Plane curves: singularities, intersections, and Plücker relations[1]

GEOMETRICAL TANGENTS AT SINGULARITY. Let $C \subset \mathbb{P}_2$ be a curve given by an equation $F(x) = 0$ of degree d and $p \in C$ be a (singular) point of multiplicity $m \geqslant 2$. Then all the polars $C_p^{(\nu)}$ (which would be given by equations[2] $\widetilde{F}(p^{d-\nu}, x^\nu) = 0$) are trivial for $0 \leqslant \nu \leqslant (m-1)$ and m-th degree polar $C_p^{(m)}$ (given by $\widetilde{F}(p^{d-m}, x^m) = 0$) is non trivial but singular: its Taylor expansion near p

$$\widetilde{F}(p^{d-m}, (p+tq)^m) = \sum_{\mu=1}^{m} t^\mu \binom{m}{\mu} \widetilde{F}(p^{d-m+\mu}, q^\mu) = t^m\, \widetilde{F}(p^{d-m}, q^m)$$

contains just one term and a line (p,q) either is a component of $C_p^{(m)}$ (when $\widetilde{F}(p^{d-m}, q^m) = 0$) or intersects $C_p^{(m)}$ only at p with multiplicity m (when $F(p^{d-m}, q^m) \neq 0$). So, $C_p^{(m)}$ splits into union of m lines (pq_i), where q_i are the roots of $\widetilde{F}(p^{d-m}, q^m) = 0$ considered as degree m equation on q, where q runs through any fixed line $\ell \not\ni p$. (Of course, some of (pq_i) may coincide when the roots became multiple.) The lines (pq_i) are called *geometrical* tangent lines to C at p.

Geometrically, generic line (pq), through p, intersects C at p with multiplicity m, because the Taylor expansion $F(p+tq) = \binom{d}{m} \cdot t^m \cdot \widetilde{F}(p^{d-m}, q^m) + \cdots$ starts with non-zero m-th degree term. The geometric tangents (pq_i) are the lines whose intersection multiplicity with C at p jumps w. r. t. the generic value. Algebraically, this means that $\widetilde{F}(p^{d-m}, x^m) = \xi_1(x)\, \xi_2(x) \cdots \xi_m(x)$ is the product of m linear forms $\xi_1, \xi_2, \ldots, \xi_s$ whose zeros are the geometric tangents (pq_i) (again, some of them may coincide).

Example 10.1: the simplest singularities. Given a curve $C \subset \mathbb{P}_2$, an m-typle point $p \in C$ is called *an m-typle node* (or *an m-typle selfintersection*) if there are m distinct geometrical tangents $\ell_1, \ell_2, \ldots, \ell_m$ to C through p. Geometrically, this means that C has m mutually transversal branches through p. The difference $(\ell_i, C)_p - m - 1$ is called *an order of the contact* between ℓ_i and the corresponding branch of C. A node is called *ordinary* if all the geometrical tangents have the second order contacts with its branches, that is $(\ell_i, C)_p = m + 1\ \forall\, i$.

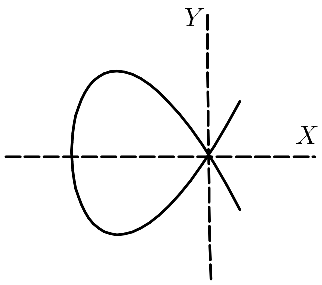

Fig. 10.1. The node $y^2 = x^2(x+1)$.

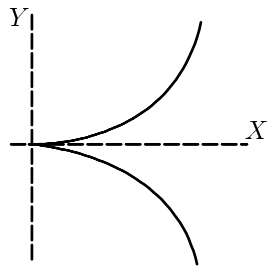

Fig. 10.2. The cusp $y^2 = x^3$.

A double point $p \in C$ is called *a cusp* (or *a selfcontact*) if the quadratic polar of p is a double line ℓ. Geometrically, this means that C has two branches which do touch each other at p. The unique geometrical tangent ℓ at p is called *a cuspidal tangent*. A local intersection number $(\ell, C)_p$ measures *an order of the selfcontact* for C at p; clearly, $(\ell, C)_p \geqslant 3$. A cusp is called *ordinary*, if $(\ell, C)_p = 3$ is minimal possible.

We say that C has *only the simplest singularities*, if the singular points of C are exhausted by ordinary double nodes and ordinary cusps. Two cubic curves with the simplest singularities are shown on the figs. 10.1–10.2. For higher degree curves, the neighborhood of the simple singularity looks similarly[3].

Exercice 10.2. Show that irreducible cubic curve has at most one (automatically simple) singularity.
HINT. A line ℓ has to be a component of a cubic C as soon as $(\ell, C) \geqslant 4$.

Example 10.3: how much is to put a singularity on a curve? Given a point $p \in \mathbb{P}_2$, then the polar map

$$S^d V^* \xrightarrow{F \mapsto \partial^{d-m} F / \partial p^{d-m}} S^m V^*$$

[1] WARNING!!! In this lecture we suppose that the ground field k is algebraically closed and has zero characteristic.

[2] recall that $\widetilde{F}(p^{d-\nu}, x^\nu) = \mathrm{pl}(F)(\underbrace{p, p, \ldots, p}_{d-\nu}, \underbrace{x, x, \ldots, x}_{\nu}) = \dfrac{\nu!}{d!} \dfrac{\partial^{(d-\nu)} F}{\partial p^{(d-\nu)}}(x) = \dfrac{(d-\nu)!}{d!} \dfrac{\partial^\nu F}{\partial x^\nu}(p)$

[3] one can show that any smooth curve in $\mathbb{P}_n$ admits a plane projection that has only the simplest singularities

is a linear epimorphism. Hence the curves whose m-degree polar coincides with a given collection of lines through p form a projective subspace of codimension $\dim(S^m V^*) - 1 = m(m+3)/2$. For example, 5 parameters in a curve equation are fixed by assuming that this curve has at a given point a cusp with a given cuspidal tangency. However, these restrictions, if come from several distinct points, are not independent, in general.

<u>Affine neighborhood of a singularity.</u> Practical computation of geometric tangents usually becomes simpler in affine chart with the origin in the singular point in question. Let C have an affine equation $f(x,y)=0$ in such a chart U. Write it as $\sum\limits_{\nu>0} f_\nu(x,y)=0$, where each $f_\nu(x,y)$ is homogeneous of degree ν, and consider a line $\ell_{\alpha:\beta}$ given parametrically as[4] $x=\alpha t\,,\ y=\beta t$. Then $C\cap\ell$ is given by the following equation on t:

$$f_m(\alpha,\beta)\,t^m + f_{m+1}(\alpha,\beta)\,t^{m+1} + \cdots + f_d(\alpha,\beta)\,t^d = 0$$

where m is the degree of lowest non trivial homogeneous component of f and each $f_\nu(\alpha,\beta)$ is actually nothing but the ν-th degree polar of p evaluated at $q=(\alpha:\beta)$. Thus, the multiplicity of p coincides with the degree of lowest non trivial homogeneous component of f and the directions $(\alpha:\beta)$ of the geometrical tangents through p are the roots of this component, i.e. satisfy the equation $\varphi_m(\alpha,\beta)=0$.

<u>Example 10.4</u>: analyzing singularities. Taking $x=\alpha t$, $y=\beta t$ in the nodal cubic equation $y^2-x^2-x^3=0$, we get the lowest term $(\beta+\alpha)(\beta-\alpha)\,t^2$, which vanishes for $(\alpha:\beta)=(1:\pm1)$; so, there are two distinct geometrical tangents $x=\pm y$. Local intersection number $(\ell,C)_p=3$ for each tangent line ℓ, i.e. each tangent has the second order contact with its branch. The cuspidal cubic on the 10.2 has the lowest term $\beta^2\,t^2$. So, the second polar is a the double line $x=0$ with local intersection 3 with C at the origin.

As an advanced example, consider a quartic given by the polynomial $F(t)=t_0^4-t_0^3t_1+t_0^2t_2^2-t_1^2t_2^2$. Its singularities are $t\in\mathbb{P}_2$ where all partial derivatives

$$\partial F/\partial t_0 = 4\,t_0^3-3\,t_0^2t_2-2\,t_0t_2^2\,,\quad \partial F/\partial t_1=-2\,t_1t_2^2\,,\quad \partial F/\partial t_2=-t_0^3+2\,t_0^2t_2-2\,t_1^2t_2$$

vanish simultaneously. It happens at two points $a=(0:0:1)$ and $b=(0:1:0)$. Take an affine chart with $x=t_0/t_2\,,\ y=t_1/t_2$ near a. Then $F=0$ turns into $x^2-y^2-x^3+x^4=0$ with two simple geometrical tangents $x=\pm y$ at the origin. Since a local intersection number equals 3 for each tangency, a is an ordinary node. Taking a chart with $x=t_0/t_1\,,\ y=t_2/t_1$ near b, we get the equation $y^2-x^4+x^3y-x^2y^2$ whose geometrical tangent is a double line $\ell=\{x=0\}$ with $(\ell,C)_p=4$. So, b is the non ordinary cusp, where C has a selfcontact of order 4.

<u>Blow up.</u> Geometrically, the substitution $x=\alpha t$, $y=\beta t$ lifts C from $\mathbb{P}_2$ to a surface $\Gamma\subset\mathbb{P}_1\times\mathbb{P}_2$ called *a blow up* of $p\in\mathbb{P}_2$. It is described as follows. Identify a pencil of lines through p with any fixed line $\mathbb{P}_1=(ab)\not\ni p$ and consider the incidence graph $\Gamma \overset{\text{def}}{=} \{(\ell,q)\in\mathbb{P}_1\times\mathbb{P}_2 \mid \ell\ni q\}$. It is an algebraic surface in $\mathbb{P}_1\times\mathbb{P}_2$: if we put $p=(1:0:0)$, $a=(0:1:0)$, $b=(0:0:1)$, take $q=\alpha a+\beta b$, and consider $((\alpha:\beta),(x_0:x_1:x_2))$ as coordinates on $\mathbb{P}_1\times\mathbb{P}_2$, then $(x_0:x_1:x_2)\in(pq)$ is equivalent to the quadratic relation $\alpha x_2=\beta x_1$.

Projection $\Gamma \xrightarrow{\sigma} \mathbb{P}_2$ is bijective outside p, but $\sigma^{-1}(p)\simeq\mathbb{P}_1$ is the pencil of lines through p on $\mathbb{P}_2$ (see fig. 10.3). A map $((\alpha:\beta)\,,\,t)\longmapsto((\alpha:\beta)\,,\,(1:\alpha t:\beta t))\in\mathbb{P}_1\times\mathbb{P}_2$ gives a rational parameterization for some affine neighborhood of this exceptional fiber in Γ. Full preimage $\sigma^{-1}(C)$, of a curve $C\subset\mathbb{P}_2$ passing through p, consists of 2 components: $\sigma^{-1}(p)\simeq\mathbb{P}_1$ and and some curve whose equation (in terms of parameters $(\alpha:\beta;t)$ on Γ) is a result of the substitution $x=\alpha t$, $y=\beta t$ in the affine equation for C.

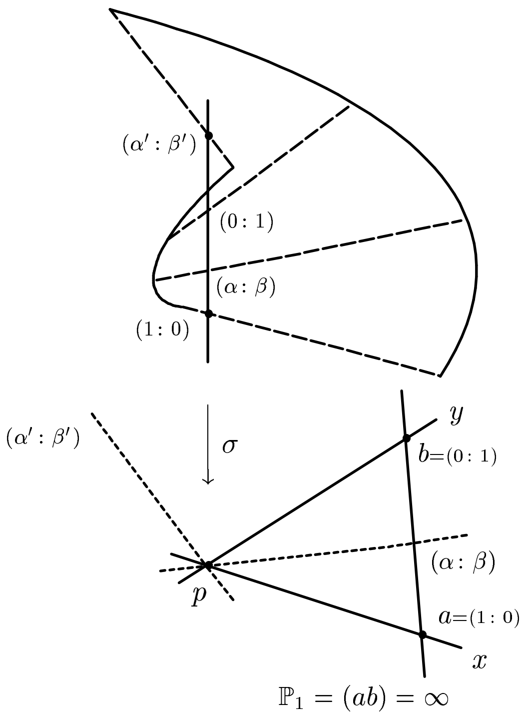

Fig. 10.3. Blow up.

<u>Local intersection multiplicity of two plane curves.</u> Consider two curves $C_1,C_2\subset\mathbb{P}_2$ given by homogeneous equations $F=0$, $G=0$ of degrees n, m without common divisors. Let $u\in C_1\cap C_2$. We fix two points p, v such that the line (pu) satisfies the conditions:

$$(pu)\cap C_1\cap C_2=\{u\} \qquad \& \qquad v\notin(pu) \qquad\qquad (28)$$

[4] as in the beginning of this lecture, line $\ell_{\alpha:\beta}$ has a form $(p+tq)$, where $q=(\alpha:\beta)\in U_\infty\simeq\mathbb{P}_1$ is runing through the infinite projective line of the chart U

and will use the triple $\{p, u, v\}$ as a basis for $\mathbb{P}_2$. Let a point $q(x) = u + x\,v$ tend to u (as $x \to 0$) along the line (pu). We restrict the both curves onto a varying line $\ell(x) = (p, q(x))$ and write $\alpha_0(x), \dots, \alpha_n(x)$ for the points $C_1 \cap \ell(x)$ and $\beta_0(x), \dots, \beta_m(x)$ for the points $C_2 \cap \ell(x)$ (see fig. 10.4). These points are the roots of two homogeneous polynomials $f_x(t_0, t_1) = F(t_0 p + t_1 q(x))$, $g_x(t_0, t_1) = G(t_0 p + t_1 q(x))$ in $t = (t_0 : t_1)$ with coefficients depending on the parameter x. When x varies, the pints $\alpha_\nu(x)$, $\beta_\mu(x)$ draw the branches of C_1 and C_2. Let $\beta_1(x), \dots, \beta_r(x)\,;\ \alpha_1(x), \dots, \alpha_s(x)$ be all the branches that come to u as $x \to 0$.

Natural geometric definition[5] of local intersection multiplicity $(C_1, C_2)_u$ puts it to be equal to the sum of orders of rs infinitesimals $\alpha_i(x) - \beta_j(x)$ w.r.t. x as $x \to 0$. Since the sum of orders of $\alpha_i(x) - \beta_j(x)$ coincides with the order of zero of the resultant R_{f_x, g_x} (considered as a polynomial in x) at $x = 0$, we can treat $(C_1, C_2)_u$ pure algebraically as the multiplicity of x in the prime factorization of R_{f_x, g_x} in $k[x]$. Such the definition is correct (i.e. does not depend on a choice of p, v satisfying (28)), because of the following

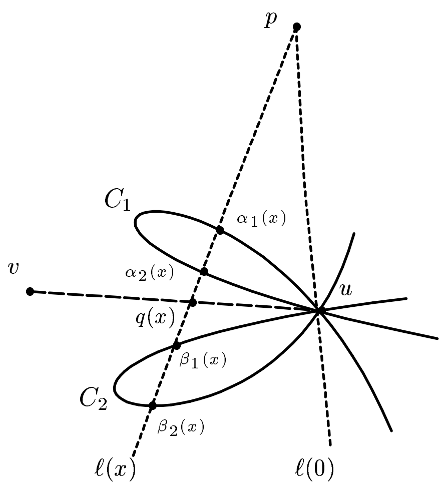

Fig. 10.4. Zeuthen's rule.

Lemma 10.5. *Consider the resultant $R_{f,g}(p, q) \in k[p, q]$, of two binary forms $f(t_0, t_1) = F(t_0 p + t_1 q)$, $g(t_0, t_1) = G(t_0 p + t_1 q)$, as a polynomial in $p = (p_0, p_1, p_2)$, $q = (q_0, q_1, q_2)$. Then its irreducible factorization in $k[p, q]$ has a form*[6]*:*

$$R_{f,g}(p, q) = const \cdot \prod_{w \in C_1 \cap C_2} \det \begin{pmatrix} w_0 & w_1 & w_2 \\ p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix}^{m_w} \tag{29}$$

where the multiplicities m_w can be calculated by the Zeuthen rule with any choice of p, v satisfying (28).

Proof. Denote the determinants in the right side of (29) by $D_w(p, q)$. Geometrically, $D_w(p, q) = 0$ is an irreducible quadric, which consists of all pairs $p, q \in \mathbb{P}_2$ whose joining line (pq) belongs to the pencil of lines through w. If $w \in C_1 \cap C_2$, then $R(p, q)$ vanishes along the quadric $D_w(p, q) = 0$. Hence, by Hilbert's Nullstellensatz, any such $D_w(p, q)$ divides the resultant. Wise verse, if $R(p, q) = 0$, then the restrictions of F, G onto the line (p, q) have a common root, i.e. (p, q) pass through some $w \in C_1 \cap C_2$ and $\prod\limits_{w \in C_1 \cap C_2} D_w(p, q)$ vanishes at this (p, q). So, this product vanishes everywhere along $V(R(p, q))$ and, again by Hilbert, $R(p, q)$ should divide some power of $\prod\limits_{w \in C_1 \cap C_2} D_w(p, q)$. To check the Zeuthen rule, fix p and $q = q(x)$ as on the fig. 10.4 and note that in (29) only $D_w(p, q(x))$ with $w = u$ vanishes at $x = 0$, because of (28); and this one $D_u(p, q(x)) = \det(u, p, u + x\,v) = x\,\det(u, p, v)$ is proportional to x. $\square$

Intersection theory of plane curves. It follows immediately from the Zeuthen rule, that the local intersection multiplicities are distributive w.r.t. the curve branches, that is if C_1 has b_1 branches passing through u and C_2 has b_2 ones, then $(C_1, C_2)_u$ is the sum of $b_1 b_2$ mutual intersection indices between the branches[7]. On the other side, since each $D_w(p, q)$ in (29) is bilinear in (p, q) and $R_{f,g}(p, q)$ has bidegree (mn, mn), we get *the Bézout theorem*:

Theorem 10.6. $\sum\limits_{w \in C_1 \cap C_2} (C_1, C_2)_w = \deg C_1 \cdot \deg C_2$ *for any two plane projective curves without common components.* $\square$

Example 10.7: proper tangents and class. A tangent lane is called *proper*, if its tangency point is smooth. A number of proper tangents to C passing through a generic point $q \in \mathbb{P}_2$ is called *a class* of C and denoted by $c = c(C)$. If $\deg C = d$, then by Corollary 9.5 the tangents coming form a point $q \in \mathbb{P}_2 \setminus \mathit{Sing}(C)$ touch C at the points of $C \cap C_q^{(d-1)}$, where $C_q^{(d-1)}$ is $(d-1)$-th degree polar of q. If C is irreducible, then $C \cap C_q^{(d-1)}$ consists of $d(d-1)$ points[8] counted with multiplicities. Besides the proper tangency points, $C \cap C_q^{(d-1)}$ contains also all singular points of C, because each line trough a singularity is (non proper) tangent. So, class of irreducible curve satisfies inequality $c \leqslant d(d-1)$, which turns to equality iff C is smooth.

Example 10.8: inflections. A smooth point $p \in C$ is called *an inflection*, if $(C, T_pC)_p \geqslant 3$. An inflection is called *ordinary* (or *simple*), if this number equals 3. If $p \in C$ is an inflection, then the quadratic polar $C_p^{(2)}$ of p has the zero restriction onto $\ell = T_pC$, i.e. ℓ is a component of $C_p^{(2)}$. Note that p is a smooth point of the conic $C_p^{(2)}$, because $C_p^{(2)}$ and C have the same linear polar w.r.t. p and p is smooth on C. So, $p \in C$ is an inflection iff $C_p^{(2)} = \ell \cup \ell'$ with $\ell \cap \ell' \neq p$. The points $q \in \mathbb{P}_2$

[5] known as *Zeuthen's rule*

[6] geometrically, $R_{f,g}(p, q) = 0$ defines in the space of lines (pq) a figure whose irreducible components are pencils of lines centered at the intersection points and the multiplicities of these components are predicted by the Zeuthen rule

[7] in particular, $(n_1 C_1 + n_2 C_2, D) = n_1 (C_1, D) + n_2 (C_2, D)$, where $m_1 C_1 + m_2 C_2$ is a curve given by equation $F_1^{m_1} F_2^{m_2} = 0$ and $F_1 = 0$, $F_2 = 0$ are the equations for C_1 and C_2

[8] Note that this is not true in positive characteristic: for example, if char $(k) = 2$, then *all* tangents to the smooth conic $x_0^2 = x_1 x_2$ pass through one common point.

with degenerate quadratic polar $C_q^{(2)}$ form a curve H_C, which is called *the Hessian* of C. It is defined by the equation $\det C_q^{(2)} = 0$, which has degree $3(d-2)$ in q, where $d = \deg C$. Hence, an irreducible curve of degree $d \geqslant 3$ has at most $3d(d-2)$ inflections, which are contained in $C \cap H_C$. Again, this intersection contains also all singular points[9] of C.

Example 10.9: affine localization. Let us restrict the picture 10.4 onto affine chart where (pv) is the infinity, (uv) is the x-axis, and (up) is the y-axis. Then the line pencil through p turns to the family of vertical lines $x = \text{const}$. Consider affine equations $f(x,y) = 0$, $g(x,y) = 0$ for C_1, C_2 as (nonhomogeneous) polynomials in y with the coefficients in $k[x]$. Their resultant $R_{f,g}(x)$ is a polynomial in x and vanishes at $x = 0$. The multiplicity of this zero root coincides with $(C_1, C_2)_{(0,0)}$. If there are known some explicit analytic expressions of all the branches $y = \alpha_i(x)$ and $y = \beta_j(x)$ through x (even not algebraic, say several starting terms of the (formal fractional) power series expansions are OK), then $(C_1, C_2)_{(0,0)}$ usually can be also computed explicitly by looking at either the orders of $\alpha_i(x) - \beta_j(x)$ or the order of the resultant.

Dual curve Let $C \subset \mathbb{P}_2$ be an irreducible curve given by an equation $F(x) = 0$ of degree d. For any smooth $p \in C$ its tangent $\tau_p = T_pC$ defines a point $\tau_p^* = \operatorname{Ann}\tau_p \in \mathbb{P}^\times$ on the dual plane $\mathbb{P}_2^\times$. When p varies along C, τ_p^* also is running through some curve $C^\times \subset \mathbb{P}_2^\times$ called *a dual curve* for C. The degree of dual curve, i.e. the number of its intersection points with a generic line $\vartheta = q^\times \subset \mathbb{P}_2^x$, is nothing but the number of proper tangents to C living in a generic pencil of lines (centered at a generic point $q \in \mathbb{P}_2$). Thus $\deg(C^\times) = c(C)$.

Claim 10.10. *$C^{\times\times} = C$ (in particular, $\deg(C) = c(C^\times)$).*

Proof. A tangency $\vartheta = T_{\tau_1^*}C^\times \subset \mathbb{P}_2^\times$, at a smooth point $\tau_1^* \in C^\times$, is a limit of secant lines $\sigma = (\tau_1^*, \tau_2^*)$ as $\tau_2^* \to \tau_1^*$ (see figs. 10.5–10.6). The secant σ represents a pencil of lines through $\sigma^* = \tau_1 \cap \tau_2 \in \mathbb{P}_2$. Clearly, $\sigma^* \to p_1$ as $p_2 \to p_1$. $\square$

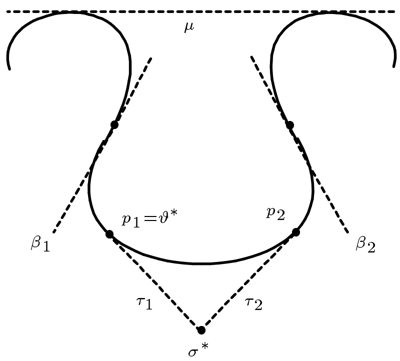

Fig. 10.5. A curve $C \subset \mathbb{P}_2$.

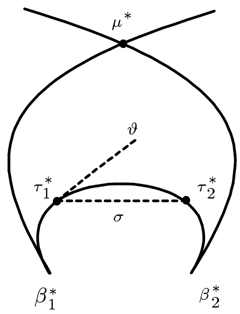

Fig. 10.6. The dual curve $C^\times \subset \mathbb{P}_2^\times$.

The figs. 10.5–10.6 also show that selfcontacts (cusps) turns into inflections and selfintersections — to *multiple tangents*[10] under the duality $C \longleftrightarrow C^\times$. In particular, if C has at most the simple singularities, then special proper tangents of $C^\times$ are exhausted by simple inflections and double tangents.

Plücker identities. Let C be a curve of class c and degree d with singularities exhausted by δ ordinary selfintersections of multiplicities $m_1, m_2, \dots, m_n$ and $\varkappa$ ordinary cusps. Then

$$c = d\,(d-1) - 3\,\varkappa - \sum_{\nu=1}^{\delta} m_\nu (m_{\nu-1}) \tag{30}$$

If we assume, in addition, that C has only ordinary inflections, then their number $\iota = \iota(C)$ equals

$$\iota = 3\,d\,(d-2) - 8\,\varkappa - 3\sum_{\nu=1}^{\delta} m_\nu (m_{\nu-1}) \tag{31}$$

This formulas are known as *the Plücker identities*. We shell prove them in the next lecture using a geometrical approach traced back to Chasles, Cayley and Brill[11]

Exercice 10.11. Let $q \notin C$ lie neither on an inflection tangency nor on a geometric tangency through a singular point of C; we write $C_q^{(d-1)}$ for $(d-1)$-th degree polar of q with respect to C. Deduce (30) from the equality $(C, C_q^{(d-1)}) = d(d-1)$ by proving that $(C, C_q^{(d-1)})_p$ equals 1 for smooth p, equals 3, if p is an ordinary cusp, and equals $m(m-1)$, if p is an ordinary m-type selfintersection.

[9]In fact, even for singular curves one can write precise equations between degree, class, number of inflections and some data describing singularities; we'll do this below for curves with simplest singularities.

[10]a proper tangency is called *multiple*, if it touch the curve in several distinct points

[11]We follow the book: J. G. Semple, L. Roth, *Introduction to algebraic geometry*, Oxford, 1949.

HINT. In the first case p is smooth on $C_q^{(d-1)}$ and $T_pC_q^{(d-1)} \neq T_pC$; in the second case p is smooth on $C_q^{(d-1)}$ again, but $T_pC_q^{(d-1)}$ coincides with the cuspidal tangency; in the third case p is an $(m-1)$-typle point on $C_q^{(d-1)}$, but each geometrical tangency of C at p is transversal to $C_q^{(d-1)}$, that is, intersects it with multiplicity $(m-1)$. Now, use the Zeuthen rule.

If both C and $C^{\times}$ have at most the simple singularities, then the Plücker relations written for the both curves turn into

$$c = d\,(d-1) - 3\,\varkappa - 2\,\delta \qquad \iota = 3\,d\,(d-2) - 8\,\varkappa - 6\,\delta \qquad d = c\,(c-1) - 3\,\iota - 2\,\beta \qquad \varkappa = 3\,c\,(c-2) - 8\,\iota - 6\,\beta$$

where β is a number of bitangents to C. Any three of d, c, $\varkappa$, δ, β, ι can be found from these equations as soon as the other three are known.