Tensor Guide

MULTILINEAR MAPS. Let $V_1, V_2, \dots, V_n$ and W be vector spaces of dimensions $d_1, d_2, \dots, d_n$ and m over an arbitrary field k with $\operatorname{char} k \neq 2$. A map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W$ is called *multilinear*, if in each argument

$$\varphi(\dots, \lambda v' + \mu v'', \dots) = \lambda\,\varphi(\dots, v', \dots) + \mu\,\varphi(\dots, v'', \dots)$$

when all the other remain to be fixed. The multilinear maps $V_1 \times V_2 \times \cdots \times V_n \longrightarrow W$ form a vector space of dimension $m \cdot \prod d_\nu$. Namely, if we fix a base $\{e_1^{(i)}, e_2^{(i)}, \dots, e_{d_i}^{(i)}\}$ for each V_i and a base $\{e_1, e_2, \dots, e_m\}$ for W, then any multilinear map φ is uniquely defined by its values at all combinations of the base vectors:

$$\varphi(\, e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \dots, e_{\alpha_n}^{(n)}\,) = \sum_\nu a_\nu^{(\alpha_1,\alpha_2,\dots,\alpha_n)} \cdot e^\nu \;\in\; W$$

As soon as $m \cdot \prod d_\nu$ numbers $a_\nu^{(\alpha_1,\alpha_2,\dots,\alpha_n)} \in k$ are given, the map φ is well defined by the multilinearity. It sends a collection of vectors $(v_1, v_2, \dots, v_n)$, where $v_i = \sum\limits_{\alpha_i=1}^{d_i} x_{\alpha_i}^{(i)}\, e_{\alpha_i}^{(i)} \in V_i$ for $1 \leqslant i \leqslant n$, to

$$\varphi(v_1, v_2, \dots, v_n) \;=\; \sum_{j=1}^{m} \Big(\sum_{\alpha_1,\alpha_2,\dots,\alpha_n} a_\nu^{(\alpha_1,\alpha_2,\dots,\alpha_n)} \cdot x_{\alpha_1}^{(1)} \cdot x_{\alpha_2}^{(2)} \cdot \;\cdots\; \cdot x_{\alpha_n}^{(n)} \Big) \cdot e^\nu \;\in\; W$$

the numbers $a_\nu^{(\alpha_1,\alpha_2,\dots,\alpha_n)}$ can be considered as elements of some «$(n+1)$-dimensional format matrix of the size $m \times d_1 \times d_2 \times \cdots \times d_n$», if you can imagine such a thing[1].

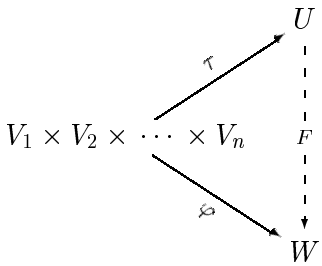
Exercice 4.1. Check that a collection $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \cdots \times V_n$ doesn't contain zero vector iff there exists a multilinear map φ (to somewhere) such that $\varphi(v_1, v_2, \dots, v_n) \neq 0$.

Exercice 4.2. Check that a multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} U$ followed by *a linear operator* $U \xrightarrow{F} W$ is the multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{F\circ\varphi} W$ as well.

TENSOR PRODUCT OF VECTOR SPACES. Let $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} U$ be a fixed multilinear map. Then for any vector space W we have a composition operator

$$\left(\begin{array}{c}\text{the space } \operatorname{Hom}(U,W) \text{ of all}\\ \text{linear operators } U \xrightarrow{F} W\end{array}\right) \quad \xrightarrow{F \longmapsto F\circ\tau} \quad \left(\begin{array}{c}\text{the space of all multilinear maps}\\ V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W\end{array}\right) \tag{5}$$

A multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} U$ is called *universal* if the composition operator (5) is an isomorphism for any vector space W. In other words, the multilinear map τ is universal, if for any W and any multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W$ there exist a unique linear operator $U \xrightarrow{F} W$ such that $\varphi = F\circ\tau$, i. e. the commutative diagram

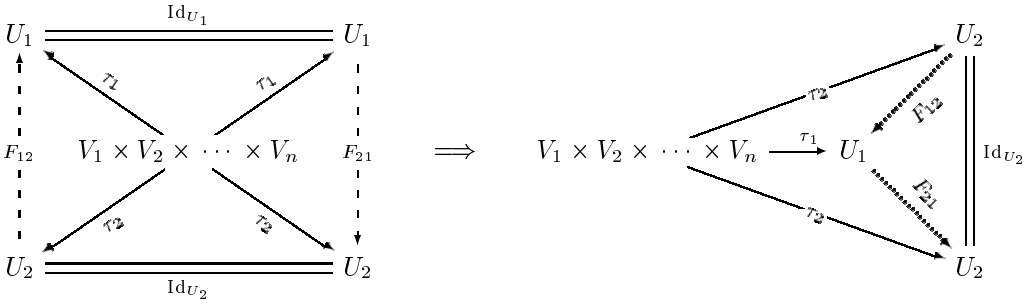


can be always closed by a unique linear dotted row.

Claim 4.3. *Let $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau_1} U_1$ и $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau_2} U_2$ be two universal multilinear maps. Then there exists a unique linear isomorphism $U_1 \xrightarrow{\iota} U_2$ such that $\tau_2 = \iota\tau_1$.*

[1] usual $d \times m$ - matrices, which present the *linear* maps $V \longrightarrow W$, have just 2-dimensional format

Proof. Since both U_1, U_2 are universal, there are unique linear operators $U_1 \xrightarrow{F_{21}} U_2$ and $U_2 \xrightarrow{F_{12}} U_1$ mounted in the diagrams



So, the composition $F_{21}F_{12} = \mathrm{Id}_{U_2}$, because of the uniqueness property in the universality of U_2. Similarly, $F_{12}F_{21} = \mathrm{Id}_{U_1}$. □

Claim 4.4. *Let $\{e_1^{(i)}, e_2^{(i)}, \dots, e_{d_i}^{(i)}\} \in V_i$ be a base (for $1 \leqslant i \leqslant n$). Denote by $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ a $(\prod d_i)$ - dimensional vector space whose base vectors are the symbols*

$$e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \ldots \otimes e_{\alpha_n}^{(n)}, \quad 1 \leqslant \alpha_i \leqslant d_i \tag{6}$$

(all possible formal «tensor products» of base vectors $e_\nu^{(\mu)}$). Then the multilinear map

$$V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} V_1 \otimes V_2 \otimes \cdots \otimes V_n$$

which sends a base vector collection $(e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_n}) \in V_1 \times V_2 \times \cdots \times V_n$ to the corresponding base vector (6) is universal.

Proof. Let $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W$ be a multilinear map and $V_1 \otimes V_2 \otimes \cdots \otimes V_n \xrightarrow{F} W$ be a linear operator. Comparing the values at the base vectors, we see that

$$\varphi = F \circ \tau \iff F(e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \ldots \otimes e_{\alpha_n}^{(n)}) = \varphi(e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \dots, e_{\alpha_n}^{(n)}) .$$

□

The Segre embedding. The vector space $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is called *a tensor product* of $V_1, V_2, \dots, V_n$. The universal multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is called *a tensor multiplication*. For a collection of vectors $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \cdots \times V_n$ the image $\tau(v_1, v_2, \dots, v_n)$ is denoted by $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ and called *a tensor product* of these vectors. All such products are called *decomposable tensors*. Of course, not all the vectors of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ are decomposable and $\operatorname{im}\tau$ isn't a vector subspace in $V_1 \otimes V_2 \otimes \cdots \otimes V_n$, because τ is *multi*linear but not linear. However, the linear span of decomposable tensors exhausts the whole of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$. Geometrically, the tensor multiplication gives a map

$$\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \cdots \times \mathbb{P}(V_n) \xrightarrow{s} \mathbb{P}(V_1 \otimes V_2 \otimes \cdots \otimes V_n)$$

called *a Segre embedding*. If $d_i = \dim V_i = m_i + 1$, then the Segre embedding is a bijection between $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2} \times \cdots \times \mathbb{P}_{m_n}$ and *a Segre subvariety* which consist of all decomposable tensors considered up to proportionality. The Segre subvariety has dimension $(\sum m_i)$, doesn't belong to any hyperplane, is ruled by n families of linear subspaces, and lives in $\mathbb{P}_N$ with $N = -1 + \prod(1 + m_i)$.

Example 4.5: the Segre embedding $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2} \hookrightarrow \mathbb{P}_{m_1+m_2+m_1m_2}$ sends $x = (x_0 \colon x_1 \colon \ldots \colon x_{m_1}) \in \mathbb{P}_{m_1}$ and $y = (y_0 \colon y_1 \colon \ldots \colon y_{m_2}) \in \mathbb{P}_{m_2}$ to the point $s(x, y) \in \mathbb{P}_{m_1+m_2+m_1m_2}$ whose $(1 + m_1)(1 + m_2)$ homogeneous coordinates are all possible products $x_i y_j$ with $0 \leqslant i \leqslant m_1$ and $0 \leqslant j \leqslant m_2$. To visualize this thing, take $\mathbb{P}_{m_1} = \mathbb{P}(V^*)$, $\mathbb{P}_{m_2} = \mathbb{P}(W)$, and $\mathbb{P}_{m_1+m_2+m_1m_2} = \mathbb{P}(\mathrm{Hom}(V, W))$, where $\mathrm{Hom}(V, W)$ is the space of all linear maps. Then the Segre map sends a pair $(\xi, w) \in V^* \times W$ to the linear map $\xi \otimes w$, which acts by the rule $v \longmapsto \xi(v) \cdot w$.

Exercice 4.6. Check that a map $V^* \times W \longrightarrow \mathrm{Hom}(V, W)$ which sends (ξ, w) to the operator $v \longmapsto \xi(v) \cdot w$ is *the universal bilinear map* (so, there is a canonical isomorphism $V^* \otimes W \simeq \mathrm{Hom}(V, W)$)

Exercice 4.7. Check that for $\xi = (x_0, x_1, \ldots, x_{m_1}) \in V^*$ and $w = (y_0, y_1, \ldots, y_{m_2}) \in W$ operator $\xi \otimes w$ has the matrix $a_{ij} = x_j y_i$.

Since any operator $\xi \otimes w$ has 1-dimensional image, the corresponding matrix has rank 1. On the other side, any rank 1 matrix has proportional columns. Hence, the corresponding operator has 1-dimensional image, say spaned by $w \in W$, and takes $v \longmapsto \xi(v)\, w$, where the coefficient $\xi(v) \in k$ depends on v linearly. So, the image of the Segre embedding consists of all rank 1 operators up to proportionality. In particular, it may be presented as the intersection of quadrics $\det \begin{pmatrix} a_{ij} & a_{ik} \\ a_{\ell j} & a_{\ell k} \end{pmatrix} = a_{ij}a_{\ell k} - a_{ik}a_{\ell j} = 0$ (i. e. by vanishing all 2×2 - minors for the matrix $(a_{\mu\nu})$).

Tensor algebra of a vector space. If $V_1 = V_2 = \dots = V_n = V$, then $V^{\otimes n} \stackrel{\text{def}}{=} \underbrace{V \otimes V \otimes \dots \otimes V}_{n}$ is called *an n-th tensor power* of V. All tensor powers are combined in the infinite dimensional non commutative graded algebra $\mathsf{T}^\bullet V = \bigoplus\limits_{n \geqslant 0} V^{\otimes n}$, where $V^{\otimes 0} \stackrel{\text{def}}{=} k$.

Exercice 4.8. Using the universality, show that there are canonical isomorphisms

$$\left(V^{\otimes n_1} \otimes V^{\otimes n_2}\right) \otimes V^{\otimes n_3} \simeq V^{\otimes n_1} \otimes \left(V^{\otimes n_2} \otimes V^{\otimes n_3}\right) \simeq V^{\otimes (n_1+n_2+n_3)}$$

which make the vector's tensoring to be well defined associative multiplication on $\mathsf{T}^\bullet V$.

Algebraically, $\mathsf{T}^\bullet V$ is what is called «a free associative k-algebra generated by V» ([2]). Practically, this means that as soon as a basis $\{e_1, e_2, \dots, e_d\} \subset V$ is fixed, then $\mathsf{T}^\bullet V$ turns into the space of the formal finite linear combinations of words which consist of the letters e_i separated by $\otimes$. The words are multiplied by writing after one other consequently. This multiplication is extended on the linear combinations of the words by the usual distributivity rules.

Duality. The spaces $V^{\otimes n} = \underbrace{V \otimes V \otimes \dots \otimes V}_{n}$ and $V^{*\otimes n} = \underbrace{V^* \otimes V^* \otimes \dots \otimes V^*}_{n}$ are canonically dual to each other. The pairing between $v = v_1 \otimes v_2 \otimes \dots \otimes v_n \in V^{\otimes n}$ and $\xi = \xi_1 \otimes \xi_2 \otimes \dots \otimes \xi_n \in V^{*\otimes n}$ is given by *a full contraction*

$$\langle\, v\,,\, \xi\,\rangle \stackrel{\text{def}}{=} \prod_{i=1}^{n} \xi_i(v_i)\,. \tag{7}$$

Let $\{e_1, e_2, \dots, e_n\} \subset V$ and $\{\xi_1, \xi_2, \dots, \xi_n\} \subset V^*$ be dual bases, i. e. $\xi_i(e_j) = 0$ for $i \neq j$ and $\xi_i(e_i) = 1$. Then the base words $\{\, e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_r}\,\}$ and $\{\, \xi_{j_1} \otimes \xi_{j_2} \otimes \dots \otimes \xi_{j_s}\,\}$ form the dual bases for $\mathsf{T}^\bullet V$ and $\mathsf{T}^\bullet V^*$ with respect to the full contraction. So, $V^{\otimes n *} \simeq V^{*\otimes n}$. On the other side, the space $(V^{\otimes n})^*$ is naturally identified with the space of all multilinear forms $\underbrace{V \times V \times \dots \times V}_{n} \longrightarrow k$, because $V^{\otimes n}$ is universal. So, there exists a canonical isomorphism between $V^{*\otimes n} = \underbrace{V^* \otimes V^* \otimes \dots \otimes V^*}_{n}$ and the space of multilinear forms in n arguments from V. It sends a tensor $\xi = \xi_1 \otimes \xi_2 \otimes \dots \otimes \xi_n \in V^{*\otimes n}$ to the form $(v_1, v_2, \dots, v_n) \longmapsto \prod\limits_{i=1}^{n} \xi_i(v_i)$.

Partial contractions. Let $\{1, 2, \dots, p\} \xhookleftarrow{I} \{1, 2, \dots, m\} \xhookrightarrow{J} \{1, 2, \dots, q\}$ be two injective (not necessary monotonous) maps. We write i_ν and j_ν for $I(\nu)$ and $J(\nu)$ respectively and consider $I = (i_1, i_2, \dots, i_m)$ and $J = (j_1, j_2, \dots, j_m)$ as two ordered (but not necessary monotonous) index collections of the same cardinality. A linear operator

$$\underbrace{V^* \otimes V^* \otimes \dots \otimes V^*}_{p} \otimes \underbrace{V \otimes V \otimes \dots \otimes V}_{q} \xrightarrow{c_J^I} \underbrace{V^* \otimes V^* \otimes \dots \otimes V^*}_{p-m} \otimes \underbrace{V \otimes V \otimes \dots \otimes V}_{q-m}$$

which sends $\xi_1 \otimes \xi_2 \otimes \dots \otimes \xi_p \otimes v_1 \otimes v_2 \otimes \dots \otimes v_q$ to $\prod\limits_{\nu=1}^{m} \xi_{i_\nu}(v_{j_\nu}) \cdot \bigotimes\limits_{i \notin \mathrm{im}\,(I)} \xi_i \otimes \bigotimes\limits_{j \notin \mathrm{im}\,(J)} v_j$ is called *a partial contraction* in the indexes I and J.

Example 4.9: the contraction between a vector and a multilinear form. Consider a multilinear form $\varphi(v_1, v_2, \dots, v_n)$ as the tensor from $V^{*\otimes n}$ and contact it in the first index with a vector $v \in V$. The result belongs to $V^{*\otimes(n-1)}$ and gives a multilinear form in $(n-1)$ arguments. This form is denoted by $i_v\varphi$ and called an *inner product* of v and φ.

Exercice 4.10. Check that $i_v\varphi(w_1, w_2, \dots, w_{n-1}) = \varphi(v,\, w_1, w_2, \dots, w_{n-1}\,)$, i. e. the inner multiplication by v is just the fixation of v in the first argument.

Linear span of a tensor. Let $U, W \subset V$ be any two subspaces. Writing down the standard monomial bases, we see immediately that $(U \cap W)^{\otimes n} = U^{\otimes n} \cap W^{\otimes n}$ in $V^{\otimes n}$. So, for any $t \in V^{\otimes n}$ there is a *minimal* subspace $\mathit{span}(t) \subset V$ whose n-th tensor power contains t. It is called *a linear span* of t and coincides with the intersection of all $W \subset V$ such that $t \in W^{\otimes n}$. To describe $\mathit{span}(t)$ more constructively, for any injective (not necessary monotonous) map

$$J = (j_1, j_2, \dots, j_{n-1}) \,:\, \{1,\, 2,\, \dots,\, (n-1)\} \xhookrightarrow{} \{1,\, 2,\, \dots,\, n\}$$

[2] If you like it, make the following formal exercise: deduce from the universality that for any associative k-algebra A and vector space map $V \xrightarrow{f} A$ there exists a unique algebra homomorphism $\mathsf{T}^\bullet V \xrightarrow{\alpha} A$ such that $\alpha|_V = f$

consider a linear map $V^{*\otimes(n-1)} \xrightarrow{c_t^J} V$ defined by complete contraction with t: it sends a decomposable tensor $\varphi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_{n-1}$ to a vector obtained by coupling ν-th factor ξ_ν, of φ, with j_ν-th factor of t for all $1 \leqslant \nu \leqslant (n-1)$, i. e.

$$c_t^J(\varphi) = c^{(1,2,\dots,(n-1))}_{(j_1,j_2,\dots,j_{n-1})}(\varphi \otimes t)\ .$$

Claim 4.11. *span*$(t) \subset V$ *is spanned by the images* $c_t^J\left(V^{*\otimes(n-1)}\right) \subset V$ *taken for all possible* J.

Proof. Let *span*$(t) = W \subset V$. Then $t \in W^{\otimes n}$ and $\mathrm{im}\left(c_t^J\right) \subset W\ \forall J$. It remains to prove that W is annihilated by any linear form $\xi \in V^*$ which annihilate all the subspaces $\mathrm{im}\left(c_t^J\right)$. Suppose the contrary: let $\xi \in V^*$ have non zero restriction on W but annihilate all $c_t^J\left(V^{*\otimes(n-1)}\right)$. Then there exist a base $\{w_1, w_2, \dots, w_k\}$ for W and a base $\{\xi_1, \xi_2, \dots, \xi_d\}$ for V^* such that: $\xi_1 = \xi$, the restrictions of $\xi_1, \xi_2, \dots, \xi_k$ onto W form the base of W^* dual to $\{w_\nu\}$, and $\xi_{k+1}, \dots, \xi_d$ annihilate W. Now, for any J and $\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_{n-1}}$ we have

$$0 = \left\langle\, \xi\,,\, c_t^J\left(\xi_{i_1} \otimes \xi_{i_2} \otimes \cdots \otimes \xi_{i_{n-1}}\right) \right\rangle = \left\langle\, \xi_{i_{j_1^{-1}}} \otimes \cdots \otimes \xi_{i_{j_{s-1}^{-1}}} \otimes \xi \otimes \xi_{i_{j_{s+1}^{-1}}} \otimes \cdots \otimes \xi_{i_{j_n^{-1}}}\,,\, t \right\rangle \tag{8}$$

where $s = \{1, 2, \dots, n\} \setminus \mathrm{im}\,(J)$ and $J^{-1} = (j_1^{-1}, j_2^{-1}, \dots, j_n^{-1})$ is the inverse to J map from $\mathrm{im}\,(J) \subset \{1, 2, \dots, n\}$ to $\{1, 2, \dots, (n-1)\}$. Note that each basic monomial of $V^{*(n-1)}$ containing as a factor $\xi_1 = \xi$ can appear as the first operand in the right side of (8). But if we expand t trough the basic monomials $w_{i1} \otimes w_{i2} \otimes \cdots \otimes w_{in}$, then the coefficients of this expansion can be computed as full contractions of t with the corresponding elements $\xi_{i1} \otimes \xi_{i2} \otimes \cdots \otimes \xi_{in}$ from the dual base for $W^{*\otimes n}$. By (8), such a contraction equals zero as soon one of ξ_{i_α} equals $\xi_1 = \xi$, which is dual to w_1. So, *span*(t) is contained in the linear span of $w_2, \dots, w_k$ but this contradicts our assumption. □

Symmetry properties. A multilinear map $\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\varphi} W$ is called *symmetric* if it doesn't change its value under any permutations of the arguments. If the value of φ is stable under the even permutations and changes the sign under the odd ones, then φ is called *skew symmetric*. Since the composition operator (5) preserves the symmetry properties, for the (skew)symmetric φ the composition operator (5) turns into

$$\left(\begin{array}{c}\text{the space } \mathrm{Hom}(U,W) \text{ of all}\\ \text{linear operators } U \xrightarrow{F} W\end{array}\right) \xrightarrow{F \longmapsto F \circ \varphi} \left(\begin{array}{c}\text{the space of all (skew)symmetric}\\ \text{multilinear maps } \underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\psi} W\end{array}\right) \tag{9}$$

A (skew)symmetric multilinear map $\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\varphi} U$ is called *universal* if (9) is an isomorphism for any W. In the symmetric case the universal target space is denoted by S^nV and called n-th *symmetric power* of V. In the skew symmetric case it is called n-th *exterior power* of V and denoted by Λ^nV.

Exercice 4.12. Show that, if exist, S^nV and Λ^nV are unique up to unique isomorphism commuting with the universal maps.

Symmetric algebra $S^\bullet V$ of V is a factor algebra of the free associative algebra $\mathsf{T}^\bullet V$ by a commutation relations $vw = wv$. More precisely, denote by $\mathcal{I}_{\mathrm{sym}} \subset \mathsf{T}^\bullet V$ a linear span of all tensors

$$\cdots \otimes v \otimes w \otimes \cdots \quad - \quad \cdots \otimes w \otimes v \otimes \cdots\ ,$$

where the both terms are decomposable, have the same degree, and differ only in order of v, w. Clearly, $\mathcal{I}_{\mathrm{sym}}$ is a double-sided ideal in $\mathsf{T}^\bullet V$ generated by a linear span of all the differences $v \otimes w - w \otimes v \in V \otimes V$. The factor algebra $S^\bullet V \stackrel{\mathrm{def}}{=} \mathsf{T}^\bullet V / \mathcal{I}_{\mathrm{sym}}$ is called *a symmetric algebra* of the vector space V. By the construction, it is commutative[3]. Since $\mathcal{I}_{\mathrm{sym}} = \bigoplus\limits_{n \geqslant 0} (\mathcal{I}_{\mathrm{sym}} \cap V^{\otimes n})$ is the direct sum of its homogeneous components, the symmetric algebra is graded: $S^\bullet V = \bigoplus\limits_{n \geqslant 0} S^nV$, where $S^nV \stackrel{\mathrm{def}}{=} V^{\otimes n} / (\mathcal{I}_{\mathrm{sym}} \cap V^{\otimes n})$.

Claim 4.13. *The tensor multiplication followed by the factorization map:*

$$\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\tau} V^{\otimes n} \xrightarrow{\pi} S^n(V) \tag{10}$$

gives the universal symmetric multilinear map.

[3] Again, if you like it, prove that for any commutative k-algebra A and a vector space map $V \xrightarrow{f} A$ there exists a unique homomorphism of commutative algebras $S^\bullet V \xrightarrow{\alpha} A$ such that $\alpha|_V = f$

Proof. *Any* multilinear map $V \times V \times \cdots \times V \xrightarrow{\varphi} W$ is uniquely decomposed as $\varphi = F \circ \tau$, where $V^{\otimes n} \xrightarrow{F} W$ is linear. F is factored through π iff $F(\cdots \otimes v \otimes w \otimes \cdots) = F(\cdots \otimes w \otimes v \otimes \cdots)$, i. e. iff $\varphi(\dots, v, w, \dots) = \varphi(\dots, w, v, \dots)$ □

The graded components S^nV are called *symmetric powers* of V and the map (10) is called *a symmetric multiplication*. If a base $\{e_1, e_2, \dots, e_d\} \subset V$ is fixed, then S^nV is naturally identified with the space of all homogeneous polynomials of degree n in e_i. Namely, consider the polynomial ring $k[e_1, e_2, \dots, e_d]$ (whose «variables» are the base vectors e_i) and identify V with the space of all linear homogeneous polynomials in e_i.

Exercice 4.14. Check that the multiplication map

$$\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{(\ell_1, \ell_2, \dots, \ell_n) \longmapsto \prod_{\nu=1}^{n} \ell_\nu} \begin{pmatrix} \text{the homogeneous polynomials} \\ \text{of degree } n \text{ in } e_i \end{pmatrix}$$

is universal and show that $\dim S^nV = \binom{d+n-1}{n}$.

Exterior algebra $\Lambda^\bullet V$ of V is a factor algebra of the free associative algebra $\mathsf{T}^\bullet V$ by a skew commutation relations $vw = -wv$. More precisely, consider a double-sided ideal $\mathcal{I}_{\text{skew}} \subset \mathsf{T}^\bullet V$ generated by all sums $v \otimes w + w \otimes v \in V \otimes V$ and put $\Lambda^\bullet V \stackrel{\text{def}}{=} \mathsf{T}^\bullet V / \mathcal{I}_{\text{skew}}$. Exactly as in the symmetric case, the ideal $\mathcal{I}_{\text{skew}}$ is *homogeneous*: $\mathcal{I}_{\text{skew}} = \bigoplus\limits_{n \geqslant 0} (\mathcal{I}_{\text{skew}} \cap V^{\otimes n})$, where $(\mathcal{I}_{\text{skew}} \cap V^{\otimes n})$ is the linear span of all sums

$$\cdots \otimes v \otimes w \otimes \cdots \quad + \quad \cdots \otimes w \otimes v \otimes \cdots$$

(the both items have degree n and differ only in the order of v, w). So, the factor algebra $\Lambda^\bullet V$ is graded by the subspaces $\Lambda^n V \stackrel{\text{def}}{=} V^{\otimes n} / (\mathcal{I}_{\text{skew}} \cap V^{\otimes n})$.

Exercice 4.15. Prove that the tensor multiplication followed by the factorization

$$\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\tau} V^{\otimes n} \xrightarrow{\pi} \Lambda^n(V) \tag{11}$$

gives the universal skew symmetric multilinear map.

The map (11) is called *an exterior* or *skew* multiplication. The skew product of vectors $(v_1, v_2, \dots, v_n)$ is denoted by $v_1 \wedge v_2 \wedge \cdots \wedge v_n$. By the construction, it changes the sign under the transposition of any two consequent terms. So, under any permutation of terms the skew product is multiplied by the sign of the permutation.

Exercice 4.16. For any $U, W \subset V$ check that $S^nU \cap S^nW = S^n(U \cap U)$ in S^nV and $\Lambda^nU \cap \Lambda^nW = \Lambda^n(U \cap U)$ in Λ^nV.

Grassmannian polynomials. Let $\{e_1, e_2, \dots, e_d\} \subset V$ be a basis. Then the exterior algebra $\Lambda^\bullet V$ is identified with *a grassmannian polynomial ring* $k\langle e_1, e_2, \dots, e_d \rangle$ whose «variables» are the base vectors e_i which *skew commute*, that is, $e_i \wedge e_j = -e_j \wedge e_i$ for all i, j. More precisely, it is linearly spanned by the grassmannian monomials $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$. It follows from skew commutativity that $e_i \wedge e_i = 0$ for all i, that is, a grassmannian monomial vanishes as soon as it becomes of degree more then 1 in some e_i. So, any grassmannian monomial has a unique representation $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$ with $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant d$.

Claim 4.17. *The monomials* $e_I \stackrel{\text{def}}{=} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$, *where* $I = (i_1, i_2, \dots, i_n)$ *runs through the increasing* n*-element subsets in* $\{1, 2, \dots, d\}$, *form a base for* Λ^nV. *In particular,* $\Lambda^nV = 0$ *for* $n > \dim V$, $\dim \Lambda^nV = \binom{d}{n}$, *and* $\dim k\langle e_1, e_2, \dots, e_d \rangle = 2^d$.

Proof. Consider $\binom{d}{n}$-dimensional vector space U whose base consists of the symbols ξ_I, where $I = (i_1, i_2, \dots, i_n)$ runs through the increasing n-element subsets in $\{1, 2, \dots, d\}$. Define a skew symmetric multilinear map

$$V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\alpha} U \ : \ (e_{j_1}, e_{j_2}, \dots, e_{j_n}) \longmapsto \operatorname{sgn}(\sigma) \cdot \xi_I \ ,$$

where $I = (j_{\sigma(1)}, j_{\sigma(2)}, \dots, j_{\sigma(n)},)$ is an *increasing* collection obtained from $(j_1, j_2, \dots, j_n)$ by a (unique) permutation σ. This map is universal. Indeed, for any skew symmetric multilinear map $\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\varphi} W$ there exists at most one linear operator $U \xrightarrow{F} W$ such that $\varphi = F \circ \alpha$, because it has to act on the base as $F(\xi_I) = \varphi(e_{i_1}, e_{i_2}, \dots, e_{i_n}))$ for all increasing $I = (i_1, i_2, \dots, i_n)$. On the other side, such F really decomposes φ, because $F(\alpha(e_{j_1}, e_{j_2}, \dots, e_{j_n})) = \varphi(e_{j_1}, e_{j_2}, \dots, e_{j_n}))$ for all *not increasing* base collections $(e_{j_1}, e_{j_2}, \dots, e_{j_n}) \subset \underbrace{V \times V \times \cdots \times V}_{n}$ as well. By the universality, there exists a canonical isomorphism between U and Λ^nV which sends ξ_I to $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} = e_I$. □

Exercice 4.18. Check that $f(e) \wedge g(e) = (-1)^{\deg(f) \cdot \deg(g)}\, g(e) \wedge f(e)$ for all homogeneous $f(e), g(e) \in k\langle e_1, e_2, \dots, e_d \rangle$. In particular, each *even degree* homogeneous polynomial commutes with any grassmannian polynomial.

Exercice 4.19. Describe the center of $k\langle e_1, e_2, \dots, e_d\rangle$, i. e. all grassmannian polynomials which commute with everything.

Example 4.20: linear base change in grassmannian polynomial. Under the linear substitution $e_i = \sum\limits_{j=1}^{d} a_{ij}\,\xi_j$ the base monomials e_I are changed by the new base monomials ξ_I as follows:

$$e_I = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n} = \left(\sum_{j_1} a_{i_1 j_1}\,\xi_j\right) \wedge \left(\sum_{j_2} a_{i_2 j_2}\,\xi_j\right) \wedge \dots \wedge \left(\sum_{j_n} a_{i_n j_n}\,\xi_j\right) =$$

$$= \sum_{1\leqslant j_1<j_2<\dots<j_n\leqslant n} \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\, a_{i_1 j_{\sigma(1)}} a_{i_2 j_{\sigma(2)}} \dots a_{i_n j_{\sigma(n)}}\, \xi_{j_1} \wedge \xi_{j_2} \wedge \dots \wedge \xi_{j_n} = \sum_J a_{IJ}\,\xi_J\ ,$$

where a_{IJ} is $(n\times n)$-minor of (a_{ij}) placed at $(i_1, i_2, \dots, i_n)$ rows and $(j_1, j_2, \dots, j_n)$ columns, and J runs through all increasing index collections of *the length* $\#J = n$.

Exercice 4.21. Let $|I| = \sum\limits_{\nu} i_\nu$ denote *a weight* of the increasing index collection $I = (i_1, i_2, \dots, i_n)$ of length $\#I = n$.

Check that

$$e_I \wedge e_{\widehat{I}} = (-1)^{|I|+\frac{1}{2}\#I(1+\#I)} \cdot e_1 \wedge e_2 \wedge \dots \wedge e_d \tag{12}$$

for any two *complementary* index collections I and $\widehat{I} \stackrel{\text{def}}{=} \{1, 2, \dots, n\} \setminus I$.

Example 4.22: Laplace relations via grassmannian polynomials. Let us take two complementary index collections I and $\widehat{I} \stackrel{\text{def}}{=} \{1, 2, \dots, n\} \setminus I$ and do a base change $e_i = \sum\limits_{j=1}^{d} a_{ij}\,\xi_j$ in the identity (12). Its left side $e_I \wedge e_{\widehat{I}}$ turns to

$$\left(\sum_{\substack{K:\\ \#K=\#I}} a_{IK}\,\xi_K\right) \wedge \left(\sum_{\substack{L:\\ \#L=(d-\#I)}} a_{L\widehat{I}}\,\xi_L\right) = (-1)^{\frac{1}{2}\#I(1+\#I)} \sum_{\substack{K:\\ \#K=\#I}} (-1)^{|K|} a_{IK} a_{\widehat{I}\widehat{K}}\,\xi_1 \wedge \xi_2 \wedge \dots \wedge \xi_d\ ,$$

where $\widehat{K} = \{1, 2, \dots, d\} \setminus K$. The right side of (12) gives $(-1)^{\frac{1}{2}\#I(1+\#I)}(-1)^{|I|}\det(a_{ij})\xi_1 \wedge \xi_2 \wedge \dots \wedge \xi_d$. So, for any any collection I of rows in any square matrix (a_{ij}) the following relation holds:

$$\sum_{\substack{K:\\ \#K=\#I}} (-1)^{|K|+|I|} a_{IK}\,\widehat{a}_{IK} = \det(a_{ij})\ , \tag{13}$$

where $\widehat{a}_{IK} \stackrel{\text{def}}{=} a_{\widehat{I}\widehat{K}}$ denotes the $(d-n)\times(d-n)$ - minor which is complementary[4] to a_{IK} and the summation runs over all $(n\times n)$ - minors a_{IK} contained in the rows $(i_1, i_2, \dots, i_n)$.

If we take $I \cap J \neq \varnothing$, then starting from $e_I \wedge e_J = 0$ instead of (12) we get by the same calculation the relation

$$\sum_{\substack{K:\\ \#K=\#I}} (-1)^{|K|+|I|} a_{IK}\,\widehat{a}_{JK} = 0\ , \tag{14}$$

The identities (13) and (14) are known as *Laplace relations*. Let us fix, say lexicografical, order on the set of indices I and arrange all $(n\times n)$-minors a_{IJ} as $\binom{d}{n}\times\binom{d}{n}$ - matrix $A^{(n)} \stackrel{\text{def}}{=} (a_{IJ})$. If we denote by $\widehat{A}^{(n)}$ a matrix whose (IJ)-entry equals $\big((-1)^{|I|+|J|}\widehat{a}_{JI}\big)$, then all the Laplace relations are expressed by the single matrix equality $A^{(n)}\cdot\widehat{A}^{(n)} = \det(a_{ij})\cdot E$.

Example 4.23: Reduction of grassmannian quadratic forms. Any homogeneous grassmannian polynomial of degree 2 can be written as

$$\xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4 + \dots + \xi_{r-1} \wedge \xi_r \tag{15}$$

in some base (over any field k). Namely, we can suppose[5] that our grassmannian quadratic form is

$$q(e) = e_1 \wedge (\alpha_2 e_2 + \dots + \alpha_n e_n) + (\text{terms without } e_1)$$

where $\alpha_2 \neq 0$ and $\xi_2 \stackrel{\text{def}}{=} \alpha_2 e_2 + \dots + \alpha_n e_n$ does not contain e_1, that is it can be included in the new base $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ with $\zeta_i = e_i$ for $i \neq 2$. After the substitution $e_2 = \alpha_2^{-1}(\xi_2 - \alpha_3\zeta_3 - \dots - \alpha_n\zeta_n)$, $e_i = \zeta_i$ for $i \neq 2$, we can write q as $q(\zeta) = \zeta_1 \wedge \zeta_2 + \zeta_2 \wedge (\beta_3\zeta_3 + \dots + \beta_n\zeta_n) + (\text{terms without } \zeta_1 \text{ and } \zeta_2)$. So, in the next new base: $\{\xi_1, \xi_2, \dots, \xi_n\}$ with $\xi_1 = \zeta_1 - \beta_3\zeta_3 - \dots - \beta_n\zeta_n$, $\xi_i = \zeta_i$ for $i \neq 1$ our q turns to

$$q(\zeta) = \xi_1 \wedge \xi_2 + (\text{terms without } \xi_1 \text{ and } \xi_2)$$

and this procedure can be repeated inductively for the remaining terms.

Exercice 4.24. Let $A = (a_{ij})$ be a skew symmetric matrix (i. e. $a_{ij} = -a_{ji}$) and $q(e) = \sum\limits_{ij} a_{ij}\,e_i \wedge e_j$ be a grassmannian quadratic form. Show that in the representation (15) the number r doesn't depend on the base choice and equals $\operatorname{rk} A$. (In particular, $\operatorname{rk} A$ is always even.)

[4] i. e. sitting in the complementary rows and columns

[5] may be after appropriate renumbering of the base vectors