Algebraic manifolds

<u>LOCALIZATION.</u> Let $U \subset X$ be an open subset of an affine algebraic variety and $u \in U$. A function $U \xrightarrow{f} k$ is called *regular* at u, if there are $p, q \in k[X]$ such that $q(u) \neq 0$ and $f(x) = p(x)/q(x)\ \forall\, x \in \mathscr{D}(q) \cap U$. Functions $U \xrightarrow{f} k$ which are regular at any $u \in U$ form a commutative ring denoted by $\mathscr{O}_X(U)$ or by $\Gamma(U, \mathscr{O}_X)$. It is called *a ring of local regular functions* on $U \subset X$.

<u>Claim 13.1</u>. *Let X be irreducible and $h \in k[X]$. Then any $f \in \mathscr{O}_X(\mathscr{D}(h))$ can be written as $f(x) = r(x)/h^d(x)$ for appropriate $r \in k[X]$, $d \in \mathbb{N}$ (in particular, taking $h \equiv 1$, we get $\mathscr{O}_X(X) = k[X]$).*

Proof. If $f \in \mathscr{O}_X(\mathscr{D}(h))$, then $\forall\, u \in \mathscr{D}(h)$ there are $p_u, q_u \in k[X]$ such that $q_u(u) \neq 0$ and $f(x) = p_u(x)/q_u(x)\quad \forall\, x \in \mathscr{D}(q_u) \cap \mathscr{D}(h)$. So, $\bigcap\limits_{u \in U} V(q_u) \subset V(h)$ and, by Hilbert, there are some $u_1, u_2, \dots, u_m \in \mathscr{D}(h)$ such that $h^d = \sum q_{u_\nu} g_\nu$ for appropriate $g_1, g_2, \dots, g_m \in k[X]$ and $d \in \mathbb{N}$. At the same time, $f(x)\, q_{u_\nu}(x) = p_{u_\nu}(x)\quad \forall\, \nu$ and $\forall\, x \in \mathscr{D}(h)$ including $\mathscr{D}(h) \cap V(q_{u_\nu})$. Indeed, let $q_{u_\nu}(w) = 0$ for some $w \in \mathscr{D}(h)$; rewriting $f = p_{u_\nu}/q_{u_\nu}$ as p_w/q_w with $q_w(w) \neq 0$ we get $p_{u_\nu}(x)\, q_w(x) = q_{u_\nu}(x)\, p_w(x)\quad \forall\, x \in \mathscr{D}(h \cdot q_{u_\nu} \cdot q_w)$ and hence, by Exercice 12.13, for any $x \in X$ at all; in particular, $p_{u_\nu}(w) = q_{u_\nu}(w)\, p_w(w)/q_w(w) = 0$. So, $f\, h^d = \sum f\, q_{u_\nu}\, g_\nu = \sum p_{u_\nu}\, g_\nu \in k[X]$. □

<u>Corollary 13.2</u>. *Each $\mathscr{D}(f) = Spec_m\, k[X][f^{-1}]$ is an affine algebraic variety and inclusion $\mathscr{D}(f) \hookrightarrow X$ is a regular map whose pull back is $k[X] \hookrightarrow k[X][f^{-1}]$.* □

<u>STRUCTURE SHEAF.</u> The correspondence $\mathscr{O}_X\ :\ U \longmapsto \mathscr{O}_X(U)$ is called *a structure sheaf* of an *affine algebraic manifold* X. If $U = \bigcup W_i$ is an union of open sets, then $U \xrightarrow{f} k$ is regular iff each its restriction $f|_{W_i}$ is regular on W_i. Conversely, a collection of functions $W_i \xrightarrow{f_i} k$ such that $f_i \equiv f_j$ on $W_i \cap W_j$ gives a unique regular function $f \in \mathscr{O}_X(\cup W_i)$ whose restriction onto W_i is f_i for all i. Although Corollary 13.2 shows that open sets are *locally* affine, general open $U \subset X$ is *not* an affine algebraic variety and there is no bijection between $Spec_m \mathscr{O}_X(U)$ and U.

Exercice 13.3. Let $U = \mathbb{A}^n \setminus O$ be the complement to the origin. Show that $\mathscr{O}_{\mathbb{A}^n}(U) = k[\mathbb{A}^n]$ for $n \geqslant 2$.
HINT. Use the covering $U = \bigcup \mathscr{D}(x_i)$ and Claim 13.1.

<u>ALGEBRAIC MANIFOLDS.</u> Let X be a topological space. An open subset $U \subset X$ is called *an algebraic affine chart* on X, if there exists an affine algebraic variety X_U and *a homeomorphism* $X_U \xrightarrow{\varphi_U} U$. Two algebraic charts $X_U \xrightarrow{\varphi_U} U$ and $X_W \xrightarrow{\varphi_W} W$ on X are called *compatible*, if their *transition map* $\varphi_{WU} = \varphi_W \circ \varphi_U^{-1}$, which identifies $\varphi_U^{-1}(U \cap W) \subset X_U$ with $\varphi_W^{-1}(U \cap W) \subset X_W$, is a regular isomorphism of algebraic open sets, i.e. its pull back $\Gamma\left(\varphi_W^{-1}(U \cap W)\,, \mathscr{O}_{X_W}\right) \xrightarrow{\varphi_{WU}^*} \Gamma\left(\varphi_U^{-1}(U \cap W)\,, \mathscr{O}_{X_U}\right)$ is a well defined isomorphism of k-algebras. A (finite) open covering $X = \bigcup U_\nu$ by mutually compatible algebraic charts is called a (finite) *algebraic atlas* on X. Two algebraic atlases are called *equivalent*, if their union is an algebraic atlas too. A topological space X equipped with a class of equivalent (finite) algebraic atlases is called *an algebraic manifold* (of finite type).

Exercice 13.4. Check that projective spaces and Grassmannians are algebraic manifolds of finite type as well as any zero set of a collection of multihomogeneous polynomials on $\mathbb{P}_{n_1} \times \mathbb{P}_{n_2} \times \cdots \times \mathbb{P}_{n_m}$.

<u>Example 13.5</u>: a direct product $X \times Y$ of algebraic manifolds X, Y is an algebraic manifold too. Its atlas consists of all pairwise products $U \times W$, where $U \subset X$, $W \subset X$ are affine algebraic charts on X, Y.

<u>Example 13.6</u>: separability. The standard atlas on $\mathbb{P}_1$ consists of two charts $\varphi_i : \mathbb{A}_1 \xrightarrow{\sim} U_i \subset \mathbb{P}_1$, $i = 0, 1$, and $\varphi_0^{-1}(U_0 \cap U_1) = \varphi_1^{-1}(U_0 \cap U_1) = \{t \in \mathbb{A}^1 \mid t \neq 0\}$ is the complement to the origin. The charts $U_{0,1} = \mathbb{A}^1$ are glued together along $\mathbb{A}^1 \setminus \{O\}$ via transition map $\varphi_{01} : t \mapsto 1/t$. If we replace it by the identity map $\widetilde{\varphi}_{01} : t \mapsto t$, then we get an other manifold called «$\mathbb{A}^1$ with doubled origin», which looks like . Such the pathology is known as a *nonseparability*. It has appeared, because the last gluing rule is not «complete» and could be extended from $\mathbb{A}^1 \setminus \{O\}$ to a larger set. This can be formalized as follows. Two inclusions $U_0 \hookleftarrow U_0 \cap U_1 \hookrightarrow U_1$ give an embedding $U_0 \cap U_1 \hookrightarrow U_0 \times U_1$. In the case of $\mathbb{P}_1$, this is an inclusion $(\mathbb{A}^1 \setminus O) \hookrightarrow \mathbb{A}^2$ given as $x = t\,;\ y = 1/t$; it identifies $U_0 \cap U_1$ with *a closed* subset $V(xy - 1) \subset \mathbb{A}^2 = U_0 \times U_1$. In the second case, the embedding $U_0 \cap U_1 \hookrightarrow U_0 \times U_1 = \mathbb{A}^2$ is given as $x = t\,;\ y = t$ and has a *nonclosed* image $\Delta \setminus \{(0,0)\}$, where $\Delta = V(x - y) \subset \mathbb{A}^2$ is the diagonal. An algebraic manifold X is called *separable*, if an image of the canonical embedding $U \cap W \hookrightarrow U \times W$ is closed for any two affine charts $U, W \subset X$. Since this image is nothing more than an intersection of the diagonal $\Delta \subset X \times X$ with the affine chart $U \times W$ on $X \times X$, a manifold X is separable iff the diagonal $\Delta \subset X \times X$ is closed. For example, $\mathbb{A}^n$ and $\mathbb{P}_n$ are separable, because the diagonals in $\mathbb{A}^n \times \mathbb{A}^n$ and in $\mathbb{P}_n \times \mathbb{P}_n$ are given, accordingly, by the equations $x_i = y_i$ and $x_i y_j = x_j y_i$.

<u>REGULAR FUNCTIONS AND MORPHISMS.</u> Let $U \subset X$ be an open set. A function $U \xrightarrow{f} k$ is called *regular*, if each point $u \in U$ has an affine neighborhood $X_W \xrightarrow{\varphi_W} W \ni u$ such that $f \circ \varphi \in \mathscr{O}_{X_W}\left(\varphi^{-1}(U \cap W)\right)$. In other

words, a local function on X is regular, if it induces a local regular functions on each affine chart. Regular functions $U \longrightarrow k$ form a commutative ring $\mathscr{O}_X(U)$; a correspondence $U \longmapsto \mathscr{O}_X(U)$ is called *a structure sheaf* on X. More generally, a map of algebraic manifolds $X \xrightarrow{\varphi} Y$ is called *a regular morphism*, if its pull back is well defined homomorphism of k-algebras: $\mathscr{O}_Y(U) \xrightarrow{\varphi^*} \mathscr{O}_X(\varphi^{-1}U)$ for any open $U \subset Y$. For example, a set of regular morphisms $X \longrightarrow \mathbb{A}^1$ coincides with $\mathscr{O}_X(X)$.

RATIONAL MAPS. A regular[1] morphism $U \xrightarrow{\varphi} Y$, which is defined only on some open dense $U \subset X$, is called *a rational map* from X to Y. One should be careful in composing rational maps, because an image of the first map may be completely outside the domain where the second is defined.

Example 13.7: a projection $\mathbb{A}^{n+1} \xrightarrow{\pi} \mathbb{P}_n$ sending a point $A \in \mathbb{A}^n$ to the line $(OA) \in \mathbb{P}_n$ is a rational surjection defined on $U = \mathbb{A}^n \setminus O$. In terms of the standard affine chart $\mathbb{A}^n \xrightarrow{\varphi_i} U_i = \{(t_0, t_1, \dots, t_n) \in \mathbb{P}_n \mid t_i = 1\}$, the pull back homomorphism $\mathscr{O}_{\mathbb{P}_n}(U_i) \xrightarrow{\pi^*} \mathscr{O}_{\mathbb{A}^{n+1}}(\pi^{-1}(U_i))$ sends $f(x_1, x_2, \dots, x_n) \in k[x_1, x_2, \dots, x_n] = \mathscr{O}_{\mathbb{P}_n}(U_i)$ to the rational function $\widetilde{f}(t_0, t_1, \dots, t_n) = f(t_0/t_i, \dots, t_{i-1}/t_i, t_{i+1}/t_i, \dots, t_n/t_i) \in \mathscr{O}_{\mathbb{A}^{n+1}}(\mathscr{D}(t_i)) = \mathscr{O}_{\mathbb{A}^{n+1}}(\pi^{-1}(U_i))$.

CLOSED SUBMANIFOLDS. Any closed subset $Z \subset X$ of an algebraic manifold X has natural structure of algebraic manifold. Namely, for any affine chart U the intersection $Z \cap U$ is a closed subset of U, that is an affine algebraic set $Spec_m\,(\mathscr{O}_X(U)/\mathscr{I}_Z(U))$, where $\mathscr{I}_Z(U) = \{f \in \mathscr{O}_X(U) \mid f|_Z \equiv 0\}$ is the ideal of $Z \cap U$ on U. The correspondence $U \longmapsto \mathscr{I}_Z(U)$ is called *the ideal sheaf* of the closed submanifold $Z \subset X$. This is a subsheaf of the structure sheaf. It consists of all local regular functions vanishing along Z. A regular morphism of arbitrary algebraic manifolds $X \xrightarrow{\varphi} Y$ is called *a closed embedding*, if $\varphi(X) \subset Y$ is a closed submanifold and φ gives an isomorphism between X an $\varphi(X)$. One can say that an algebraic manifold X is affine iff it admits a closed embedding into affine space. Similarly, a manifold X is called *projective*, if it admits a closed embedding $X \hookrightarrow \mathbb{P}_m$ for some m.

Example 13.8: closed submanifold $X \subset Y$ is separable as soon Y is, because the diagonal in $X \times X$ is a preimage of the diagonal in $Y \times Y$ under an embedding $X \times X \hookrightarrow Y \times Y$. In particular, any affine or projective manifold is separable and has a finite type.

Example 13.9: graph of morphism. Let $X \xrightarrow{\varphi} Y$ be a regular morphism. A preimage of the diagonal $\Delta \subset Y \times Y$ under an induced morphism $X \times Y \xrightarrow{\varphi \times \mathrm{Id}_Y} Y \times Y$ is called *a graph* of φ and is denoted by Γ_φ. Geometrically, $\Gamma_\varphi = \{(x, f(x)\} \subset X \times Y$. It is closed, if Y is separable. For example, a graph of a regular morphism of affine manifolds $Spec_m\,(A) \xrightarrow{\varphi} Spec_m\,(B)$ is given in $A \otimes B$ by the equation system $1 \otimes f = \varphi^*(f) \otimes 1$, where f runs through B and $B \xrightarrow{\varphi^*} A$ is the pull back of φ.

Example 13.10: family of closed submanifolds. Any regular morphism $X \xrightarrow{\pi} Y$ may be considered as a family of closed submanifolds $X_y = \pi^{-1}(y) \subset X$ parameterized by the points $y \in Y$. If $X \xrightarrow{\pi} Y$, $X' \xrightarrow{\pi'} Y$ are two families with the same base, then a regular morphism $X \xrightarrow{\varphi} X'$ is called *a morphism of families* (or a morphism *over* Y), if it sends X_y to X'_y for any $y \in Y$, that is if $\pi = \pi' \circ \varphi$. A family $X \xrightarrow{\pi} Y$ is called *constant* or *trivial*, if it is isomorphic over Y to a direct product $X_0 \times Y \xrightarrow{\pi_Y} Y$ for some manifold X_0.

Example 13.11: blow up a point $p \in \mathbb{P}_n$. Lines passing through a given point $p \in \mathbb{P}_n$ form a projective space $E \simeq \mathbb{P}_{n-1}$, which can be identified with any hyperplane $H \subset \mathbb{P}_n$ such that $p \notin H$. An incidence graph $\mathcal{B}_p = \{(\ell, q) \in E \times \mathbb{P}_n \mid q \in \ell\}$ is called *a blow up* of $p \in \mathbb{P}_n$. If $n \geqslant 2$, then the projection $\sigma_p : \mathcal{B}_p \twoheadrightarrow \mathbb{P}_n$ is bijective outside $q = p$, but a preimage $\sigma_p^{-1}(p) \subset \mathcal{B}_p$ coincides with E; it is called *an exceptional divisor*. The second projection $\varrho_E : \mathcal{B}_p \twoheadrightarrow E$ fibers $\mathcal{B}_p$ as *a line bundle* over E whose fiber at a point $\ell \in E$ coincides with the line $\ell \subset \mathbb{P}_n$ itself; this line bundle is called *a tautological line bundle* over E. If we fix homogeneous coordinates $(t_0 : t_1 : \dots : t_n)$ on $\mathbb{P}_n$ such that $p = (1 : 0 : \cdots : 0)$ and identify E with the hyperplane $H = \{(0 : q_1 : \cdots : q_n)\} \subset \mathbb{P}_n$, then $(q, t) \in \mathcal{B}_p$ iff $\mathrm{rk} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & q_1 & \cdots & q_n \\ t_0 & t_1 & \cdots & t_n \end{pmatrix} = 3$, that is $q_i t_j = q_j t_i$ for all $1 \leqslant i < j \leqslant n$. So, $\mathcal{B}_p$ is closed submanifold of $H \times \mathbb{P}_n$.

CLOSED MORPHISMS. A regular morphism $X \xrightarrow{\varphi} Y$ is called *closed*, if $\varphi(Z) \subset Y$ is closed for any closed $Z \subset X$. Of course, any closed embedding is closed. Claim 12.17 says that any finite morphism of affine manifolds is closed. The projection $\mathbb{P}_m \times \mathbb{A}^n \longrightarrow \mathbb{A}^n$ is also closed by Example 8.27.

Corollary 13.12. *If X is a projective manifold, then the projection $X \times Y \longrightarrow Y$ is closed for any manifold Y.*

Proof. Taking an affine chart on Y, we can suppose that Y is affine, i. e. that $X \times Y$ is a closed submanifold of $\mathbb{P}_m \times \mathbb{A}^n$. Then our projection is the restriction of the closed map $\mathbb{P}_m \times \mathbb{A}^n \longrightarrow \mathbb{A}^n$ onto the closed subset $X \times Y \subset \mathbb{P}_m \times \mathbb{A}^n$. □

[1] as above, the regularity means that the pull back $\mathscr{O}_Y(W) \xrightarrow{\varphi^*} \mathscr{O}_X(\varphi^{-1}(W))$ is a well defined k-algebra homomorphism

Corollary 13.13. *If X is projective and Y is separable, then any morphism $X \xrightarrow{\varphi} Y$ is closed.*

Proof. Since Y is separable, the graph $\Gamma_\varphi \subset X \times Y$ is closed. $Z \times Y$ is also closed in $X \times Y$ for any closed $Z \subset X$. But $\varphi(Z)$ is the image of $\Gamma_\varphi \cap (Z \times Y)$ under the projection $X \times Y \longrightarrow Y$. $\square$

Corollary 13.14. *If X is a connected projective manifold, then $\mathcal{O}_X(X) = k$. Moreover, each regular map from X to any affine manifold contracts X into one point.*

Proof. Let us identify $k = \mathbb{A}^1$ with an affine chart on $\mathbb{P}_1$ and consider a global regular function $X \xrightarrow{f} k$ as a regular morphism $X \xrightarrow{f} \mathbb{P}_1$. Since $f(X) \subsetneq \mathbb{P}_1$ is closed and connected, it is one point. In particular, if $X \xrightarrow{\varphi} \mathbb{A}^n$ is regular, then each coordinate form $x_i \in k[\mathbb{A}^n]$ takes a constant value along $\varphi(X)$. $\square$

Finite morphisms of manifolds. A regular morphism of arbitrary algebraic manifolds $X \xrightarrow{\varphi} Y$ is called *finite*, if $W = \varphi^{-1}(U)$ is an affine chart on X for any affine chart $U \subset Y$ and the restriction $W \xrightarrow{\varphi_W} U$ is a finite morphism of affine algebraic varieties. It follows from Claim 12.17 that any finite morphism is closed and a restriction of a finite morphism onto any closed submanifold $Z \subset X$ is a finite morphism as well. Moreover, if X is irreducible, then each proper closed $Z \subset X$ goes to a *proper* closed subset of Y.

Example 13.15: a projection of any projective manifold $X \subsetneq \mathbb{P}_n$ from any point $p \notin X$ onto any hyperplane $H \not\ni p$ is a finite morphism. To check this, let us fix the coordinates as in Example 13.11 and follow the notations of that example. Consider a standard affine chart on H, say $U_{q_n} = \{q = (0 : u_1 : \cdots : u_{n-1} : 1)\} \subset H$. Its preimage $Y = \pi_p^{-1}(U_{q_n}) \subset X$ under the projection from p lies inside the cone over U_{q_n} with the punctured vertex p (because $p \notin X$). The blow up maping σ_p identifies this punctured cone with the affine space $\mathbb{A}^n = U_{q_n} \times \mathbb{A}^1$ via the substitution $t = \vartheta p + q_u$, where $t = (t_0 : t_1 : \ \dots \ : t_n)$ is the homogeneous coordinate on $\mathbb{P}_n$, $p = (1 : 0 : \ \dots \ : 0)$, $q_u = (0 : u_1 : \cdots : u_{n-1} : 1) \in U_{q_n}$. If X is given by a system of homogeneous equations $f_\nu(t) = 0$, then Y is given in affine coordinates (u, t) by equations

$$f_\nu(\vartheta p + q_u) = \alpha_0^{(\nu)}(u)\,\vartheta^m + \alpha_1^{(\nu)}(u)\,\vartheta^{m-1} + \cdots + \alpha_m^{(\nu)}(u) = 0\,. \tag{33}$$

So, Y is affine and it remains to check that $k[Y] = k[u][\vartheta]/(f_\nu(\vartheta p + q_u))$ is a finitely generated $k[u]$-module. To this aim it is enough to find in the ideal spanned by equations (33) an integer equation with $\alpha_0(u) \equiv 1$, because the factorization through such an equation already gives a finitely generated $k[u]$-module. But this is possible, since the leading coefficients $\alpha_0^{(\nu)}(u)$ have no common zero in U_{q_n}. Indeed, if such a zero u_0 exists, then the homogeneous version of (33): $\alpha_0^{(\nu)}(u_0)\,\vartheta_0^m + \alpha_1^{(\nu)}(u_0)\,\vartheta_0^{m-1}\vartheta_1 + \cdots + \alpha_m^{(\nu)}(u_0)\vartheta_1^m = 0$ has the root $(\vartheta_0 : \vartheta_1) = (1 : 0)$. But these homogeneous equations, obtained by the substitution $t = \vartheta_0\, p + \vartheta_1\, q_u$, describe the intersection $X \cap (pq_{u_0})$ and this root gives the point p itself, but $p \notin X$!

Exercice 13.16. Check that a composition of finite morphisms is finite and prove that any projective manifold admits a finite surjective morphism onto a projective space.

Corollary 13.17. *Each affine manifold X admits a finite surjective morphism φ onto an affine space.*

Proof. Let $X \subsetneq \mathbb{A}^n$, where $\mathbb{A}^n$ is embedded in $\mathbb{P}_n$ as the standard chart U_0. We write H_∞ for $\mathbb{P}_n \setminus U_0$ and $\overline{X} \subset \mathbb{P}_n$ for the projective closure of X. A projection of $\overline{X}$ from any point $p \in H_\infty \setminus \left(\overline{X} \cap H_\infty\right)$ onto any hyperplane $L \not\ni p$ induces a finite morphism from $X = \overline{X} \setminus \left(\overline{X} \cap H_\infty\right)$ to $\mathbb{A}^{n-1} = L \setminus (L \cap H_\infty)$. If it is non surjective, we repeat this procedure. $\square$

Dimension. Maximal $n \in \mathbb{N}$ such that there exists a chain of irreducible closed subsets:

$$\{x\} = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_{n-1} \subsetneq X_n \subset X\,. \tag{34}$$

is called *a dimension* of an algebraic manifold X at a given point $x \in X$; it is denoted by $\dim_x X$. Certainly, $X_n = X$, if X is irreducible. If X is reducible, $\dim_x X$ coincides with the maximal dimension of irreducible components passing through x.

Exercice 13.18. Show that $\dim_p X = \dim_p U$ for any affine chart $U \ni p$.

Hint. Let $X_1, X_2 \subset X$ be two closed irreducible subsets and $U \subset X$ be an open set such that both $X_1 \cap U$, $X_2 \cap U$ are nonempty. Then $X_1 = X_2 \iff X_1 \cap U = X_1 \cap U$, because of $X_i = \overline{X_i \cap U}$.

Claim 13.19. *Let $X \xrightarrow{\varphi} Y$ be a finite morphism of irreducible manifolds, $x \in X$, and $y = \varphi(x) \in Y$. Then $\dim_x X \leqslant \dim_y Y$ and the equality holds iff $\varphi(X) = Y$.*

Proof. By the above exercise we can suppose X, Y to be affine. Then Claim 12.17 implies that each chain (34) in X produces a chain $\cdots \subsetneq \varphi(X_i) \subsetneq \varphi(X_{i+1}) \subsetneq \cdots$ of closed irreducible submanifolds in Y. Wise versa, if $\varphi(X) = Y$, then given a chain (34) in Y, for each i we can choose an irreducible component of $\varphi^{-1}(X_i)$ mapped surjectively onto X_i. This gives a chain (34) in X. $\square$

Corollary 13.20. $\dim_p \mathbb{A}^n = n$ *at any* $p \in \mathbb{A}^n$.

Proof. Clearly, $\dim \mathbb{A}^0 = 0$. Suppose inductively that $\dim \mathbb{A}^{n-1} = n - 1$. Since any proper closed $X \subset \mathbb{A}^n$ has a finite projection on $\mathbb{A}^{n-1}$, Claim 13.19 implies $\dim_p X \leqslant (n-1)$ for any p, i. e. $\dim_p \mathbb{A}^n \leqslant n$. On the other hand, there is a chain (34) of affine subspaces passing through p, so $\dim_p \mathbb{A}^n \geqslant n$. $\square$

Corollary 13.21. *Let X be an irreducible affine manifold and $X \xrightarrow{\varphi} \mathbb{A}^m$ be a surjective finite morphism. Then $\dim_p X = m$ at each $p \in X$. In particular, m doesn't depend on a choice of φ and $\dim_p X$ is the same for all $p \in X$.* □

Exercice 13.22. Prove that $\dim(X \times Y) = \dim X + \dim Y$ for any irreducible manifolds X, Y.

Exercice 13.23. Let $V(f) \subset \mathbb{A}^n$ be given by irreducible $f \in k[x_1, x_2, \dots, x_n]$. Show that $\dim V(f) = (n-1)$.
Hint. Find a surjective finite projection $V(f) \longrightarrow \mathbb{A}^{n-1}$ (comp. with Exercice 13.16 and Corollary 13.17).

Claim 13.24. *If X is irreducible, then $\dim_p V(f) = \dim_p(X) - 1$ for any non constant $f \in \mathcal{O}_X(X)$ $\forall p \in V(f)$.*

Proof. We can assume that X is affine. Fix some finite surjection $X \xrightarrow{\pi} \mathbb{A}^m$ and consider, as in the proof of Claim 12.18, the induced map $\varphi = \pi \times f : X \xrightarrow{x \mapsto (\pi(x), f(x))} \mathbb{A}^m \times \mathbb{A}^1$. It maps X finitely and surjectively onto affine hypersurface $V(\mu_f) \subset \mathbb{A}^m \times \mathbb{A}^1$ given by the minimal polynomial $\mu_f(u,t) = t^n + \alpha_1(u)\,t^{n-1} + \dots + \alpha_n(u) \in k[u_1, u_2, \dots, u_m][t]$, of f over $k(\mathbb{A}^m)$, and $V(f) = \varphi^{-1}\left(H \cap V(\mu_f)\right)$, where $H \subset \mathbb{A}^m \times \mathbb{A}^1$ is the hyperplane $t = 0$. Inside $H = \mathbb{A}^n$, the intersection $H \cap V(\mu_f)$ is a hypersurface given by the equation $\alpha_n(u) = 0$. So, we have a finite surjection $V(f) \xrightarrow{\varphi|_{V(f)}} V(a_n) \subset \mathbb{A}^n$ and can apply Exercice 13.23 and Claim 13.19. □

Corollary 13.25. *For any X and $f \in k[X]$ the inequality $\dim_p V(f) \geqslant \dim_p(X) - 1$ holds at any $p \in V(f)$.* □

Claim 13.26. *Let $X_1, X_2 \subset \mathbb{A}^n$ be closed submanifolds and $x \in X_1 \cap X_2$. Then $\dim_x(X_1 \cap X_2) \geqslant \dim_x(X_1) + \dim_x(X_2) - n$.*

Proof. Write $X_1 \xhookrightarrow{\varphi_1} \mathbb{A}^n$, $X_2 \xhookrightarrow{\varphi_2} \mathbb{A}^n$ for the closed embeddings. Then $X_1 \cap X_2$ is the preimage of the diagonal $\Delta \subset \mathbb{A}^n \times \mathbb{A}^n$ under the map $X_1 \times X_2 \xhookrightarrow{\varphi_1 \times \varphi_2} \mathbb{A}^n \times \mathbb{A}^n$. So, it is given inside $X_1 \times X_2$ by the pull backs of n equations $x_i = y_i$ defining Δ inside $\mathbb{A}^n \times \mathbb{A}^n$. It remains to apply Corollary 13.25. □

Corollary 13.27. *If $\dim(X_1) + \dim(X_2) \geqslant n$ for some closed $X_1, X_2 \subset \mathbb{P}_n$, then $X_1 \cap X_2 \neq \varnothing$.*

Proof. Let $\mathbb{P}_n = \mathbb{P}(V)$. Consider in $\mathbb{A}^{n+1} = \mathbb{A}(V)$ affine cones[2] X_1', X_2'' consisting of lines passing through the origin $O \in \mathbb{A}^{n+1}$ and forming X_1 and X_2 By the previous claim $\dim_O(X_1' \cap X_2'') \geqslant \dim_O(X_1) + 1 + \dim_O(X_2) + 1 - n - 1 \geqslant 1$. So, $X_1' \cap X_2''$ is not exhausted by O. □

Theorem 13.28. *Let $X \xrightarrow{\varphi} Y$ be a dominant morphism of irreducible manifolds. Then $\dim_x \varphi^{-1}(\varphi(x)) \geqslant \dim X - \dim Y$ for any $x \in X$ and there exists non empty open $U \subset Y$ such that $\dim \varphi^{-1}(y) = \dim X - \dim Y$ for all $y \in U$.*

Proof. Projecting finitely an affine chart containing $\varphi(x)$ onto $\mathbb{A}^m$, we reduce the first part of the theorem to the case $Y = \mathbb{A}^m = \mathit{Spec}_m\, k[u_1, u_2, \dots, u_m]$, $\varphi(x) = 0$. Then $\varphi^{-1}(0)$ is an intersection of m hypersurfaces $V(\varphi^*(u_i))$ in X and the required inequality follows inductively from Corollary 13.25. For the second part, we can suppose that $X = \mathit{Spec}_m A$, $Y = \mathit{Spec}_m B$ are affine and φ is the projection $Y \times \mathbb{A}^m \xrightarrow{\pi} Y$ restricted onto closed submanifold $X \subset Y \times \mathbb{A}^m$ (see the decomposition (32) from Exercice 12.16). We are going to apply Corollary 13.17 fiberwise over Y. Consider the closure $\overline{X} \subset Y \times \mathbb{P}_m$ and choose a hyperplane $H \subset \mathbb{P}_m$ and a point $p \in \mathbb{P}_m \setminus H$ such that the section $Y \times \{p\} \subset Y \times \mathbb{P}_m$ is not contained in $\overline{X}$. The fiberwise projection from p onto H is well defined over an open subset $U \subset Y$ complementary to $\overline{\pi}\Big(\left(Y \times \{p\}\right) \cap \overline{X}\Big)$, where $Y \times \mathbb{P}_m \xrightarrow{\overline{\pi}} Y$ is the projection along $\mathbb{P}_m$. So, replacing Y by its open subset, we get a finite morphism $X \longrightarrow Y \times \mathbb{A}^{m-1}$. After a number of such replacements we can suppose that φ is a finite surjection $X \twoheadrightarrow Y \times \mathbb{A}^n$ followed by the projection $Y \times \mathbb{A}^n \twoheadrightarrow Y$; it clearly has fibers of dimension $n = \dim X - \dim Y$. □

Corollary 13.29. [3] *$\forall k \in \mathbb{Z}$ sets $X_k \overset{\text{def}}{=} \{ x \in X \mid \dim_x \varphi^{-1}(\varphi(x)) \geqslant k \}$ are closed in X for any morphism $X \xrightarrow{\varphi} Y$ of algebraic manifolds.*

Proof. We can suppose that X, Y are irreducible. If $k \leqslant \dim(X) - \dim(Y)$, then $X_k = X$ by the above theorem. For $k > \dim(X) - \dim(Y)$ we can replace Y by $Y' = Y \setminus U$, where U is the same as in Theorem 13.28, and X — by $X' = \varphi^{-1}(Y')$. Clearly, $X_k \subset X'$ and we can repeat the arguments decreasing the dimensions of X, Y. □

Exercice 13.30. Show that isolated points of fibers of a morphism $X \xrightarrow{\varphi} Y$ form an open subset in X.

Corollary 13.31. *If $X \xrightarrow{\varphi} Y$ is closed, than $Y_k \overset{\text{def}}{=} \{ y \in Y \mid \dim \varphi^{-1}(y) \geqslant k \}$ is closed in Y $\forall k \in \mathbb{Z}$.* □

Corollary 13.32. *Let $X \xrightarrow{\varphi} Y$ be a closed morphism with irreducible fibers of the same dimension. If Y is irreducible, then X is irreducible as well.*

Proof. Let $X = X' \cup X''$. Since each fiber $\varphi^{-1}(y)$ is irreducible, it completely belongs to one of X', X''. Applying Corollary 13.31 to the restriction $X' \xrightarrow{\varphi|_{X'}} Y$, we see that a set of all fibers completely contained in X' is mapped to some closed subset $Z' \subset Y$. Similarly, all fibers completely contained in X'' are also mapped to some closed $Z'' \subset Y$. So, $Y = Z' \cup Z''$ but both Z', Z'' should be proper as soon as X', X'' were proper. □

[2] they have the same equations as X_1, X_2 but these equations are considered now as affine rather than homogeneous
[3] it is known as *Chevalley's theorem on semi-continuity*