Polarizations and contracions for (skew) polynomials[1]

(SKEW) SYMMETRIC TENSORS. A symmetric group $\mathfrak{S}_n$ acts on $V^{\otimes n}$ permuting factors in the decomposable tensors: $\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_n) \overset{\text{def}}{=} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \quad \forall\, \sigma \in \mathfrak{S}_n$. Subspaces

$$\mathsf{Skew}^n V = \{\, t \in V^{\otimes n} \mid \sigma(t) = \operatorname{sgn}(\sigma) \cdot t \quad \forall\, \sigma \in \mathfrak{S}_n \,\}$$
$$\mathsf{Sym}^n V = \{\, t \in V^{\otimes n} \mid \sigma(t) = t \quad \forall\, \sigma \in \mathfrak{S}_n \,\}$$

are called the spaces of *skew symmetric* and *symmetric* tensors.

Claim 5.1. *If* char $(k) = 0$*, then, restricting the canonical factorization maps:* $V^{\otimes n} \xrightarrow{\pi_{skew}} \Lambda^n V$, $V^{\otimes n} \xrightarrow{\pi_{sym}} S^n V$ *, onto the spaces of (skew) symmetric tensors, we get canonical isomorphisms:*

$$\mathsf{Skew}^n V \xrightarrow{\pi_{skew}} \Lambda^n V \qquad \text{and} \qquad \mathsf{Sym}^n V \xrightarrow{\pi_{sym}} S^n V\,. \tag{16}$$

Proof. In the skew symmetric case, a base of $\mathsf{Skew}^n V$ is formed by the tensors

$$e_{\langle i_1, i_2, \dots, i_n \rangle} \overset{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \cdot e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}} \otimes \cdots \otimes e_{i_{\sigma(n)}}$$

(sum of all the tensor monomials sent to the basic Grassmannian monomial $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$ by π_{skew}). So, $\pi_{\text{skew}}\left(e_{\langle i_1, i_2, \dots, i_n \rangle}\right) = n!\, e_I$. In the symmetric case, let us write $e_{[m_1, m_2, \dots, m_d]}$ for the sum of all tensor monomials containing m_1 factors e_1, m_2 factors e_2, … , m_d factors e_d, where $\sum\limits_{\nu} m_\nu = n$. These monomials form one $\mathfrak{S}_n$-orbit, which consists of $\frac{n!}{m_1!\, m_2! \cdots m_d!}$ elements and collects all the decomposable tensors sent to $e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$ by π_{sym}. As above, the tensors $e_{[m_1, m_2, \dots, m_d]}$ form a base for $\mathsf{Sym}^n V$ and $\pi_{\text{sym}}\left(e_{[i_1, i_2, \dots, i_n]}\right) = \frac{n!}{m_1!\, m_2! \cdots m_d!} e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$. □

Exercice 5.2. Verify that the above sums $e_{\langle i_1, i_2, \dots, i_n \rangle}$ and $e_{[m_1, m_2, \dots, m_d]}$ really give the bases for $\mathsf{Sym}^n V$ (over any field of an arbitrary characteristic). Also note that if char $(k) > 0$ divides n, then all these basic (skew) symmetric tensors are annihilated by factorization through (skew) symmetric relations.

Exercice 5.3. Verify that if char $(k) = 0$, then $V^{\otimes n} = \mathcal{I}^{(n)}_{\text{skew}} \oplus \mathsf{Skew}^n V = \mathsf{Sym}^n V \oplus \mathcal{I}^{(n)}_{\text{sym}}$, where the projection $V^{\otimes n} \twoheadrightarrow \mathsf{Sym}^n V$ along $\mathcal{I}^{(n)}_{\text{sym}}$ is given by *the symmetrization map*

$$\operatorname{sym}_n \,:\, \tau \longmapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma(\tau)$$

and the projection $V^{\otimes n} \twoheadrightarrow \mathsf{Skew}^n V$ along $\mathcal{I}^{(n)}_{\text{skew}}$ given by *the alternation map*

$$\operatorname{alt}_n \,:\, \tau \longmapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \cdot \sigma(\tau)\,.$$

POLARIZATION OF (SKEW) POLYNOMIALS. The inverse maps to the isomorphisms (16) take

$$\begin{aligned} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} &\longmapsto \frac{1}{n!} \cdot e_{\langle i_1, i_2, \dots, i_n \rangle} \\ e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d} &\longmapsto \frac{m_1!\, m_2! \cdots m_d!}{n!} \cdot e_{[m_1, m_2, \dots, m_d]}\,. \end{aligned} \tag{17}$$

The both maps are called *complete polarizations* of (skew) polynomials and are denoted by $f \longmapsto \operatorname{pl}(f)$.

Example 5.4: (skew) polynomials and (skew) symmetric multilinear forms. Full polarization $\operatorname{pl}(f)$ of a (skew) homogeneous degree n polynomial f of one argument on V can be considered as a multilinear form of n arguments on V. It sends $(v_1, v_2, \dots, v_n)$ to the full contraction $\widetilde{f}(v_1, v_2, \dots, v_n) \overset{\text{def}}{=} \langle\, v_1 \otimes v_2 \otimes \cdots \otimes v_n \,,\, \operatorname{pl}(f) \,\rangle$ and has the same symmetry properties as f, because for all $\sigma \in \mathfrak{S}_n$, $t \in V^{\otimes n}$, $\xi \in V^{*\otimes n}$ we have $\langle\, \sigma(t) \,,\, \sigma(\xi) \,\rangle = \langle\, t \,,\, \xi \,\rangle$, which implies $\langle\, \sigma(t) \,,\, \xi \,\rangle = \langle\, t \,,\, \sigma^{-1}(\xi) \,\rangle$.

Exercice 5.5. Check that for a symmetric quadratic form $q(x) \in S^2 V^*$ we have

$$\widetilde{q}(x,y) = \frac{q(x+y) - q(x-y)}{4} = \frac{q(x+y) - q(x) - q(y)}{2} = \frac{1}{2} \sum_{\nu=1}^{\dim V} y_\nu \frac{\partial q}{\partial x_\nu}\,.$$

[1] WARNING!!! In this lecture we suppose that the ground field k is algebraically closed and has zero characteristic.

Since any multilinear form φ may be presented via full contraction

$$\varphi(v_1, v_2, \dots, v_n) = \langle\, v_1 \otimes v_2 \otimes \cdots \otimes v_n \,,\, \xi \,\rangle$$

with some $\xi \in V^{*\otimes n}$, the complete polarization identifies S^nV^* and Λ^nV^* with the spaces of all symmetric and skew symmetric multilinear forms $\underbrace{V \times V \times \cdots \times V}_{n} \longrightarrow k$, in n arguments on V.

Exercice 5.6. In symmetric case, show that a homogeneous degree n polynomial $f(x)$ (in one argument $x \in V$) coincides with the restriction of the corresponding symmetric multilinear form $\widetilde{f}(x_1, x_2, \dots, x_n)$ (in n arguments $x_\nu \in V$) onto the diagonal: $f(x) = \widetilde{f}(x, x, \dots, x)$.

Example 5.7: duality on polynomials Using the complete polarization and the full contraction between $V^{\otimes n}$ and $V^{*\otimes n}$ we obtain (over a field of zero characteristic) a natural non degenerate pairing between $\Lambda^n(V)$ and $\Lambda^n(V^*)$ as well as between $S^n(V)$ and $S^n(V^*)$. Namely, for two (skew) polynomials f, in $e_i \in V$, and ξ, in $x_i \in V^*$, we put $\langle\, f \,,\, \xi \,\rangle \stackrel{\text{def}}{=} \langle\, \mathrm{pl}(f) \,,\, \mathrm{pl}(\xi) \,\rangle$.

Exercice 5.8. Let $\{e_1, e_2, \dots, e_n\} \subset V$ and $\{x_1, x_2, \dots, x_n\} \subset V^*$ be dual bases. Check that:

$$\langle\, e_I \,,\, x_J \,\rangle = \begin{cases} 1/n! & \text{, for } I = J \\ 0 & \text{, for } I \neq J \end{cases} \tag{18}$$

$$\langle e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d} \,,\, x_1^{\ell_1} x_2^{\ell_2} \cdots x_d^{\ell_d} \rangle = \begin{cases} \frac{m_1!\, m_2! \cdots m_d!}{n!} & \text{, if } m_\nu = \ell_\nu \ \forall \nu \\ 0 & \text{, otherwise} \end{cases} \tag{19}$$

Partial derivatives (symmetric case) For any vector $v \in V$ and any polynomial $f \in S^nV^*$ the contraction $c_1^i(\mathrm{pl}(f) \otimes v) \in V^{*n-1}$ does not depend on the choice of index i in $\mathrm{pl}(f)$ to be contracted, because $\mathrm{pl}(f) \in \mathsf{Sym}^n V^*$ is symmetric. Its projection to $S^{n-1}V^*$ is called *a polar* of v w.r.t. f and is denoted by $\mathrm{pl}_v f$.

Exercice 5.9. Show that $\deg(f) \cdot \mathrm{pl}_v f = \partial_v f$, where ∂_v is the derivative in v-direction, which takes f to $\sum\limits_{i=1}^{\dim V} v_i \frac{\partial f}{\partial x_i}$ (here $v = \sum v_i e_i$, $\{e_i\}$ is a base for V and $\{x_i\}$ is a dual base for V^*).

Exercice 5.10. Show that $n! \cdot \widetilde{f}(v_1, v_2, \dots, v_n) = \partial_{v_1} \partial_{v_2} \cdots \partial_{v_n} f$ for any $f \in S^nV^*$, $v_1, v_2, \dots, v_n \in V$.

In particular, $\partial_v f$ does not depend on a base choice, is bilinear in v, f and satisfies the following symmetry property:

$$m_2!\; \partial_{v_1}^{m_1} f(v_2) = (m_1 + m_2)!\; \widetilde{f}(\underbrace{v_1, v_1, \dots, v_1}_{m_1}, \underbrace{v_2, v_2, \dots, v_2}_{m_2}) = m_1!\; \partial_{v_2}^{m_2} f(v_1)\;, \tag{20}$$

where $(m_1 + m_2) = n = \deg f$. In particular, the both, left and right sides, are bihomogeneous of bidegree (m_1, m_2) in (v_1, v_2).

Exercice 5.11. Show that *multiple polars*: $\mathrm{pl}_{\,v_1, v_2, \dots, v_m} f \stackrel{\text{def}}{=} \mathrm{pl}_{v_1} \mathrm{pl}_{v_2} \cdots \mathrm{pl}_{v_1} f$ are symmetric and multilinear in v_i and linear in f, that is come from a linear map $S^mV \otimes S^nV^* \longrightarrow S^{n-m}V^*$, which sends $v_1 v_2 \cdots v_m \otimes f$ to $\mathrm{pl}_{\,v_1, v_2, \dots, v_m} f(w) = \widetilde{f}(\, v_1, v_2, \dots, v_m\,, w, w, \dots, w)$.

Example 5.12: Taylor formula. For $f \in S^nV^*$ the value $f(v + w)$ can be computed as the full contraction of f with $(v + w)^m = \sum\limits_m \binom{n}{m} v^m w^{n-m} \in S^nV$. This can be arranged as the *Taylor formula*:

$$f(v + w) = \sum_{\nu=0}^{n} \frac{1}{\nu!}\, \partial_w^\nu f(v)\;. \tag{21}$$

Example 5.13: *span*(f) for $f \in S^nV^*$, that is a minimal subspace $W \subset V^*$ such that $f \in S^nW$, coincides with *span*$(\mathrm{pl}(f))$ described in Claim 4.11 as an image of the contraction map $S^{n-1}V \otimes S^nV^* \longrightarrow V^*$. So, in terms of partial derivatives, *span*(f) is generated by the linear forms

$$\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_{n-1}}} f(x) \tag{22}$$

obtained from f by all possible $(n-1)$-fold differentiations.

Veronese variety in $\mathbb{P}_N = \mathbb{P}(S^nV^*)$, where $N = \binom{n}{d} - 1$ and $\dim V = d + 1$, consists of pure n-th powers of linear forms, that is, of $f(x) = \xi^n(x)$, where $\xi \in \mathbb{P}(V^*)$. It has a rational parameterization given by the Veronese map

$$\mathbb{P}(V^*) \xhookrightarrow{\;\xi \longmapsto \xi^n\;} \mathbb{P}(S^nV^*)\;.$$

Example 5.13 presents the Veronese variety by a system of quadratic equations. Namely, $f(x) = \xi^n(x)$ for some $\xi \in V^*$ iff *span*(f) is 1-dimensional subspace generated by ξ. So, f is pure n-th power iff any two linear forms (22) are proportional to each other. If we arrange the coefficients of two linear forms as the rows of some $2 \times (d+1)$-matrix, then their proportionality means that all 2×2-minors of this matrix vanish.

Example 5.14: Veronese's curve. Let $\dim U = 2$, $\mathbb{P}_1 = \mathbb{P}(U^*)$, $\mathbb{P}_n = \mathbb{P}(S^n U^*)$, $\{t_0, t_1\}$ be a base of U^*, and $\binom{n}{i} t_0^i t_1^{n-i}$ for $0 \leqslant i \leqslant n$ be the corresponding base of $S^n V^*$. Then the Veronese embedding $\mathbb{P}_1 \hookrightarrow \mathbb{P}_n$ sends a linear form $(\alpha_0 t_0 + \alpha_1 t_1)$ to

$$(\alpha_0 t_0 + \alpha_1 t_1)^n = \sum_i \alpha_0^i \alpha_1^{n-i} \cdot \binom{n}{i} t_0^i t_1^{n-i}$$

Its image is a rational curve $C_n \subset \mathbb{P}_n$ called *Veronese curve* or *rational normal curve* of degree n. If we use the coefficients $(a_0 : a_1 : \ \ldots \ : a_n)$ and $(\alpha_0 : \alpha_1)$ as homogeneous coordinates for a polynomial $\sum\limits_i a_i \cdot \binom{n}{i} t_0^i t_1^{n-i} \in S^n U^*$ and for a linear form $\alpha_0 t_0 + \alpha_1 t_1 \in \mathbb{P}(U^*)$, then C_n will be presented parametrically as $a_i = \alpha_0^i \alpha_1^{n-i}$.

On the other hand, for binary form $f(t) = \sum\limits_i a_i \cdot \binom{n}{i} t_0^i t_1^{n-i}$ the linear forms (22) are exhausted by

$$\frac{\partial^i}{\partial t_0^i} \frac{\partial^{n-1-i}}{\partial t_1^{i-1-n}} f(t) = a_i t_0 + a_{i+1} t_1 \quad , \quad \text{where} \quad 0 \leqslant i \leqslant (n-1) \, .$$

So, $f \in C_n$ iff $\det \begin{pmatrix} a_i & a_{i+1} \\ a_{j-1} & a_j \end{pmatrix} = a_i a_j - a_{i+1} a_{j-1} = 0$ for all $0 \leqslant i < j \leqslant n$, i. e. iff

$$\operatorname{rk} \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_1 & a_2 & \ldots & a_n \end{pmatrix} = 1 \, .$$

For $n = 2$ we get the Veronese quadric $a_0 a_2 = a_1^2$ in $\mathbb{P}_2$. For $n = 3$ we have a rational cubic curve, which is called *a twisted cubic*, given as an intersection of 3 quadrics $a_0 a_2 = a_1^2$, $a_1 a_3 = a_2^2$, $a_0 a_3 = a_1 a_2$.

Exercice 5.15. Draw the picture and discover that the first two quadrics are simple cones with vertices at $(0:0:0:1)$ and at $(1:0:0:0)$; the third quadric is Segre's one and is ruled by two line families; two cones have a common line element $a_1 = a_2 = 0$, which joins the vertices but does not lie on the Segre quadric; the line element $a_0 = a_1 = 0$ of the first cone and the one $a_2 = a_3 = 0$ of the second do lie on the Segre quadric in the same ruling family. So, any two of 3 quadrics are intersected along the twisted cubic and one more line, that is, the Veronese curve can not be given by use of 2 equations only!

Partial derivatives (skew-symmetric case). The contraction $c_1^i(\mathrm{pl}(f) \otimes v) \in V^{*n-1}$ between a vector $v \in V$ and a skew polynomial $f \in \Lambda^n V^*$ slightly depends on the choice of index i: it changes the sign when i is incremented or decremented by 1, because $\mathrm{pl}(f) \in \mathsf{Skew}^n V^*$ is skew symmetric. We choose $i = 1$ and write $\mathrm{pl}_v f$ for the projection of $c_1^1(\mathrm{pl}(f) \otimes v)$ into $\Lambda^{n-1} V^*$. Now consecutive polarizations anti commute: $\mathrm{pl}_v \mathrm{pl}_w f = -\mathrm{pl}_w \mathrm{pl}_v f$, i. e. the multilinear map

$$\underbrace{V \times V \times \cdots \times V}_{m} \times \Lambda^n V^* \xrightarrow{(v_1, v_2, \ldots, v_m \ , f) \mapsto \mathrm{pl}_{v_1} \mathrm{pl}_{v_2} \ldots \mathrm{pl}_{v_m} f} \Lambda^{n-m} V^*$$

comes from the linear map $\Lambda^m V \otimes \Lambda^n V^* \longrightarrow \Lambda^{n-m} V^*$.

Exercice 5.16. Show that $\deg f \cdot \mathrm{pl}_v f = \partial_v f$, where $\partial_v = \sum v_i \frac{\partial}{\partial x_i}$, as in Exercice 5.9, and verify the following properties of skew partial derivatives:
a) $\partial_v \partial_w = -\partial_w \partial_v$ (in particular, $\partial_v^2 = 0 \quad \forall \, v \in V$);
b) $\widetilde{f}(v_1, v_2, \ldots, v_n) = \frac{1}{n!} \partial_{v_1} \partial_{v_2} \cdots \partial_{v_n} f$ for any $f \in \Lambda^n V^*$, $v_1, v_2, \ldots, v_n \in V$.
c) $\partial_v (f_1 \wedge f_2) = (\partial_v f_1) \wedge f_2 - (-1)^{\deg f_1} f_1 \wedge (\partial_v f_2)$, in particular,

$$\partial_{e_{i_\nu}} x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} = (-1)^{\nu - 1} x_{i_1} \wedge \cdots \wedge x_{i_{\nu-1}} \wedge x_{i_{\nu+1}} \wedge \cdots \wedge x_{i_n} \, .$$

Example 5.17: *span*(f) for grassmannian polynomial $f \in \Lambda^n V$ is linearly generated by partial derivatives

$$\partial_J f = \partial_{x_{j_{n-1}}} \partial_{x_{j_{n-2}}} \ldots \partial_{x_{j_1}} f = n! \left\langle x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_{n-1}} \, , \, \mathrm{pl}(f) \right\rangle$$

where $J = (j_1, j_2, \ldots, j_{n-1}) \subset \{1, 2, \ldots, d\}$ runs through all $\binom{d}{n-1}$ increasing ordered collections of $(n-1)$ indexes and $\{x_i\} \in V^*$ form a dual base to some base $\{e_i\} \in V$. If $f = \sum\limits_I \alpha_I \, e_I$ w. r. t. the last base, then

$$\partial_J f = n! \cdot \sum_{i \notin J} (-1)^{n - p(i,J)} \, \alpha_{J \sqcup \{i\}} \, e_i \tag{23}$$

where $p(i, J)$ is the number of place where i stays in the increasing permutation I of $J \sqcup \{i\}$ (because x_I is the only monomial in $\Lambda^n V^*$ whose complete contraction with e_I is non zero, see (18)).

Example 5.18: Plücker relations. A skew polynomial $f \in \Lambda^n V$ is called *completely decomposable*, if $f = v_1 \wedge v_2 \wedge \cdots \wedge v_n$ is a product of n linear factors, or equivalently, if $\dim span(f) = n$ is minimal possible.

Exercice 5.19. Show that f is completely decomposable iff $f \wedge v = 0\ \forall v \in span(f)$.

So, like in Example 5.13 — Example 5.14, the set of completely decomposable polynomials $f \in \Lambda^n V$ is described by a system of quadratic equations $(\partial_J f) \wedge f = 0$. By (23), a basic monomial $e_K \in \Lambda^{n+1} V$ appears in

$$(\partial_J f) \wedge f = n! \cdot \left(\sum_{i \notin J} (-1)^{p(i,J)}\, \alpha_{J \sqcup \{i\}}\, e_i \right) \wedge \left(\sum_I a_I e_I \right)$$

as $n! \cdot \sum_{\alpha=1}^{n+1} (-1)^{n-p(k_\alpha,J)} \cdot \alpha_{J\sqcup\{k_\alpha\}} \alpha_{K\setminus\{k_\alpha\}} \cdot e_{k_\alpha} \wedge e_{K\setminus\{k_\alpha\}}$, i.e. with the coefficient, which up to a constant factor equals

$$P_{JK}(f) \stackrel{\text{def}}{=} \sum_{i \in K \setminus (K \cap J)} (-1)^{p(i,J,K)}\, \alpha_{J \sqcup i}\, \alpha_{K \setminus i}\ , \tag{24}$$

where $p(i, J, K)$ is the sum of the place numbers where i stays in the increasing version of $J \sqcup \{i\}$ and in K.

Exercice 5.20. Show that this coefficient vanishes, if $J \subset K$

System of quadratic equations $P_{JK}(f) = 0$, which defines the variety of completely decomposable Grassmannian polynomials, is known as the *Plücker relations*. Note that these quadratic equations are not «independent», even not pairwise different (see Example 6.10 below).