Plane drawings: line pencils and conics[1].

<u>PROJECTIVE DUALITY.</u> For any $0 \leqslant m \leqslant (n-1)$ there is a canonical bijection between the m-dimensional projective subspaces in $\mathbb{P}_n = \mathbb{P}(V)$ and the $(n-1-m)$-dimensional ones in $\mathbb{P}_n^\times \overset{\text{def}}{=} \mathbb{P}(V^*)$. It sends a subspace $L = \mathbb{P}(U)$ to the subspace $L^\times \overset{\text{def}}{=} \mathbb{P}(\operatorname{Ann}(U))$, where $\operatorname{Ann}(U) \overset{\text{def}}{=} \{\, \xi \in V^* \mid \xi(u) = 0 \ \ \forall u \in U \,\}$ is *an anihilator* of U. Since under the natural identification $V^{**} = V$ we have $\operatorname{Ann}\operatorname{Ann} U = \{\, v \in V \mid \xi(v) = 0 \ \ \forall \xi \in \operatorname{Ann} U \,\} = U$, then $L^{\times\times} = L$. The correspondence $L \leftrightarrow L^\times$ is called *a projective duality*. It inverts the inclusions[2] and the linear incidence: $L_1, L_2, \dots, L_r$ are contained in some m-dimensional subspace L iff $L_1^\times, L_2^\times, \dots, L_r^\times$ contain some $(n-m-1)$-dimensional subspace $L^\times$ ([3]). The projective duality translates the geometry on $\mathbb{P}_n$ to the one on $\mathbb{P}_n^\times$ and back. For example, in $\mathbb{P}_2$-case we have the following dictionary:

a line $\ell \subset \mathbb{P}_2$	$\longleftrightarrow$	a point $\ell^\times \in \mathbb{P}_2^\times$
the points p of the above line ℓ	$\longleftrightarrow$	the lines $p^\times$ passing through the above point $\ell^\times$
the line passing through two points $p_1, p_2 \in \mathbb{P}_2$	$\longleftrightarrow$	the intersection point for two lines $p_1^\times, p_2^\times \subset \mathbb{P}_2^\times$
the points p of some conic Q	$\longleftrightarrow$	the tangent lines $p^\times$ of some conic $Q^\times$
the tangent lines ℓ of Q	$\longleftrightarrow$	the points $\ell^\times$ of $Q^\times$

Exercice 3.1. Explain the last two items by proving that the tangent spaces of a non singular quadric $Q \subset \mathbb{P}_n$ correspond to the points of some non singular quadric $Q^\times \subset \mathbb{P}_n^\times$. Show also that Q and $Q^\times$ have inverse Gram matrices in dual bases of V and V^*.

HINT. If $\xi = (\xi_0, \xi_1, \dots, \xi_n) \in V^*$ and $x = (x_0, x_1, \dots, x_n) \in V$ in dual coordinate systems, then $\xi(x) = \xi \cdot {}^t x$. Let Q be given by $x \cdot A \cdot {}^t x = 0$. Then $T_x Q = \mathbb{P}(\operatorname{Ann} \xi) \iff \xi = x \cdot A \iff x = \xi \cdot A^{-1}$. Hence,

$$x \in Q \iff x \cdot A \cdot {}^t x = 0 \iff \left(\xi \cdot A^{-1}\right) \cdot A \cdot {}^t\!\left(\xi \cdot A^{-1}\right) = 0 \iff \xi \cdot A^{-1}\,{}^t\xi = 0 \iff \xi \in Q^\times ,$$

where $Q^\times$ is given by the Gram matrix A^{-1}.

<u>Corollary 3.2.</u> *Any 5 lines without triple intersections in $\mathbb{P}_2$ are tangent to a unique non singular conic.*

Proof. This assertion is projectively dual to the Proposition 2.16. Namely, let $\ell_i \in \mathbb{P}_2$ be the given lines. There exists a unique conic $Q^\times$ passing through the 5 points $\ell_i^\times \in \mathbb{P}_2^\times$. Then $\ell_i = \ell_i^{\times\times}$ are tangent to the dual conic $Q = Q^{\times\times} \subset \mathbb{P}_2$. $\square$

<u>PROJECTIVE LINEAR ISOMORPHISMS $\mathbb{P}_1 \longrightarrow \mathbb{P}_1$ VIA CONICS.</u> Consider a non singular conic Q and a line ℓ and write $Q \xrightarrow{\pi_\ell^p} \ell$ for the bijective map given by projection from a point $p \in Q$ extended into p by sending $p \longmapsto \ell \cap T_p Q$.

Exercice 3.3. Show that this bijection is given by some rational algebraic functions, which express coordinates of the corresponding points through each other.

HINT. To express the coordinates of $t = \pi_\ell^p(x)$ through x, you take special coordinates where ℓ is given by $x_0 = 0$ and $p = (1:0:0)$; then $t = (0 : x_1 : x_2)$. In order to get the inverse expression, only two skills are quite enough: linear equations solving and finding the second root for a quadratic equation with the first root known. The both procedures have a rational output.

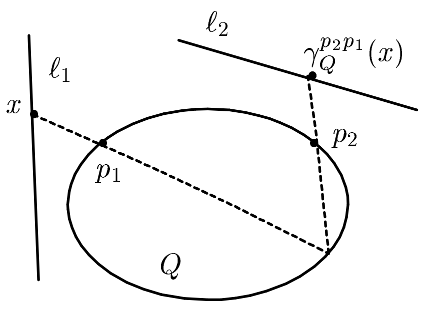

Fig. 3.1. Composing projections.

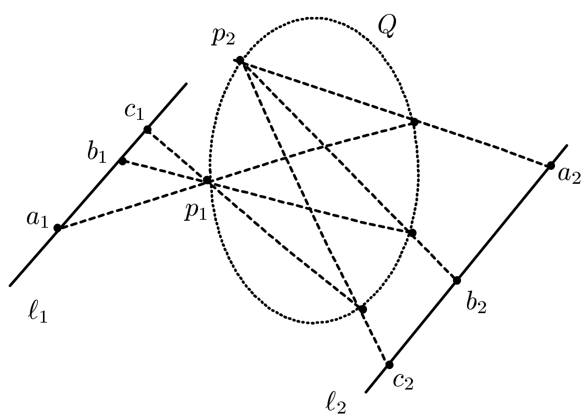

Fig. 3.2. How to find p_1, p_2.

[1] WARNING: in this lecture we always suppose that $\operatorname{char} k \neq 2$

[2] i. e. $L_1 \subset L_2 \iff L_1^\times \supset L_2^\times$

[3] for example, 3 points are collinear iff the dual 3 hyperplanes have common subspace of codimension 2

So, if ℓ_1, ℓ_2 are two lines and p_1, p_2 are two distinct points on some non singular conic Q, then the composition $\gamma_Q^{p_2 p_1} \stackrel{\text{def}}{=} \pi_{\ell_2}^{p_2} \circ \left(\pi_{\ell_1}^{p_1}\right)^{-1}$ gives the projective linear isomorphism $\ell_1 \xrightarrow{\sim} \ell_2$ (see fig. 3.1). In fact, any projective linear isomorphism $\ell_1 \xrightarrow{\gamma} \ell_2$ can be presented[4] as $\gamma_Q^{p_2 p_1}$ for some Q and $p_1, p_2 \in Q$. Indeed, if γ sends, say $a_1, b_1, c_1 \in \ell_1$ to $a_2, b_2, c_2 \in \ell_2$, we pick *any* p_1, p_2 such that no 3 out of 5 points p_1, p_2, $(a_1\, p_1) \cap (a_2\, p_2)$, $(b_1\, p_1) \cap (b_2\, p_2)$, $(c_1\, p_1) \cap (c_2\, p_2)$ are collinear (see fig. 3.2) and draw Q through these 5 points. Then $\gamma = \gamma_Q^{p_2 p_1}$, because both have the same action on 3 points a_1, b_1, c_1.

<u>DRAWING A CONIC BY THE RULER.</u> Let 5 distinct points p_1, p_2, a, b, c lay on a non singular conic Q. Denote the line $(a\,c)$ by ℓ_1, the line $(b\,c)$ by ℓ_2, and the intersection point $(a\,p_2) \cap (b\,p_1)$ by O (see fig. 3.3). Then the projective linear isomorphism $\ell_1 \xrightarrow{\gamma_Q^{p_2 p_1}} \ell_2$ coincides with the simple projective linear isomorphism $\ell_1 \xrightarrow{\gamma_O} \ell_2$, which takes $x \in \ell_1$ to $\gamma_O(x) = (x\,O) \cap \ell_2$ (indeed, the both send $c \longmapsto c$, $a \longmapsto d$, $e \longmapsto b$, where $d = (a\,p_2) \cap \ell_2$ and $e = (b\,p_1) \cap \ell_1$ — see fig. 3.3).

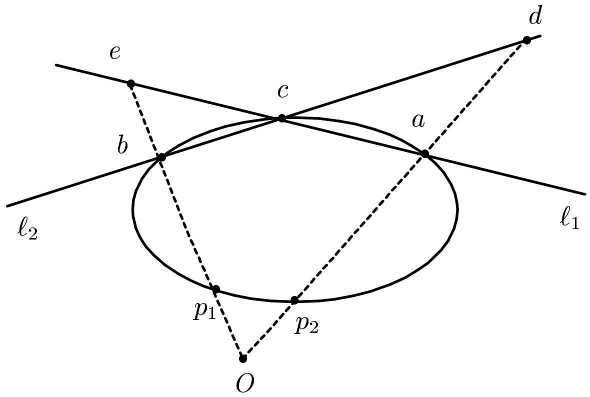

Fig. 3.3. Remarkable coincidence.

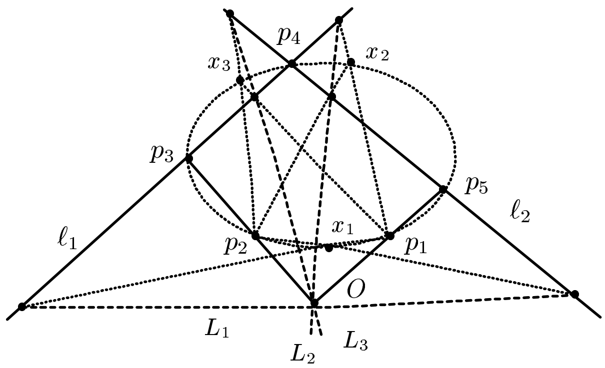

Fig. 3.4. Tracing a conic.

This simple remark allows us to trace, using only the ruler, a dense point set on the conic passing through 5 given points $p_1, p_2, \dots, p_5$ (see fig. 3.4). Namely, let $\ell_1 = (p_3 p_4)$, $\ell_2 = (p_4 p_5)$, $O = (p_1 p_6) \cap (p_2 p_3)$. Then any line $L \ni O$ gives two intersection points $\ell_1 \cap L$ and $\ell_2 \cap L$. These points are sent to each other by the projective linear isomorphism $\gamma_O = \gamma_Q^{p_2 p_1}$. So, if we draw lines through p_1 and $\ell_1 \cap L$, and trough p_2 and $\ell_2 \cap L$, then the intersection point x of these two lines has to lay on Q. On the fig. 3.4 the points x_1, x_2, x_3 are constructed by this way starting from the lines L_1, L_2, L_3 passing through O.

<u>Proposition 3.4.</u> [5] *A hexagon $p_1, p_2, \dots, p_6$ is inscribed into a non singular conic iff the points*[6] $(p_1 p_2) \cap (p_4 p_5)$, $(p_2 p_3) \cap (p_5 p_6)$, $(p_3 p_4) \cap (p_6 p_1)$ *are collinear (see fig. 3.5).*

Proof. Draw the conic Q through 5 of p_i except for p_4 and put $\ell_1 = (p_1 p_6)$, $\ell_2 = (p_1 p_2)$, $y = (p_5 p_6) \cap (p_2 p_3)$, $x = (p_3 p_4) \cap \ell_1$, $z = (xy) \cap \ell_2$. Then $z \in \ell_2$ is the image of $x \in \ell_1$ under the projective linear isomorphism $\gamma_y = \gamma^{p_5 p_3} : \ell_1 \xrightarrow{\sim} \ell_2$ like before. In particular, the intersection point $(p_3 p_4) \cap (p_5 z)$ lays on Q. Hence, $p_4 \in Q$ iff $p_4 = (p_3 p_4) \cap (p_5 z)$. $\square$

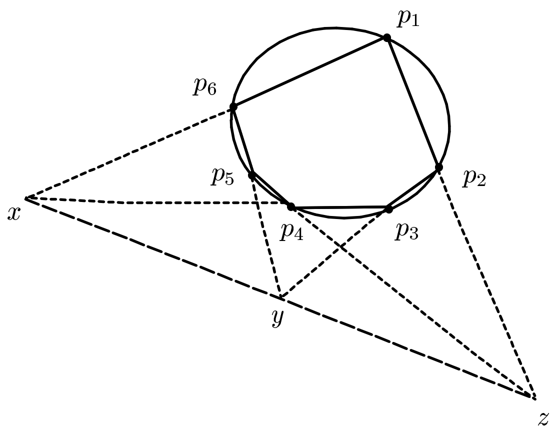

Fig. 3.5. Inscribed hexagon.

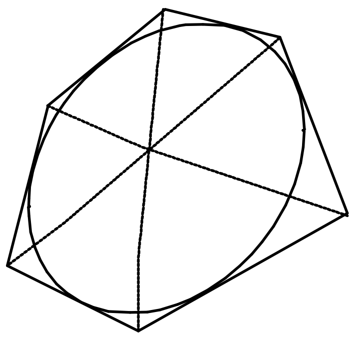

Fig. 3.6. Circumscribed hexagon.

<u>Corollary 3.5.</u> [7] *A hexagon $p_1, p_2, \dots, p_6$ is circumscribed around a non singular conic iff its main diagonals* $(p_1 p_4)$, $(p_2 p_5)$,

[4] in infinitely many different ways

[5] It is known as *B. Pascal's theorem*

[6] i. e. 3 intersection points of the line pairs passing through the opposite sides of the hexagon

[7] It is known as *Ch. J. Brianchon's theorem*

(p_3p_6) *are intersecting at one point (see fig. 3.6).*

Proof. This is just the projectively dual version of the Pascal theorem. □

LINEAR ISOMORPHISMS OF PENCILS. A family of geometrical figures is referred as *a pencil*, if it is naturally parameterized by the projective line. For example, all lines passing through a given point $p \in \mathbb{P}_2$ form a pencil, because their equations run trough the line $p^\times \in \mathbb{P}_2^\times$ by projective duality. More generally, there is a pencil of hyperplanes $H \subset \mathbb{P}_n$ passing through a given subspace $L \subset \mathbb{P}_n$ of codimension 2. Such a pencil is denoted by $|h - L|$ (read: «all hyperplanes containing L») or by $L^\times \in \mathbb{P}_n^\times$. Given two such pencils, say $L_1^\times$, $L_2^\times$ and 3 points $a, b, c \in \mathbb{P}_n \setminus (L_1 \cup L_2)$ such that 3 hyperplanes from $L_i^\times$ passing through them are distinct in the both pencils, then these 3 points define a projective linear isomorphism $L_1^\times \xrightarrow{\gamma_{abc}} L_2^\times$ that sends 3 hyperplanes of the first pencil passing through a, b, c to the corresponding ones from the second pencil.

Example 3.6: linear identification of two pencils $p_1^\times$ and $p_2^\times$ on $\mathbb{P}_2$ is given by any 3 points a, b, c such that any 2 of them are not collinear with p_1 or p_2. It sends $(p_1a) \longmapsto (p_2a)$, $(p_1b) \longmapsto (p_2b)$, $(p_1c) \longmapsto (p_2c)$. Let Q be the (unique!) conic passing trough 5 points p_1, p_2, a, b, c. There are two different cases (see fig. 3.7–3.8).

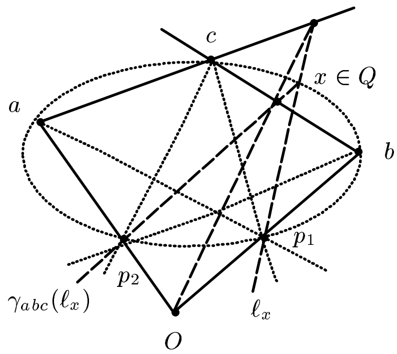

Fig. 3.7. Elliptic isomorphism of pencils.

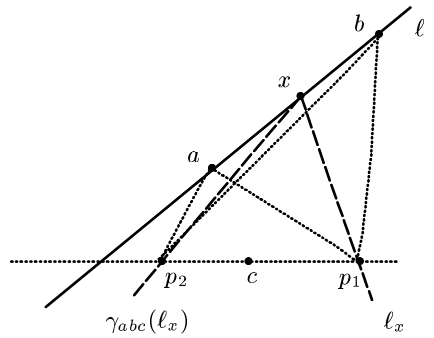

Fig. 3.8. Parabolic isomorphism of pencils.

(A) Elliptic case: Q is non singular, i. e. all 5 points are linearly general. In this case *the incidence graph*[8] of γ_{abc} coincides with Q, because the points of Q give the projective linear isomorphism $p_1^\times \longrightarrow p_2^\times$ that has the same action on a, b, c. Moreover, the above discussions let us draw the line $\gamma_{abc}(\ell_x)$ for a given $(\ell_x \in p_1^\times)$ by the ruler as follows (see fig. 3.7). First mark the point $O = (p_1b) \cap (p_2c)$; then find the intersection point $\ell_x \cap (a\,c)$, join it with O by a line, and mark the point where this line crosses $(b\,c)$; then the line $\gamma_{abc}(\ell_x)$ goes through this marked point.

(B) Parabolic case: Q is reducible, i. e. splits in two lines: (p_1p_2) and, say, $\ell = (a\,b)$. This happens when $c \in (p_1p_2)$ (recall that no 2 points from a, b, c are collinear with any of p_i). In this case the incidence graph for γ_{abc} coincides with the line ℓ (see fig. 3.8).

Dualizing these examples, we get the geometrical classification of projective linear isomorphisms $\ell_1 \xrightarrow{\sim} \ell_2$ between two given lines on $\mathbb{P}_2$.

Corollary 3.7. *There are exactly two types of projective linear isomorphisms $\ell_1 \xrightarrow{\sim} \ell_2$. Elliptic isomorphisms γ_Q correspond bijectively to the non singular conics Q touching both ℓ_1 and ℓ_2. Such γ_Q sends $x \longmapsto y$ iff the line $(x\,y)$ is tangent to Q (see fig. 3.9). Parabolic isomorphisms γ_L are parameterized by the points $L \in \mathbb{P}_2 \setminus (\ell_1 \cup \ell_2)$. Such γ_L sends $x \longmapsto y$ iff the line $(x\,y)$ pass through L (see fig. 3.10).* □

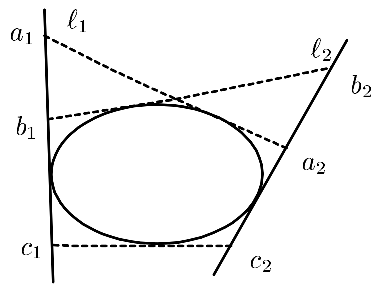

Fig. 3.9. Elliptic isomorphism of lines.

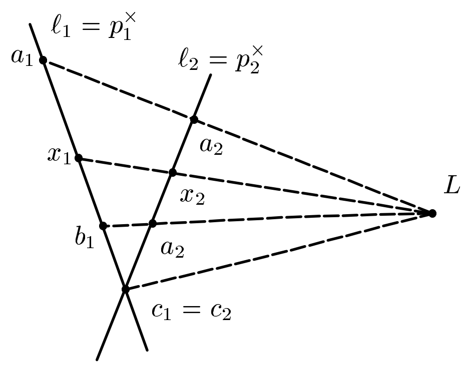

Fig. 3.10. Parabolic isomorphism of lines.

[8] i. e. a curve traced by the intersection points $\ell \cap \gamma_{abc}(\ell)$ while ℓ runs through $p_1^\times$

Towards Poncelet's porism. Given two non singular conics Q, Q', we can try to draw an n-gone simultaneously inscribed in Q' and circumscribed about Q: starting from some point $p_1 \in Q'$ draw a tangent line from p_1 to Q until it meets Q' in p_2, then draw a tangency from p_2 e. t. c. Poncelet's theorem says that if this procedure comes back to $p_n = p_1$ after n steps, then the same holds for any choice of the starting point p_1 maybe except for some finite set. The next two corollaries explain Poncelet porism for triangles (i. e. for $n = 3$)

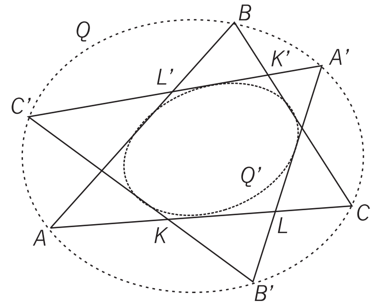

Fig. 3.11. Inscribed-circumscribed triangles.

Corollary 3.8. *Two triangles ABC and $A'B'C'$ are both inscribed into the same conic Q iff they are both circumscribed around the same conic Q'.*

Proof. We check only «$\Rightarrow$» implication, then the opposite implication comes by projective duality. Consider two lines $\ell = (AB)$, $\ell' = (A'B')$ and elliptic projective linear isomorphism $\ell \xrightarrow{\gamma} \ell'$ composed as the projection of ℓ onto Q from B' followed by the projection of Q onto ℓ' from B (see fig. 3.11). Since it takes $A \mapsto L'$, $C \mapsto K'$, $K \mapsto C'$, $L \mapsto A'$, all the sides of the both triangles should touch the conic associated with γ via Corollary 3.7. $\square$

Corollary 3.9. *Given two conics Q, Q' such that there exists a triangle ABC inscribed into Q and circumscribed around Q', then any point $A' \in Q$ is a vertex of a triangle $A'B'C'$ inscribed into Q and circumscribed around Q'.*

Proof. Take any $A' \in Q$ and pick $B', C' \in Q$ such that the lines $(A'B')$, $(A'C')$ are tangent to Q' (see fig. 3.11 again). By the previous corollary, both ABC and $A'B'C'$ are circumscribed around some conic, which must coincide with Q', because there exist a unique conic touching 5 lines (AB), (BC), (CA), $(A'B')$, $(A'C')$. $\square$

Exercice 3.10. Make Corollary 3.2 more precise by finding necessary and sufficient condition on 5 lines in $\mathbb{P}_2$ for existence of a unique non singular conic touching all of them.

Hint. This is projectively dual to Proposition 2.16

Making a projective isomorphism by the ruler. If an isomorphism $\ell_1 \xrightarrow{\gamma} \ell_2$ is given by its action on some 3 points, say: $a_1 \longmapsto a_2$, $b_1 \longmapsto b_2$, $c_1 \longmapsto c_2$, then we can find the image $\gamma(x)$ of any $x \in \ell_1$ by the ruler. In parabolic case this is trivial (see fig. 3.10). In elliptic case the drawing algorithm is projectively dual to the one discussed in Example 3.6 (A). Namely, draw the line $O^\times = (b_1a_2)$; then pass the line through x and $(a_1a_2) \cap (c_1c_2)$ and mark its intersection point with $O^\times$; now $\gamma(x)$ is the intersection of ℓ_2 with the line passing through the last marked point and $(b_1b_2) \cap (c_1c_2)$ (compare fig. 3.7 and fig. 3.12).

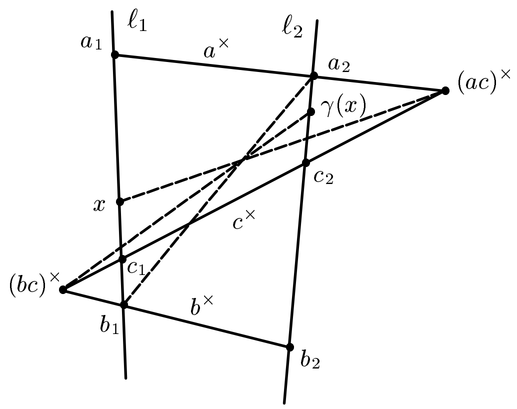

Fig. 3.12. Finding $\gamma_Q(x)$.

Exercice 3.11. Let $Q \subset \mathbb{P}_2$ be non singular conic considered together with some rational parameterization $\mathbb{P}_1 \xrightarrow{\sim} Q$. Show that for any two points $p_1, p_2 \in Q$ and a line $\ell \subset \mathbb{P}_2$ a map $Q \xrightarrow{\gamma} Q$ given by prescription: $x \overset{\gamma}{\longmapsto} y \iff \pi_\ell^{p_1} x = \pi_\ell^{p_2} y$ is induced by some linear automorphism of $\mathbb{P}_1$ (i. e. by some linear fractional reparameterization). Find the images of p_1, p_2 and the fixed points of the above map. Show that any bijection $Q \xrightarrow{\sim} Q$ induced by a linear automorphism of $\mathbb{P}_1$ can be (not uniquely) realized geometrically by two points $p_1, p_2 \in Q$ and a line $\ell \subset \mathbb{P}_2$ in the way described above. Is it possible, using only the ruler, to find (some) p_1, p_2, ℓ for a bijection $Q \xrightarrow{\sim} Q$ given by its action on 3 points $a, b, c, \in Q$?

Hint. Try $p_2 = a$.

Exercice 3.12*. Given a non singular conic Q and three points A, B, C, draw (using only the ruler) a triangle inscribed in Q with sides passing through A, B, C. How many solutions may have this problem?

Hint. Start «naive» drawing from any $p \in Q$ and denote by $\gamma(p)$ your return point after passing trough A, B, C. Is $p \longmapsto \gamma(p)$ a projective isomorphism of kind described in Exercice 3.11?

Exercice 3.13*. Formulate and solve projectively dual problem to the previous one.