheta_3)$. Since $\ell, \ell', \ell'' \subset S$, the equation (35) has a form $x_1x_2(x_1-x_2)+x_3\cdot q(x)$ with some quadratic $q(x)$, i.e. the only elements of G possibly not divisible by ϑ_3 are $L_{11} \equiv x_2 \pmod{\vartheta_3}$ and $Q_1 \equiv -x_2^2/2 \pmod{\vartheta_3}$. So, $D(\vartheta_2,\vartheta_3) \equiv -L_{00}Q_1^2 \pmod{\vartheta_3^2}$ and this term is of order 1 in t_3 as soon $x_1x_2^2$ and $x_0^2x_2$ do come in (35) with non zero coefficients. But the first is the only monomial that gives non zero contribution into $\partial F/\partial x_1$ computed at $e_2 \in S$ and the second — in $\partial F/\partial x_2$ at $e_0 \in S$; so, they do.

In the second case, we can fix a base in order to have $\ell' = (e_0e_2)$, $\ell'' = (e_1e_2)$ (given by $x_3 = x_1 = 0$ and $x_3 = x_0 = 0$), and the same root $\vartheta = (1:0)$. Now, (35) takes $x_0x_1x_2 + x_3\cdot q(x)$ non zero modulo ϑ_3 entry of G is only $L_{01} \equiv x_2/2 \pmod{\vartheta_3}$. So, $D(\vartheta_2,\vartheta_3) \equiv -L_{01}^2R \pmod{\vartheta_3^2}$, which is of order 1 in t_3 if $x_2^2x_3$ and $x_0x_1x_2$ really do appear in (35). The second does, because otherwise F is divisible by x_3. The first is the only monomial that gives non zero contribution into $\partial F/\partial x_3$ computed at $e_2 \in S$.

The rest assertions follow immediately from Corollary 14.9, Claim 14.8 and remark that any line in $\mathbb{P}_3$ intersects any plane. □

Claim 14.11. *Any 4 mutually skew lines on S do not lay on a common quadric and always there exist either 1 or 2 (but no more!) lines on S intersecting all these 4 lines.*

Proof. Any line passing through 4 distinct points of S has to lay on S. If 4 lines on S belong to a line family ruling a quadric[2] then the second ruling line family of this quadric, that is the quadric itself, lays on S, i.e. S is reducible. Now it remains to apply Exercice 2.20. □

CONFIGURATION OF 27 LINES. Take 2 skew lines a, $b \subset S$ and construct 5 pairs of lines ℓ_i, ℓ_i' predicted by Claim 14.10 applied to $\ell = a$. Let us write ℓ_i for those which do meet b and ℓ_i' for those which do not. Then there are 5 more lines ℓ_i'' coupled with ℓ_i by Claim 14.10 applied to $\ell = b$. Each ℓ_i'' meets b but neither a nor ℓ_j with $j \neq i$; so, it intersects all ℓ_j' with $j \neq i$. Any line $c \subset S$, different from 17 just constructed, is skew to a, b but meets either ℓ_i or ℓ_i' for each i. By Claim 14.11, all lines meeting $\geqslant 4$ of ℓ_i's are exhausted by a, b. If c meets $\leqslant 2$ out of ℓ_i's, then, up to a permutation of indices, it meets ℓ_1', ℓ_2', ℓ_3' and, say, either ℓ_4' or ℓ_5; but in the both cases a and ℓ_5'' are the only such the lines. So, c has to intersect precisely 3 out of ℓ_i.

Claim 14.12. *Remaining lines $c \subset S$ are in 1–1 correspondence with 15 triples $\{i, j, k\} \subset \{1, 2, 3, 4, 5\}$.*

Proof. There is at most one line c intersecting a given triple of ℓ_i's — this is the second possible line besides a meeting all these ℓ_i's and the rest ℓ_j''s (all 5 are mutually skew). On the other hand, by Claim 14.10, for each i there are precisely 10 lines on S intersecting ℓ_i: 4 of them are a, b, ℓ_i', ℓ_i'' and other 6 have to intersect exactly 2 of the rest four ℓ_j's. So, we have 1–1 correspondence between these 6 lines and $6 = \binom{4}{2}$ choices of pairs of ℓ's. □

Corollary 14.13. *Each smooth cubic surface $S \subset \mathbb{P}_3$ contains precisely 27 lines; their incidence combinatorics is the same for all S.* □

Exercice 14.14. Let $\mathfrak{G} \subset \mathfrak{S}_{27}$ be a group of all permutations of the 27 lines preserving all the incidence relations between them; find the order of $\mathfrak{G}$. (ANSWER: $|\mathfrak{G}| = 51\,840 = 2^7\cdot 3^4\cdot 5$)

Exercice 14.15*. Consider the field of 4 elements: $\mathbb{F}_4 \stackrel{\text{def}}{=} \mathbb{F}_2[\omega]/(\omega^2+\omega+1)$, where $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Like for $\mathbb{R} \subset \mathbb{C}$, the extension $\mathbb{F}_2 \subset \mathbb{F}_4$ has a conjugation automorphism: $z \longmapsto \overline{z} \stackrel{\text{def}}{=} z^2$, which lives $\mathbb{F}_2$ fixed and permutes two roots of $\omega^2+\omega+1$. Show that a factor group of the *unitary* 4×4 - matrices with entries in $\mathbb{F}_4$ modulo the scalar matrices is canonically embedded as a (normal) subgroup of index 2 into the group $\mathfrak{G}$ from Exercice 14.14.

HINT. This group preserves the Fermat cubic C_F (see Exercice 14.7) whose equation over $\mathbb{F}_4$ turns to Hermitian form $\sum x_i\overline{x}_i$.

[2] comp. with Corollary 2.19