Working example: Veronese curves[1]

<u>LINEAR SPAN OF VERONESE CURVE.</u> Recall (see Example 5.14) that the Veronese curve $C_n \subset \mathbb{P}_n$ is the image of Veronese's map

$$\mathbb{P}_1 = \mathbb{P}(U^*) \overset{v_n}{\hookrightarrow} \mathbb{P}_n = \mathbb{P}(S^n U^*) \ ,$$

which takes a linear form $\alpha = \alpha_0 t_0 + \alpha_1 t_1 \in U^*$ to its n-th power:

$$v_n(\alpha) = \alpha^n = \sum \binom{n}{i} \ \alpha_0^i \alpha_1^{n-i} \, t_0^i t_1^{n-i} \ .$$

As in Example 5.14, we write polynomials $f \in S^n U^*$ in the form $f = \sum \binom{n}{i} \ a_i \, t_0^i t_1^{n-i}$ and use a_i as homogeneous coordinates on $\mathbb{P}_n = \mathbb{P}(S^n U^*)$.

Given an arbitrary hyperplane $\pi = \{a \in \mathbb{P}_n \mid \sum A_i a_i = 0\}$, the intersection $C_n \cap \pi$ consists of all $a = a(\alpha)$ that satisfy the equation $\sum\limits_i A_i \alpha_0^i \alpha_1^{n-i} = 0$ whose left side is non zero polynomial of degree n. So, a hyperplane section of Veronese's curve always consists[2] of n points counted with appropriate multiplicities (typically, of n *distinct* points). In particular, the linear span of any $(n+1)$ Veronese's curve points gives the whole of $\mathbb{P}_n$.

<u>Corollary 7.1.</u> [3] *To prove that some linear in f assertion holds for all polynomials f, it is enough to verify it only for all powers of all linear forms.* □

Exercice 7.2. Use Arongold principle to give another proof for Taylor formula (21).

<u>Example 7.3: Projecting twisted cubic.</u> Let us describe all plane projections of the twisted cubic $C_3 \subset \mathbb{P}_3 = \mathbb{P}(S^3 U^*)$. Up to a projective isomorphism, the projection does not depend on the choice of a target plane as soon as the center is fixed, because the projection of one target plane onto another gives an linear isomorphism between the projection images. Let $p = p(t) = \ell_1(t)\ell_2(t)\ell_3(t) \in \mathbb{P}_3 = \mathbb{P}(S^3 V^*)$ be a projection center. After some parameter change we can suppose that either $\ell_1 = \ell_2 = \ell_3 = t_0^2$, (this means that $p \in C$), or $\ell_1 = \ell_2 = t_0$, $\ell_3 = t_1$ ([4]), or $\ell_1 = (t_0 + t_1)$, $\ell_2 = (t_0 + \omega\, t_1)$, $\ell_3 = (t_0 + \omega^2 t_1)$, where $\omega = \sqrt[3]{1} \neq 1$ (that is, $p(t) = t_0^3 + t_1^3$ has 3 distinct roots).

In the first case $p = (1:0:0:0)$; take a target plane to be $a_0 = 0$ with the coordinates $(x_0 : x_1 : x_2) = (a_1 : a_2 : a_3)$. Then the projection is given by parametric equations $(x_0 : x_1 : x_2) = (\alpha_0^2 : \alpha_0\alpha_1 : \alpha_1^2)$ and coincides with the plane Veronese conic $x_0 x_2 = x_1^2$.

In the second case $p = (0:1:0:0)$; take a target plane to be $a_1 = 0$ with the coordinates $(x_0 : x_1 : x_2) = (a_0 : a_2 : a_3)$. Then the projection is given by parametric equations $(x_0 : x_1 : x_2) = (\alpha_0^3 : \alpha_0\alpha_1^2 : \alpha_1^3)$ and in the affine chart $\{x_0 = 1\}$ it turns into $x = \alpha^2$, $y = \alpha^3$, where $x = x_1/x_0$, $y = x_2/x_0$, and $\alpha = \alpha_1/\alpha_0$. So, we get a curve $y^2 = x^3$ or, in the homogeneous coordinates, $x_2^3 = x_3^2 x_0$. This curve is called *a cuspidal cubic*, because of the singularity form at the origin.

In the third case $p = (1:0:0:1)$; take a target plane $\pi = \{a_2 = 0\}$ with the coordinates $(x_0 : x_1 : x_2) = ((a_0 - a_1) : a_1 : a_2)$ (the first three coordinates w.r.t. the base $\{ t_0^3 , 3\, t_0^2 t_1 \, , \, 3\, t_0 t_1^2 \, , \, t_0^3 + t_1^3 \})$. So, the projection from $p = t_0^3 + t_1^3$ gives the parameterized curve $(x_0 : x_1 : x_2) = \big((\alpha_0^3 - \alpha_1^3) : \alpha_0^2\alpha_1 : \alpha_0\alpha_1^2\big)$. In affine chart $x_0 = 1$ we get, like above, $x = \alpha/(1-\alpha^3)$, $y = \alpha^2/(1-\alpha^3)$ or $xy = x^3 - y^3$. This curve has a self intersection point at the origin and is called *a nodal* cubic.

<u>GEOMETRIC DESCRIPTION OF RATIONAL CURVES.</u> A plane curve $C \subset \mathbb{P}_2$ is called *rational* if there are 3 coprime homogeneous polynomials $p_0(t)$, $p_1(t)$, $p_2(t)$ of the same degree in $t = (t_0 : t_1)$ such that a map $\mathbb{P}_1 \xrightarrow{\alpha \mapsto (p_0(\alpha):p_1(\alpha):p_2(\alpha))} \mathbb{P}_2$ gives (maybe, after removing some finite sets of points from $\mathbb{P}_1$ and C) a bijection between $\mathbb{P}_1$ and C .

Exercice 7.4. Intersecting C with lines, show that $\deg C = \deg p_i$.

When $\deg p_i = d$, a map $(t_0^d : t_0^{d-1} t_1 : \ldots : t_0 t_1^{d-1} : t_1^d) \longmapsto (p_0(t) : p_1(t) : p_2(t))$ defines a projection of the Veronese curve $C_d \subset \mathbb{P}_d$ into some plane $\mathbb{P}_2 \subset \mathbb{P}_d$. So, we have

<u>Claim 7.5.</u> *Each rational plane curve of degree d is an appropriate projection of the Veronese curve $C_d \subset \mathbb{P}_d$.* □

<u>Corollary 7.6.</u> *A smooth plane cubic curve is not rational.*

[1] WARNING!!! In this lecture we suppose that the ground field k is algebraically closed and has zero characteristic.

[2] recall that we suppose the ground field to be algebraically closed

[3] it is known as *Arongold principle*

[4] geometrically, this means that p lies on a tangent line to C_3

Proof. Rational cubic curve is a plane projection of the twisted cubic $C_3 \subset \mathbb{P}_3$. But such a projection is either a conic or a singular cubic. $\square$

Simplices inscribed into Veronese curve. For $1 \leqslant i \leqslant n$ let $p_i = \xi_i^n$ be any n distinct points on the Veronese curve $C_n \subset \mathbb{P}_n = \mathbb{P}(S^n U^*)$. For each i consider the pencil of hyperplanes passing through $(n-2)$-dimensional face $(p_1, \dots p_{i-1}, p_{i+1}, \dots, p_n)$ opposite to p_i in the $(n-1)$ dimensional simplex $(p_1, p_2, \dots, p_n)$. These n pencils are parameterized uniformly by the points of $\mathbb{P}_1^\times = \mathbb{P}(U)$ as follows. For any $\xi \in \mathbb{P}(U^*)$ denote by $\widehat{\xi} \in \mathbb{P}(U)$ the annihilator[5] $\mathrm{Ann}\,(\xi)$ and for each i take the product $\tau_i = \widehat{\xi}_1, \cdots, \widehat{\xi}_{i-1}\widehat{\xi}_{i+1}, \cdots, \widehat{\xi}_1 \in S^{n-1}U$. Define a plane $\pi_i(u) \subset \mathbb{P}(S^n U^*)$, which corresponds to $u \in \mathbb{P}(U)$ in i-th pencil, as the annihilator of $u\tau_i \in S^n U$. This means that

$$\pi_i(u) = \left\{ f(t) \in S^n U^* \mid \widetilde{f}(\widehat{\xi}_1, \dots, \widehat{\xi}_{i-1}, u, \widehat{\xi}_{i+1}, \dots, \widehat{\xi}_n) = 0 \right\}$$

where $\widetilde{f}$ is the full polarization of f considered as a symmetric multilinear form on U. In particular, for $f(t) = \zeta^n(t) \in C_n$ we have

$$(\zeta^n)^\sim (\widehat{\xi}_1, \dots, \widehat{\xi}_{i-1}, u, \widehat{\xi}_{i+1}, \dots, \widehat{\xi}_n) = \zeta(u) \prod_{\nu \neq i} \zeta\left(\widehat{\xi}_\nu\right)$$

So, for any $u \in \mathbb{P}(U)$ the plane $\pi_i(u)$ pass through all $p_\nu = \xi_\nu^n$ with $\nu \neq i$ and through the point $p = \xi^n \in C_n$ whose ξ annihilate u (i. e. such that $u = \widehat{\xi}$). In other words,

$$C_n = \bigcup_u \pi_1(u) \cap \pi_2(u) \cap \dots \cap \pi_n(u) \tag{25}$$

Since a linear isomorphism between projective lines is uniquely defined by the images of any 3 distinct points, we get the following corollary.

Claim 7.7. *The Veronese curve is uniquely recovered from any its $(n+3)$ distinct points $a, b, c, p_1, p_2, \dots, p_n$ as follows. Consider n hyperplane pencils through the $(n-2)$-dimensional faces of the inscribed simplex $(p_1, p_2, \dots, p_n)$ and parameterize them uniformly by $u \in \mathbb{P}_1$ in such a way that the hyperplanes passing through a, b, c appear in each pencil when $u = 0, 1, \infty$. Then C_n coincides with the incidence graph (25), where u runs through the parameter line $\mathbb{P}_1$.* $\square$

Projective linear group action. The group $PGL_2(k)$ acts on $\mathbb{P}(S^n U^*)$ via linear variable changes $(t_0, t_1) \longmapsto (at_0 + bt_1, ct_0 + dt_1)$. Projective automorphisms $\mathbb{P}(S^n U^*) \xrightarrow{\sim} \mathbb{P}(S^n U^*)$ which are induced by these substitutions send the Veronese curve to itself. We call them *reparameterizations* of the Veronese curve.

Claim 7.8. *Let $p_1, p_2, \dots, p_n, a, b, c \in \mathbb{P}_n = \mathbb{P}(S^n U^*)$ be any $n+3$ points with no $(n+1)$ on the same hyperplane. Then there exists a projective linear isomorphism $\mathbb{P}_n \xrightarrow{\sim} \mathbb{P}_n$ that sends these points onto Veronese curve C_n; this isomorphism is unique up to a reparameterization of the Veronese curve.*

Proof. For each $i = 1, 2, \dots, n$ identify $\mathbb{P}_1 = \mathbb{P}(U)$ with a pencil of hyperplanes through $p_1, \dots p_{i-1}, p_{i+1}, \dots, p_n$ by sending $u = e_0, e_1, (e_1 - e_0)$ to the hyperplanes that contain a, b, c and denote by $\pi_i(u)$ the u-th hyperplane in the i-pencil. Let the hyperplane $(p_1, p_2, \dots, p_n)$ appear in i-th pencil when $u = u_i$. Then $u_1, u_2, \dots, u_n \in \mathbb{P}(U)$ are mutually distinct by the following reason. Consider 2-dimensional plane $\Pi = (a, b, c)$ and denote by q_1 and q_2 its intersection with i-th and j-th $(n-2)$-dimensional faces of $(p_1, p_2, \dots, p_n)$. Sitting in this plane, we will see the picture shown on 7.1. Our i-th and j-th pencils of hyperplanes are represented inside Π by the pencils of lines passing through q_1, q_2 and the hyperplane $(p_1, p_2, \dots, p_n)$ is represented by the line $(q_1 q_2)$ (compare this construction with drawings from the **lecture 2**).

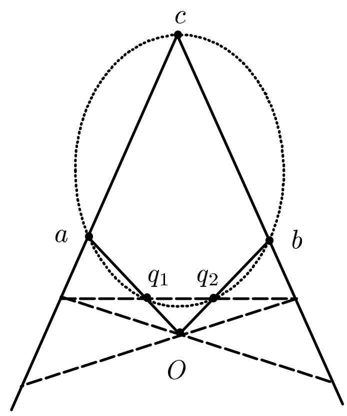

Fig. 7.1. Comparing u_i and u_j.

Exercice 7.9. Show that no 3 of q_1, q_2, a, b, c are collinear (in particular, q_1, q_2 are distinct).

Hint. In the contrary case some $(n+1)$ of $p_1, p_2, \dots, p_n, a, b, c$ would lie on a hyperplane

There are two ways to identify the parameter line $\mathbb{P}(U)$ with the pencil of lines passing through $O = (aq_1) \cap (bp_2)$: we can send u either to the line through $\pi_i(u) \cap (bc)$ or to the line through $\pi_j(u) \cap (ac)$. These parameterizations should coincide, because they attach the same u to the lines through a, b, c. The lines which correspond to $u = u_i$ and $u = u_j$ join O with $\pi_i(u_i) \cap (bc) = (q_1 q_2) \cap (bc)$ and with $\pi_j(u_j) \cap (ac) = (q_1 q_2) \cap (ac)$. Since O, q_1 and q_2 are not collinear, these 2 lines are distinct, i. e. $u_i \neq u_j$.

Now, denote by Γ the incidence graph (25) build from our current pencils of hyperplanes. Let $u_i = \mathrm{Ann}\,(\xi_i)$ for $\xi_1, \xi_2, \dots, \xi_n \in U^*$. By the Lemma 1.11, there exists a unique projective linear automorphism $\mathbb{P}_n \xrightarrow{\sim} \mathbb{P}_n$ which sends

[5] if $\xi = \alpha_0 t_0 + \alpha_1 t_1$, then $\widehat{\xi} = \alpha_1 e_0 - \alpha_0 e_1$, where $\{e_o, e_1\} \subset U$ is the base of U dual to $\{t_0, t_1\} \subset U^*$

$p_i \longmapsto \xi_i^n$ for $1 \leqslant i \leqslant n$, $a \longmapsto t_1^n$, and $b \longmapsto t_0^n$. It identifies Γ with the Veronese curve, because it sends the hyperplane pencil through $p_1, \dots p_{i-1}, p_{i+1}, \dots, p_n$ to the one through $\xi_1^n, \dots \xi_{i-1}^n, \xi_{i+1}^n, \xi_n^n$ in such a way that $\pi_i(u)$ goes to the hyperplane through ξ^n as soon as $\xi = \operatorname{Ann} u$. Indeed, this takes place for $u = e_1$, $u = e_0$ and $u = u_i$ when a, b and p_i go to t_0^n, t_1^n and ξ_i^n. Hence, this holds for each u and for $u = e_1 - e_0$ we have $c \longmapsto (t_0 + t_1)^n$. This proves the existence.

Uniqueness follows from the above construction as well. Namely, after appropriate parameter change we can suppose that the isomorphism in question sends a, b, c to t_0^n, t_1^n and $(t_0 + t_1)^n$. So, it induces the uniform parameterization of hyperplane pencils trough p_ν and this parameterization coincides with the above one. So, the images of $p_1, p_2, \dots, p_n$ are uniquely recovered as $\xi_i = \operatorname{Ann}(u_i)$. $\square$