Working example: Grassmannians

PLÜCKER QUADRIC IN $\mathbb{P}_5$. Let V be a 4-dimensional vector space and $\mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$. Then the *Plücker quadric*

$$Q_P \stackrel{\text{def}}{=} \{\omega \in \Lambda^2 V \mid \omega \wedge \omega = 0\}$$

is a non singular quadric in $\mathbb{P}_5$. Fixing a base $\{e_0, e_1, e_2, e_3\}$ for V and induced base $e_{ij} = e_i \wedge e_j$ for $\Lambda^2 V$ and writing x_{ij} for the homogeneous coordinate along e_{ij}, we have $\left(\sum\limits_{i<j} x_{ij} \cdot e_i \wedge e_j\right) \wedge \left(\sum\limits_{i<j} x_{ij} \cdot e_i \wedge e_j\right) = 2\,(x_{01}x_{23} - x_{02}x_{13} + x_{03}x_{12}) \cdot e_0 \wedge e_1 \wedge e_2 \wedge e_3$, i.e. Q_P has the equation

$$x_{02}x_{13} = x_{01}x_{23} + x_{03}x_{12} \ .$$

In coordinateless terms, Q_P is given by quadratic form $q(\omega) = \widetilde{q}(\omega, \omega)$ whose polarization $\widetilde{q}(\omega_1, \omega_2)$ is a bilinear form on $\Lambda^2 V$ defined up to a scalar factor by the prescription

$$\omega_1 \wedge \omega_2 = \widetilde{q}(\omega_1, \omega_2) \cdot \Omega \ ,$$

where $\Omega \in \Lambda^4 V \simeq k$ is any fixed non-zero vector. This form is symmetric, because even degree Grassmannian polynomials commute: $\omega_1 \wedge \omega_2 = \omega_2 \wedge \omega_1$.

PLÜCKER EMBEDDING. By the definition, *a grassmannian* $\mathrm{Gr}(2,4)$ is a set of all lines $\ell \subset \mathbb{P}_3 = \mathbb{P}(V)$, or equivalently, a set of all 2-dimensional vector subspaces $U \subset V$. *A Plücker map*

$$\mathrm{Gr}(2,4) \xhookrightarrow{\mathfrak{u}} \mathbb{P}(\Lambda^2 V)$$

sends a 2-dimensional subspace $U \subset V$ to the 1-dimensional subspace $\Lambda^2 U \subset \Lambda^2 V$. If U is spaned by a pair of vectors u_1, u_2 (i.e. $\ell = \mathbb{P}(U)$ pass trough $u_1, u_2 \in \mathbb{P}(V)$), then $\mathfrak{u}(\ell) = \mathfrak{u}(U) = u_1 \wedge u_2$ up to proportionality.

Lemma 6.1. *Two lines $\ell_1, \ell_2 \subset \mathbb{P}_3$ are intersecting iff* $\widetilde{q}(\mathfrak{u}(\ell_1), \mathfrak{u}(\ell_2)) = \mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell_2) = 0$.

Proof. Let $\ell_1 = \mathbb{P}(U_1)$, $\ell_2 = \mathbb{P}(U_2)$. If $U_1 \cap U_2 = 0$, then $V = U_1 \oplus U_2$ and there exist a base $\{e_i\} \subset V$ such that U_1 is spaned by e_0, e_1 and U_2 is spaned by e_2, e_3. So, $\mathfrak{u}(U_1) = e_0 \wedge e_1$, $\mathfrak{u}(U_2) = e_2 \wedge e_3$ and $\mathfrak{u}(U_1) \wedge \mathfrak{u}(U_2) = e_0 \wedge e_1 \wedge e_2 \wedge e_3 \neq 0$. If $U_1 \cap U_2 \neq 0$, then taking $u_0 \subset U_1 \cap U_2$ we can write $\mathfrak{u}(U_1) = u_0 \wedge u_1$, $\mathfrak{u}(U_2) = u_0 \wedge u_2$ for some u_1, u_2. So, $\mathfrak{u}(U_1) \wedge \mathfrak{u}(U_2) = u_0 \wedge u_1 \wedge u_0 \wedge u_2 = 0$. □

Lemma 6.2. *If* $\dim V = 4$, *then $\omega \in \Lambda^2 V$ is decomposable*[1] *iff* $\omega \wedge \omega = 0$.

Proof. If ω is decomposable, say $\omega = u_1 \wedge u_2$, then $\omega \wedge \omega = u_1 \wedge u_2 \wedge u_1 \wedge u_2 = 0$, because of skew symmetry. To get the inverse, take a base $\{\xi_i\}$ such that ω turns into either $\omega = \xi_0 \wedge \xi_1 + \xi_2 \wedge \xi_3$ or $\omega = \xi_0 \wedge \xi_1$. If $\omega \wedge \omega = 0$, then the first case is impossible, because of $(\xi_0 \wedge \xi_1 + \xi_2 \wedge \xi_3) \wedge (\xi_0 \wedge \xi_1 + \xi_2 \wedge \xi_3) = 2\,\xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 \neq 0$. □

Corollary 6.3. *The Plücker map is a bijection between the grassmannian* $\mathrm{Gr}(2,4)$ *and the Plücker quadric* $Q_P \subset \mathbb{P}_5$.

Proof. $\mathfrak{u}$ is injective, because for any two lines $\ell_1 \neq \ell_2$ on $\mathbb{P}_3$ there exists a third line ℓ which intersect ℓ_1 and doesn't intersect ℓ_2. Then $\mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell) = 0$ and $\mathfrak{u}(\ell_2) \wedge \mathfrak{u}(\ell) \neq 0$, so, $\mathfrak{u}(\ell_1) \neq \mathfrak{u}(\ell_2)$. Surjectivity follows from Lemma 6.2. □

Corollary 6.4. *For any point $p = \mathfrak{u}(\ell) \in Q_P$ the intersection $Q_P \cap T_p Q_P$ consists of all $\mathfrak{u}(\ell')$ such that $\ell \cap \ell' \neq \varnothing$.*

Proof. $T_p Q_P$ is a zero set of the linear form $\widetilde{q}(\mathfrak{u}(\ell), *)$. But by Lemma 6.1, $\widetilde{q}(\mathfrak{u}(\ell), \mathfrak{u}(\ell')) = 0 \iff \ell \cap \ell' \neq \varnothing$. □

LINE NETS AND LINE PENCILS IN $\mathbb{P}_3$. A set of lines on $\mathbb{P}_3$ is called *a net* if it is represented by a plane $\pi \subset Q_P \subset \mathbb{P}_5$. If $\pi \subset Q_P$ is spaned by 3 non collinear points $p_i = \mathfrak{u}(\ell_i)$, $i = 1, 2, 3$, then

$$\pi = Q_P \cap T_{p_1} Q_P \cap T_{p_2} Q_P \cap T_{p_3} Q_P \ .$$

So, by Lemma 6.1 and Corollary 6.4 the corresponding line net consist of all lines which intersect 3 given pairwise intersecting lines. Hence, there are exactly two geometrically different line nets on $\mathbb{P}_3$:

α-net is a set of all lines passing through a given point $O \in \mathbb{P}_3$; the corresponding plain $\pi_\alpha(O) \subset Q_P$ is called *α-plane*. It is spanned by the Plücker images of any 3 non coplanar lines passing through O.

[1] recall that homogeneous polynomial is called decomposable if it is represented as a product of linear forms

β-net is a set of all lines laying on a given plane $\Pi \in \mathbb{P}_3$; the corresponding plain $\pi_\beta(\Pi) \subset Q_P$ is called *β-plane*. It is spanned by the Plücker images of any 3 lines which lay on Π and don't have a common intersection.

Note that any two planes of the same type have exactly 1-point intersection, namely:

$$\pi_\beta(\Pi_1) \cap \pi_\beta(\Pi_2) = \mathfrak{u}(\,\Pi_1 \cap \Pi_2\,)$$
$$\pi_\alpha(O_1) \cap \pi_\alpha(O_2) = \mathfrak{u}(\,(O_1 O_2)\,)$$

Two planes of different types, say $\pi_\beta(\Pi)$ and $\pi_\alpha(O)$, don't intersect each other, if $O \notin \Pi$. If $O \in \Pi$, then $\pi_\beta(\Pi) \cap \pi_\alpha(O)$ is a pencil of lines $\ell \subset \mathbb{P}_3$ such that $O \in \ell \subset \Pi$.

Exercice 6.5. Show that there are no other line pencils in $\mathbb{P}_3$, i. e. each line on $Q_P \subset \mathbb{P}_5$ has the form $\pi_\beta(\Pi) \cap \pi_\alpha(O)$ for some O and Π.

HINT. Consider the cone $C = Q_P \cap T_p Q_P$. It has a vertice at p and consists of all lines which pass through p and lay on Q_p. Fix a 3-dimensional hyperplane $H \subset T_p Q_P$ which doesn't contain p. Then $G = C \cap H$ is non singular quadric on H. So, any line passing through p has a form $(pp') = \pi_\alpha \cap \pi_\beta$, where $p' \in G$ and the planes π_α, π_β are spaned by p and two lines passing through p' on G (see 6.1).

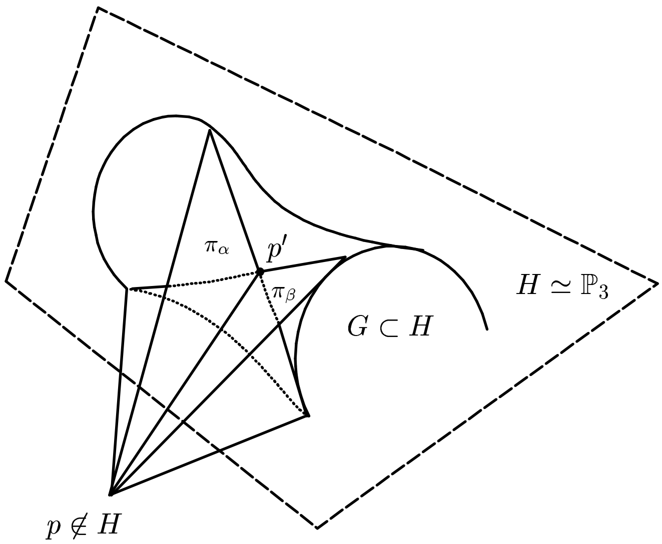


Fig. 6.1. The cone $C = Q_P \cap T_p Q_P$.

AFFINE CELL DECOMPOSITION OF $\mathrm{Gr}(2,4)$. Let $H \subset T_p Q_P$ be a 3-dimensional projective hyperplane such that $p \notin H$, as in above exercise, $C = Q_P \cap T_p Q_P$, and $G = H \cap Q_P$. Then C is the simple cone with vertex p over G (see fig. 6.1) and we have the following diagram of inclusions

$$\begin{tikzcd} & & \pi_\alpha \arrow[dr, hook] & & \\ p \arrow[r, hook] & \pi_\alpha \cap \pi_\beta \arrow[ur, hook] \arrow[dr, hook] & & C \arrow[r, hook] & Q_P \\ & & \pi_\beta \arrow[ur, hook] & & \end{tikzcd} \quad \Rightarrow \quad \mathrm{Gr}(2,4) \;=\; \mathbb{A}^0 \sqcup \mathbb{A}^1 \sqcup \begin{pmatrix} \mathbb{A}^2 \\ \sqcup \\ \mathbb{A}^2 \end{pmatrix} \sqcup \mathbb{A}^3 \sqcup \mathbb{A}^4$$

The right side decomposition is produced via replacing each stratum in the left side by the complement to all the smallest strata it contain and identifying the resulting disjoint cells with affine spaces as follows: $(\pi_\alpha \cap \pi_\beta) \setminus p \simeq \mathbb{A}^1$ (because this is a projective line without a point), $\pi_\alpha \setminus (\pi_\alpha \cap \pi_\beta) \simeq \pi_\beta \setminus (\pi_\alpha \cap \pi_\beta) \simeq \mathbb{A}^2$ (because the both are projective planes without a line), $C \setminus (\pi_\alpha \cup \pi_\beta) \simeq \mathbb{A}^1 \times \Big(G \setminus (G \cap T_{p'} G)\Big)$ (because C is a cone over G), and finally, $G \setminus (G \cap T_{p'} G) \simeq \mathbb{A}^2$ and $Q \setminus C \simeq \mathbb{A}^4$, because of the lemma below.

Lemma 6.6. *Let $Q \subset \mathbb{P}_n$ be a quadric, $p \in Q$ be a non singular point, and $H \not\ni p$ be a codimension 1 hyperplane. Then the projection from p onto H induces a bijection between $Q \setminus (Q \cap T_p Q)$ and $\mathbb{A}^{n-1} = H \setminus (H \cap T_p Q)$.*

Proof. Any non tangent line passing through p have to intersect Q precisely ones more. All such lines are 1–1 parameterized by the points of $\mathbb{A}^{n-1} = H \setminus (H \cap T_p Q)$. $\square$

Exercice 6.7*. If you have some experience in topology, show that over $\mathbb{C}$ all odd integer homologies of $\mathrm{Gr}(2,4)$ vanish and the even ones are $H_0 = H_2 = H_6 = H_8 = \mathbb{Z}$, $H_4 = \mathbb{Z} \oplus \mathbb{Z}$. Also, try to compute the homologies for the real grassmannian, where the boundary maps are non trivial.

GENERAL GRASSMANNIAN $\mathrm{Gr}(m,d)$ is defined as the set of all m-dimensional vector subspaces in a given d-dimensional vector space V. If the nature of V is important, we write $\mathrm{Gr}(m,V)$ instead of $\mathrm{Gr}(m,d)$. In the projective language, $\mathrm{Gr}(m,d)$ is the set of all $(m-1)$-dimensional projective subspaces in $\mathbb{P}_{d-1}$. If $m=1$, then $\mathrm{Gr}(m,d) = \mathbb{P}_{d-1}$. There is a canonical bijection $\mathrm{Gr}(m,V) \simeq \mathrm{Gr}(d-m,V^*)$ induced by duality. It sends $U \subset V$ to $\mathrm{Ann}\, U \subset V^*$ and wise versa.

Exercice 6.8. Let $\dim V = 4$. Fix an isomorphism $V \xrightarrow{\widehat{q}} V^*$, say presented by a non singular quadric $Q \subset \mathbb{P}(V)$, and consider an automorphism of $\mathrm{Gr}(2,V)$ given by $U \longmapsto \mathrm{Ann}\, \widehat{q}(U)$. Show that it maps the α-planes on $\mathrm{Gr}(2,4)$ to the β-planes and wise versa.

PLÜCKER EMBEDDING $\mathrm{Gr}(m,V) \xhookrightarrow{\mathfrak{u}} \mathbb{P}(\Lambda^m V)$ sends m-dimensional subspace $U \subset V$ to the 1-dimensional subspace $\Lambda^m U \subset \Lambda^m V$. If U is based by the vectors $\{u_1, u_2, \ldots, u_m\} \subset U$, then $\mathfrak{u}(U) = u_1 \wedge u_2 \wedge \cdots \wedge u_m$ up to proportionality, because taking an other base, say $v_i = \sum a_{ij} u_j$, we get

$$\mathfrak{u}(U) = v_1 \wedge v_2 \wedge \cdots \wedge v_m = \det(a_{ij}) \cdot u_1 \wedge u_2 \wedge \cdots \wedge u_m\,.$$

Lemma 6.9. *The Plücker embedding is really injective.*

Proof. If $U_1 \neq U_2$, then there exist a base in V such that some vectors $w_1, w_2, \dots, w_r$ of this base give a base for $U_1 \cap U_2$, some other $u_1, u_2, \dots, u_{m-r}$ together with $\{w_\nu\}$ give a base for U_1, some other $v_1, v_2, \dots, v_{m-r}$ together with $\{w_\nu\}$ give a base for U_2, and the rest $e_1, e_2, \dots, e_{d+r-2m}$ are complementary to $U_1 + U_2$. Let $\omega \in \Lambda^{d-m}V$ be the skew product of all v_ν and e_ν. The skew multiplication by ω $\Lambda^m V \xrightarrow{\xi \mapsto \xi\wedge\omega} \Lambda^d V \simeq k$ turns into a linear form on $\Lambda^m V$ as soon as a base vector for $\Lambda^d V$ is fixed. This linear form does vanish at $\mathfrak{u}(U_2)$ and doesn't at $\mathfrak{u}(U_1)$. □

Example 6.10: 2×2-minors of 2×4-matrices. Let $\dim V = 4$ and a base $\{e_1, e_2, e_3, e_4\} \subset V$ be fixed. Then a subspace $U \subset V$ based by u_1, u_2 can be presented as 2×4-matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$ whose rows are the coordinates of u_1, u_2. This matrix is defined by U up to the left multiplication $A \mapsto C\cdot A$ by any $C \in GL_2(k)$ (this corresponds to a base change in U). The Plücker embedding sends A to

$$u_1 \wedge u_2 = \sum_{i<j} \det \begin{pmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{pmatrix} e_i \wedge e_j \,.$$

So, the homogeneous coordinates of $\mathfrak{u}(U) \in \mathbb{P}_5$ in the base $\{e_{ij} = e_i \wedge e_j\}$ are six 2×2-minors of A. In particular, the left multiplications by $C \in GL_2$ doesn't effect on the ratios between 2×2-minors of A. An other claim: six numbers $x_1, x_2, \dots, x_6$ give a collection of 2×2-minors for some 2×4 matrix iff they satisfy (maybe, after some renumbering) the Plücker equation $x_1x_2 = x_3x_4 + x_5x_6$.

Exercice 6.11. Is there 2×4 - matrix with minors a) $\{2, 3, 4, 5, 6, 7\}$ b) $\{3, 4, 5, 6, 7, 8\}$?

Matrix notations and Plücker coordinates on $\mathrm{Gr}(m,d)$. A point $U \in \mathrm{Gr}(m,d)$ can be presented by $(m\times d)$-matrix A_U whose rows are the coordinates of some base vectors $\{u_1, u_2, \dots, u_m\} \subset U$ with respect to a fixed base $\{e_1, e_2, \dots, e_d\} \subset V$. Such a matrix is not unique and is defined by U only up to the left multiplication by any $C \in GL_m$ (this corresponds to a base change in U). So, the grassmannian $\mathrm{Gr}(m,d)$ can be considered as a factor space of $Mat_{m\times d}(k)$ by the left action of $GL_m(k)$. Under the Plücker embedding , the homogeneous coordinates of $\mathfrak{u}(U) \in \Lambda^m V$ in the standard base $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$ are equal to the maximal minors of A. They are stable under the left GL_m-action and are called *Plücker coordinates* of U.

Affine covering and affine coordinates on $\mathrm{Gr}(m,d)$. Consider the standard affine card $\mathscr{U}_I \subset \mathbb{P}(\Lambda^m V)$ given by $x_I = 1$, where x_I is the coordinate along $e_I = e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_m}$. The inverse image $\mathfrak{U}_I \stackrel{\text{def}}{=} \mathfrak{u}^{-1}(\mathscr{U}_I) \subset \mathrm{Gr}(m,d)$ consists of all U such that A_U has non zero maximal minor in the columns $(i_1, i_2, \dots, i_m)$. Any such U has *a unique* matrix representation $A_U^{(I)} = \left(a_{ij}^{(I)}\right)$ with *the identity $m\times m$-submatrix* staying in these columns. It is given by $A_U^{(I)} = A_{U,I}^{-1}\cdot A_U$, where A_U is an arbitrary matrix representation for U and $A_{U,I} \subset A_U$ is $m\times m$-submatrix formed by the columns $(i_1, i_2, \dots, i_m)$. So, the points of $\mathfrak{U}_I \subset \mathrm{Gr}(m,d)$ are 1–1 parametrized by $m\,(d-m)$ matrix elements $\left(a_{\mu\nu}^{(I)}\right)$ staying outside the columns $(i_1, i_2, \dots, i_m)$ in $A_U^{(I)}$. In other words, we have *an affine chart* $\mathbb{A}^{m(d-m)} \xrightarrow{\sim} \mathfrak{U}_I \subset \mathrm{Gr}(m,d)$ which covers an open dense subset of the grassmannian. The charts $\mathfrak{U}_I$ are called *standard* and cover the whole of $\mathrm{Gr}(m,d)$ when I runs through the length m increasing subsets in $\{1, 2, \dots, d\}$.

Exercice 6.12. Write down the explicit *transition functions* between the standard affine charts $\mathfrak{U}_{12}$ and $\mathfrak{U}_{23}$ on $\mathrm{Gr}(2,4)$.

Exercice 6.13*. If you had deal with smooth topology, check that real and complex grassmannians are the smooth (moreover, analytic) manifolds.

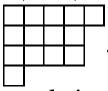
Cell decomposition. The Gauss method shows that for any $U \subset V$ there exists *a unique* base $\{u_1, u_2, \dots, u_m\}$, of U, such that the corresponding matrix $A_U = (a_{\mu\nu})$ is *a strong step matrix*, that is, $(a_{\mu\nu})$ contains the identity $m\times m$-submatrix, say in columns $(j_1, j_2, \dots, j_m)$, such that each row vanishes at the left of the unity coming from this identity submatrix (i. e. for all $i = 1, 2, \dots, m$ we have $a_{ij} = 0 \quad \forall\, j < j_i$).

Exercice 6.14. Prove that different strong step matrices give different subspaces in V.

So, there exist a bijection between $\mathrm{Gr}(m,d)$ and the set of all strong step matrices. The last one splits into disjoint union of the affine spaces. Namely, all strong step matrices that contain the identity submatrix in the fixed columns $I = (i_1, i_2, \dots, i_m)$ have exactly

$$mn - m^2 - (i_1 - 1) - (i_2 - 2) - \cdots - (i_m - m) = \dim \mathrm{Gr}(m,d) - \sum_{\nu=1}^{m} (i_\nu - \nu)$$

free entries to put there any numbers from k. Hence, topologically, $\mathrm{Gr}(m,d)$ is a disjoint union of $\binom{d}{m}$ affine cells $\mathfrak{A}_I$ enumerated by length m increasing subsets $I \subset \{1, 2, \dots, d\}$. The I-th cell is homeomorphic to the affine space and has codimension $\sum_{\nu=1}^{m} (i_\nu - \nu)$ in $\mathrm{Gr}(m,d)$.

Young diagram notations. Traditionally, the ν-th difference $(i_\nu - \nu)$ in a length m increasing subset $I \subset \{1, 2, \dots, d\}$ is denoted by $\lambda_{m+1-\nu}$ in order to have *a partition* $(d-m) \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0$ instead of the increasing collection $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant d$. By the definition, a partition λ is a not increasing collection of non negative integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$. A *length* $\ell(\lambda)$ is a number of the last non zero element in λ. A *weight* of λ is $|\lambda| \overset{\text{def}}{=} \sum\limits_\nu \lambda_\nu$. A *Young diagram* of λ is a `flushleft`'ed collection of cell rows whose lengths are $\lambda_1, \lambda_2, \dots, \lambda_m$. For example, the partition $\lambda = (5, 4, 4, 1)$ has length $\ell(\lambda) = 4$ weight $|\lambda| = 14$ and Young diagram . In other words, the partition is just the Young diagram, its weight is the number of cells, and its length is the number of rows.

Exercice 6.15. Check that there is a bijection between the length m increasing subsets $I \subset \{1, 2, \dots, d\}$ and the Young diagrams contained in the rectangle of size $m \times (d-m)$.

In terms of Young diagrams, the grassmannian $\mathrm{Gr}(m, d)$ is decomposed into the disjoint union of affine cells enumerated by the Young diagrams contained in the $m \times (d-m)$ - rectangle. λ-th cell has codimension $|\lambda|$ and is isomorphic to $\mathbb{A}^{m(d-m)-|\lambda|}$. In particular there is a unique 1-point cell, which has codimension $m\,(d-m)$ and corresponds to the rectangle itself, and a unique open dense cell of codimension zero, which corresponds to the empty diagram and coincides with the standard affine chart $\mathfrak{U}_{\{1,2,\dots,m\}}$. A topological closure of λ-th affine cell is called *a Schubert cycle* and denoted by σ_λ.

Exercice 6.16. Check that 6 Schubert cycles on the Plücker quadric $\mathrm{Gr}(2,4) \simeq Q_P \subset \mathbb{P}_5$ are: $\sigma_{00} = Q_P$; $\sigma_{22} = p = (0:0:0:0:0:1) \in \mathbb{P}_5$; $\sigma_{10} = Q_P \cap T_p Q_P$; $\sigma_{11} = \pi_\alpha(O)$, where $O = (0:0:0:1) \in \mathbb{P}_3$; $\sigma_{20} = \pi_\beta(\Pi)$, where $\Pi \subset \mathbb{P}_3$ is given by $x_0 = 0$; $\sigma_{21} = \pi_\alpha(O) \cap \pi_\beta(\Pi)$.

Remark for who studied the topology. Clearly, the Schubert cycles generate the homologies of $\mathrm{Gr}(m, d)$. Moreover, for the complex grassmannian they form the base of $H_*(\mathrm{Gr}(m, \mathbb{C}^d), \mathbb{Z})$ over $\mathbb{Z}$. It is a nice (but not simle) combinatorial problem, to express the (topological) intersections of the Schubert cycles in terms of the Schubert cycles. The corresponding technique is known as *a Schubert calculus* and is described in Griffits-Harris, Fulton-Harris and Macdonald. Roughly speaking, *the homology ring* $H_*(\mathrm{Gr}(m, \mathbb{C}^d), \mathbb{Z})$ is isomorphic to the truncated ring of symmetric polynomials via sending the Schubert cycles σ_λ to the Schur polynomials s_λ.

Exercice 6.17*. Show that $\sigma_{10}\sigma_{21} = \sigma_{20}^2 = \sigma_{11}^2 = \sigma_{22}$, $\sigma_{20}\sigma_{11} = 0$, and $\sigma_{10}^2 = \sigma_{20} + \sigma_{11}$ in the integer homology ring of complex $\mathrm{Gr}(2, 4)$.

Hint. To calculate σ_{10}^2 realize σ_{10} as $\sigma_{1,0}(\ell) = Q_P \cap T_{\mathfrak{u}(\ell)} Q_P = \{\ell'' \subset \mathbb{P}_3 \mid \ell \cap \ell'' \neq \varnothing\}$. Then, taking two *intersecting* lines ℓ and ℓ' in $\mathbb{P}_3$ we get $\sigma_{10}(\ell) \cap \sigma_{10}(\ell') = \pi_\alpha(O) \cup \pi_\beta(\Pi)$, where $O = \ell \cap \ell'$ and Π is spaned by ℓ and ℓ'.

Plücker equations The image of general Plücker embedding $\mathrm{Gr}(m, V) \xhookrightarrow{\mathfrak{u}} \mathbb{P}(\Lambda^m V)$, that is a variety of decomposable skew-symmetric quadratic polynomials, can be explicitly described by Plücker quadratic relations considered in Example 5.18 and generalizing those we written in Example 6.10, Lemma 6.2 for $\dim V = 4$. In this case, potentially, we can write 4 general Plücker relations from Example 5.18: they correspond to all possible distributions of $4 = 3 + 1$ indexes $\{1, 2, 3, 4\}$ between K and J.

Exercice 6.18. Check that they all produce the same quadratic equation $A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = 0$ on 2×2 - minors A_{ij} of 2×4 - matrix A.