Plane curves: Correspondences and Chasles - Cayley - Brill formula[1]

Blow up of $\Delta \subset \mathbb{P}_2 \times \mathbb{P}_2$. Identify $\mathbb{P}_2^{\times}$ with the set of lines on $\mathbb{P}_2$ and consider the incidence graph

$$\mathcal{B} \overset{\text{def}}{=} \{(p,q,\ell) \mid p,q \in \ell\} \subset \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^{\times}$$

It is given by two quadratic equations $\sum \vartheta_\nu x_\nu = \sum \vartheta_\nu y_\nu = 0$ on $(x,y,\vartheta) \in \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^{\times}$. Topologically, $\mathcal{B}$ is a 4-dimensional compact manifold. A projection $\mathcal{B} \xrightarrow{\sigma} \mathbb{P}_2 \times \mathbb{P}_2$ is bijective outside the diagonal $\Delta \overset{\text{def}}{=} \{(p,p)\} \subset \mathbb{P}_2 \times \mathbb{P}_2$. Each fiber $\sigma^{-1}(p,p) = \{(p,p,\ell) \mid \ell \ni p\}$ over $(p,p) \in \Delta$ is naturally identified with the line pencil through p on $\mathbb{P}_2$ and 3-dimensional submanifold $E \overset{\text{def}}{=} \sigma^{-1}(\Delta) \subset \mathcal{B}$ is called *an exceptional divisor*. Let us denote the projections of $\mathcal{B}$ onto consequent factors of $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^{\times}$ by π_1, π_2, π_3 and write $A_1 = \pi_1^{-1}(\lambda)$, $A_2 = \pi_2^{-1}(\lambda)$, $M = \pi_3^{-1}(\lambda)$ for the full preimages of a generic line λ living on these planes. Topologically, A_1, A_2, and M are 3-dimensional cycles on $\mathcal{B}$ and their homology classes don't depend on the choice of the line λ in each plane. Any 1-parametric algebraic family of «pointed» lines $(pq) \subset \mathbb{P}_2$ can be pictured by an algebraic curve $\Gamma \subset \mathcal{B}$. Topological location of such a curve is described by a triple of numbers:

$$\begin{aligned}
\alpha_1 &= \#(\Gamma \cap A_1) && \text{— a number of } p\text{-points in } \Gamma = \{(p,q,\ell)\} \text{ laying on a generic line } \lambda \subset \mathbb{P}_2;\\
\alpha_2 &= \#(\Gamma \cap A_2) && \text{— a number of } q\text{-points in } \Gamma = \{(p,q,\ell)\} \text{ laying on a generic line } \lambda \subset \mathbb{P}_2;\\
\mu &= \#(\Gamma \cap M) && \text{— a number of lines } \ell \text{ in } \Gamma = \{(p,q,\ell)\} \text{ passing through a generic point}^2\ \lambda^{\times} \in \mathbb{P}_2.
\end{aligned}$$

Strongly speaking, we should use the topological intersection indices instead of «the numbers of points». But for all Γ we will consider below there is an open dense set of lines[3] such that all corresponding A_1, A_2, M intersect Γ transversally in a finite constant number of points. We will always suppose that α_1, α_2, μ are calculated using A_1, A_2, M taken from these open dense sets[4]. The triples $(p,q,\ell) \in \Gamma$ with $p = q$, i. e. the intersection points $\Gamma \cap E$, are called *exceptional*. Typically, Γ has a finite number of exceptional points. Our goal is to equip the exceptional points with appropriate multiplicities and express the number $\xi(\Gamma)$ of «exceptional points counted with multiplicities» through α_1, α_2, μ in the following three examples.

Example 11.1: join family. Let $C_1, C_2 \subset \mathbb{P}_2$ be two curves of degrees d_1, d_2 without common components. Fix any point $u \in \mathbb{P}_2$ outside the both curves and all the lines joining pairs of their intersection points. Then

$$\Gamma = \{(p,q,\ell) \in \mathcal{B} \mid p \in C_1,\ q \in C_2,\ \ell \ni u\}$$

is a curve in $\mathcal{B}$ given by an obvious triple of algebraic equations. Its exceptional points are $(p,p,(pu))$ with $p \in C_1 \cap C_2$, that is $\xi(\Gamma)$ counts the intersections of C_1 and C_2. Further, $\alpha_1 = \alpha_2 = d_1 d_2$, because a generic line λ intersect, say C_1, in d_1 distinct points $p_1, p_2, \dots, p_{d_1}$ and for each of them $C_2 \cap (up_\nu)$ consist of d_2 points. Finally, $\mu = d_1 d_2$ too, because there is a unique line $\ell = (u\lambda^{\times})$ passing through a given $\lambda^{\times}$ and this line contains $d_1 d_2$ distinct pairs (p_i, q_j) with $p_i \in C_1 \cap \ell$, $q_j \in C_2 \cap \ell$, if $\lambda^{\times}$ is general enough.

Example 11.2: secant family. In the above example, let $C_1 = C_2 = C$ be the same irreducible curve of degree d given by an equation $F(x) = 0$ and u be a point outside the curve, all its singular tangents, and all lines joining pairs of distinct singularities. Then a closure of $\{(p,q,\ell) \in \mathcal{B} \mid p \neq q,\ p,q \in C,\ \ell \ni u\}$ is a curve given by the equations $F(x) = F(y) = \vartheta(u) = 0$ in $(x,y,\vartheta) \in \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^{x}$. Exceptional points of this curve are $(p,p,(pu))$ such that $\mathsf{mult}(C,(pu))_p \geqslant 2$, i. e. either $(pu) = T_pC$ or $p \in Sing(C)$. So, ξ counts singular points of C and proper tangents coming from c to C. By the same reasons as above, $\alpha_1 = \alpha_2 = \mu = d\,(d-1)$ in this case.

Example 11.3: tangent family. Let C be as above and Γ be a closure of $\{(p,q,\ell) \in \mathcal{B} \mid p \neq q,\ p,q \in C,\ \ell = T_pC\}$ (it is given by the equations $F(x) = F(y) = \widetilde{F}(x^{d-1}, y) = 0$). Exceptional points of Γ are (p,p,ℓ) such that $\mathsf{mult}(C,\ell)_p \geqslant 3$, i. e. either inflection tangents at smooth p or geometric tangents at singular p. So, $\xi(\Gamma)$ counts inflections and singularities. Since a simple tangent at a smooth point p intersect the curve in $(d-2)$ more points q, we have $\alpha_1 = d\,(d-2)$. Clearly, $\mu = c\,(d-2)$, because there are c proper tangents to C through generic $\lambda^{\times} \in \mathbb{P}_2$ by the definition of class. Finally, $\alpha_2 = d\,(c-2)$, because a generic line intersects C in d distinct smooth points $q_1, q_2, \dots, q_d$ and for each of them there

[1] WARNING!!! In this lecture we suppose that the ground field k is algebraically closed and has zero characteristic.

[2] here $\lambda \subset \mathbb{P}_2^{\times}$ is the pencil of lines through a point $\lambda^{\times} \in \mathbb{P}_2$

[3] in the space of all lines

[4] it will be an exercise for readers, to check the existence of such open sets of lines in each of examples below

are $(c-2)$ proper tangents $(q_j p)$ touching C at some $p \neq q_j$: when a point $q \notin C$ tends to some $q_j \in C$, precisely 2 of c tangents through q turn to $T_{q_j}C$ (see fig. 11.1; other arguments will appear in Example 11.8 and Exercice 11.9).

A correspondence on $\mathbb{P}_1$ is called *algebraic of type m-n*, if the pairs of corresponding points $(p,q) \in \mathbb{P}_1 \times \mathbb{P}_1$ form an algebraic curve $\gamma \subset \mathbb{P}_1 \times \mathbb{P}_1$ given by an irreducible bihomogeneous polynomial $g(t',t'')$ of bidegree (m,n) in $(t',t'') = \big((t'_0 : t'_1)\,,\,(t''_0 : t''_1) \big)$. This curve is called *a graph* of the correspondence. So, *images* of a given point $p \in \mathbb{P}_1$ are presented by the equation $g(p,t) = 0$ in $t \in \mathbb{P}_1$ and *preimages* — by the equation $g(t,p) = 0$. Since these equations have respectively m and n ordinary distinct roots for almost all[5] p, a generic point has n distinct images and m distinct preimages under an m-n correspondence.

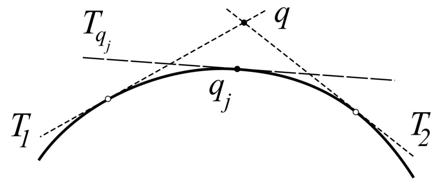

Fig. 11.1. Two proper tangent lines disappear as $q \to q_j$.

A point $e \in \mathbb{P}_1$ is called *a fixed point* of the correspondence, if it corresponds to itself[6], that is, if $g(e,e) = 0$. So, the set of all fixed points is $\gamma \cap \Delta$, where $\Delta = \{(t,t)\} \subset \mathbb{P}_1 \times \mathbb{P}_1$ is the diagonal. Since $g(t,t)$ is homogeneous of degree $m+n$ in $t = (t_0,t_1)$, any algebraic m-n correspondence has $m+n$ fixed points counted with multiplicities, where the multiplicity of a fixed point e means the local intersection multiplicity $(\gamma,\Delta)_{(e,e)}$, of the curve $g(t',t'') = 0$ and the line $t' = t''$ at the point $(e,e) \in \gamma$. In particular, it can be calculated by the Zeuthen rule applied in any affine chart $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}_2 \ni (e,e)$ as it was explained in **lecture 10**.

Claim 11.4. *Let $U \subset \mathbb{P}_1$ be an affine chart with the origin at a fixed point e of a correspondence γ, x be an affine coordinate on U, and $y_1(x)$, $y_2(x)$, …, $y_m(x)$ be all γ-images of x which tend to 0 as $x \to 0$. Then $(\gamma,\Delta)_{(0,0)}$ equals the sum of orders of infinitesimals $y_i(x) - x$ with respect to x.*

Proof. Let (x,y) be affine coordinates on $\mathbb{A}^2 = U \times U$ (see fig. 11.2). Since the both lines $x = 0$ and $y = 0$ contain just one intersection point $(0,0) \in \gamma \cap \Delta$ we can use the line pencil $x = \text{const}$ parameterized by the x-axis to calculate $(\gamma,\Delta)_{(0,0)}$ as the sum of orders of infinitesimals $\alpha_i(x) - \beta_j(x)$ where α_i and β_j run trough the intersections of a vertical line $x = \text{const}$ respectively with γ and with Δ. So, $\alpha_i(x) = y_i(x)$ and there is just one $\beta(x) = x$. □

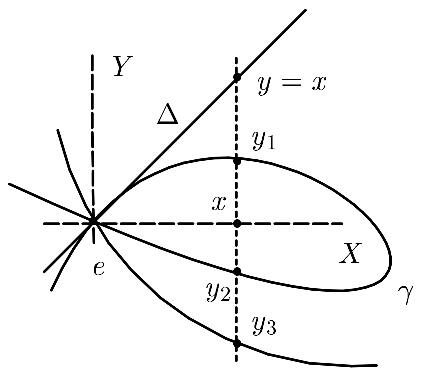

Fig. 11.2. Fixed point.

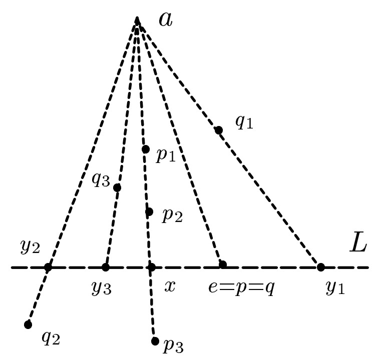

Fig. 11.3. Exceptional point.

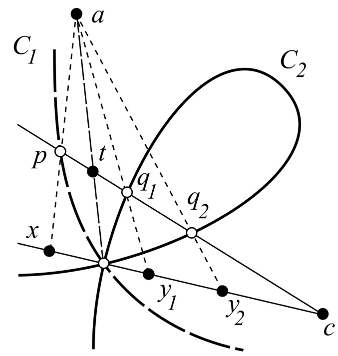

Fig. 11.4. Intersection point.

Exceptional point multiplicities. A curve $\Gamma \subset \mathcal{B}$, of pointed lines (p,q,ℓ), defines an algebraic correspondence on $\mathbb{P}_1$ as follows. Fix a point $a \in \mathbb{P}_2$ such that it lies on precisely μ distinct lines ℓ of Γ and a generic line through a contains exactly α_1 distinct p-points and exactly α_2 distinct q-points of Γ. Then consider the pencil of lines through a as $\mathbb{P}_1$ in question and say that $(ap) \longleftrightarrow (aq)$ iff $(p,q,\ell) \in \Gamma$ for some[7] ℓ. This is an algebraic α_1-α_2 correspondence, because a generic point has α_2 images and α_1 preimages, certainly. A line through a corresponds to itself under γ_Γ precisely in two cases: either it belongs to Γ, i. e. contains 2 points $p \neq q$ such that $(p,q,\ell) \in \Gamma$, or it pass through an exceptional point e such that $(e,e,\ell) \in \Gamma$ for some ℓ.

Let us define the multiplicity of an exceptional point $(e,e,\ell) \in \Gamma$ as a multiplicity of the corresponding fixed point (ae) of the correspondence γ_Γ. By Claim 11.4, it can be calculated geometrically as follows. Parameterize the pencil of lines through a by some line $L \not\ni a$, which pass through an exceptional point e (see fig 11.3), and fix on L an affine coordinate x centered at e. Let (ay_1), (ay_2), …, (ay_m), where $y_\nu \in L$, be all lines corresponding to (ax) and tending to (ae) as $x \to e$. Then the multiplicity of (pe) equals the sum of orders of infinitesimals $y_i(x) - x$ w. r. t. x as $x \to 0$.

Claim 11.5. *A total number of exceptional points counted with multiplicities equals[8] $\xi(\Gamma) = \alpha_1 + \alpha_2 - \mu$.*

[5] that is, for all p outside some finite subset on $\mathbb{P}_1$ where the both discriminants vanish
[6] of course, besides itself, a fixed point may have several other (pre) images as well
[7] of course, $\ell = (pq)$, if $p \neq q$
[8] this formula is known as *the Chasles - Cayley - Brill formula*

Proof. Since $\deg(\gamma_\Gamma) = (\alpha_1, \alpha_2)$, it has $\alpha_1 + \alpha_2$ fixed points. By the choice of a, each line (ap) such that $(p, q, (ap)) \in \Gamma$ for some $q \neq p$, $q \in (ap)$ has multiplicity 1 as a fixed point for γ_Γ. The residuary contribution of exceptional fixed points equals $\xi(\Gamma)$. $\square$

Example 11.6: exceptional point multiplicities in join family (continuation of Example 11.1). In this case the multiplicity of an exceptional point $(e, e, (ce)) \in \Gamma$ coincides with the local intersection number $(C_1, C_2)_e$ (see fig. 11.4). Indeed, use the line pencil centered at c to compute $(C_1, C_2)_e$ by the Zeuthen rule as it was explained in the previous lecture. If we take a outside all geometric tangents to the both curves at e and such that e is the only intersection point of C_1 and C_2 on (ae), then $(C_1, C_2)_e$ is a sum of orders $p - q_j$ w.r.t. t as $t \to 0$. But it is clear from fig. 11.4 that $p - q_j$ is like $x - y_j$ and t is like x as soon the both lines (ce) and (ae) do not touch the branches of C_1, C_2 at e. So, $\xi(\Gamma)$ is the sum of local intersection numbers of C_1 and C_2, i.e. $(C_1, C_2) = \xi(\Gamma) = \alpha_1 + \alpha_2 - \mu = d_1 d_2 + d_1 d_2 - d_1 d_2 = d_1 d_2$. We get the Bézout theorem.

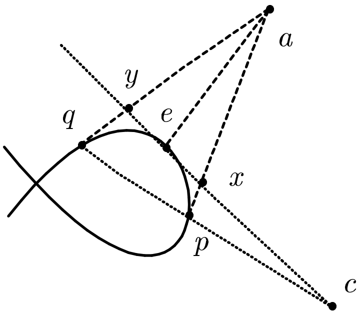

Fig. 11.5.

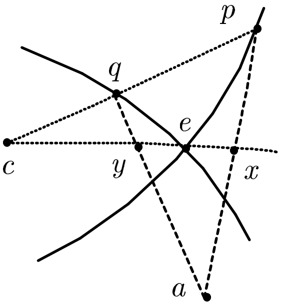

Fig. 11.6.

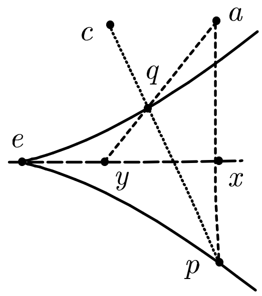

Fig. 11.7.

Example 11.7: exceptional point multiplicities in secant family (continuation of Example 11.2). There are 3 types of exceptional points here. A proper tangency $(e, e, (ce))$ (see fig. 11.5) has multiplicity 1, because any x closed to e has a unique image y coming to e when $x \to e$ and $y - x$ is like $x - e$. If e is an ordinary m-typle selfintersection, then a line through c closed to (ce) contains m p-points running through m branches of C; each such p-point produces $(m-1)$ q-points coming from other $(m-1)$ branches; so, there are $m(m-1)$ differences $y - x$ and each of them is like $(x - e)$ (see fig. 11.6) as soon (ce) does not touch any branch. If e is an ordinary cusp (see fig. 11.7, where the line pencil through a is parameterized by the cuspidal tangent), then any x closed to e produces two p-points (intersections of (ax) with two branches of C) and each of them has just one q-point (intersection of (cp) with the other branch of C); it is easy to see from fig. 11.7 that $(x - y) \sim (p - q) \sim (x - e)^{3/2}$ as $x \to e$. So, the cuspidal point contributes $2 \cdot (3/2) = 3$. Hence,

$$c + \sum_{\nu}^{\delta} m_\nu(m_\nu - 1) + 3\varkappa = \xi(\Gamma) = \alpha_1 + \alpha_2 - \mu = d(d-1) + d(d-1) - d(d-1) = d(d-1)$$

by the Chasles - Cayley - Brill formula. This gives the first Plücker identity.

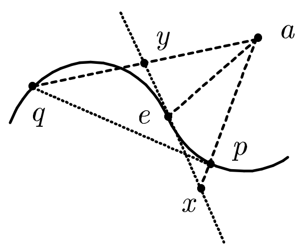

Fig. 11.8.

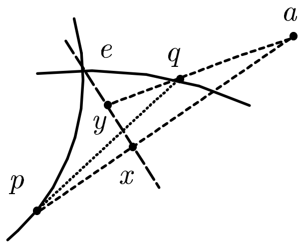

Fig. 11.9.

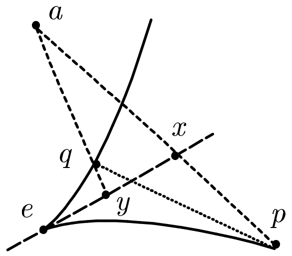

Fig. 11.10.

Example 11.8: exceptional point multiplicities in tangent family (continuation of Example 11.3). The figs. 11.8–11.10 show that each difference $(y - x)$ is like $(x - e)$ for all three types of exceptional points. An inflection point (fig. 11.8) produces a unique difference and has a multiplicity 1; each of m branches through an m-typle selfintersection (fig. 11.9) produces $(m-1)$ differences, so the multiplicity equals $m(m-1)$ here; a cusp (fig. 11.10) produces 2 differences and has multiplicity 2. Hence,

$$\iota + \sum_{\nu}^{\delta} m_\nu(m_\nu - 1) + 2\varkappa = \xi(\Gamma) = \alpha_1 + \alpha_2 - \mu = d(d-2) + d(c-2) - c(d-2) = d^2 - 4d + 2c$$

by the Chasles - Cayley - Brill formula. Since $c = d(d-1) - \sum m_\nu(m_\nu - 1) - 3\varkappa$, we get the second Plücker identity $\iota = 3d(d-2) - 3\sum m_\nu(m_\nu - 1) - 8\varkappa$.

Exercice 11.9. Let c be a smooth point of C in Example 11.2 and Example 11.7. Check that a multiplicity of an exceptional point $(c, c, T_cC) \in \Gamma$ equals 2 (so, we have really $\alpha_2 = d(c-2)$ in Example 11.3).