Projective quadrics[1].

<u>QUADRATIC AND BILINEAR FORMS.</u> The zero set $Q = (q)_0 \subset \mathbb{P}(V)$ of a non zero quadratic form $q \in S^2V^*$ is called *a projective quadric*. If $2 \neq 0$ in k, then in homogeneous coordinates any quadratic form can be written as

$$q(x) = \sum_{i,j} a_{ij}\, x_i x_j = x \cdot A \cdot {}^t\!x\ ,$$

where $x = (x_0, x_1, \dots, x_n)$ is the coordinate row, ${}^t\!x$ is its transposed column version, and $A = (a_{ij})$ is a *symmetric* matrix over k, called *a Gram matrix* of q. Hence, there exists a unique bilinear form $\widetilde{q}(u, w)$ on $V \times V$ such that $q(x) = \widetilde{q}(x, x)$. It can be calculated by the following pairwise equivalent ways

$$\widetilde{q}(x,y) = \sum_{ij} a_{ij}\, x_i y_j = x \cdot A \cdot {}^t\!y = \frac{1}{2}\sum_i y_i \frac{\partial q(x)}{\partial x_i} = \frac{1}{2}\Big(q(x+y) - q(x) - q(y)\Big) = \frac{1}{4}\Big(q(x+y) - q(x-y)\Big)$$

The form $\widehat{q}$ is called *a polarization* of q. It gives a kind of scalar product on V and the Gram matrix elements are nothing but the scalar products $a_{ij} = \widetilde{q}(e_i, e_j)$, of basic vectors $\{e_i\}$.

In an other base, say by $(e'_0, e'_1, \dots, e'_n) = (e_0, e_1, \dots, e_n) \cdot C$, the new Gram matrix A' is expressed trogh the old Gram matrix A as

$$A' = {}^t C \cdot A \cdot C\ .$$

So, *the Gram determinant* $\det q \stackrel{\text{def}}{=} \det A \in k/k^2$, which is considered up to multiplications by the squares from k, doesn't depend on the choice of base. A quadric is called *non singular* if $\det q \neq 0$. Otherwise, it is called *singular*. Two quadrics are called *isomorphic* or *projectively equivalent* if they are produced from each other by a projective automorphism (i. e. by a base change).

<u>Theorem 2.1</u>. ([2]) *Any quadratic form on V has a diagonal Gram matrix in some base.*

Proof. If $q \equiv 0$, then its Gram matrix is diagonal. If not, then $q(v) = \widetilde{q}(v, v) \neq 0$ for some $v \in V$. Take it as the first vector of the base being constructed. Note that $V = L \oplus L^\perp$, where L is one dimensional subspace spaned by v and $L^\perp = \{\, w \in V \mid \widetilde{q}(v, w) = 0 \,\}$ is its orthogonal. Indeed, any $u \in V$ is uniquely decomposed as $u = \lambda v + v^\perp$ with $v^\perp \in L^\perp$. Namely, $\lambda = \widetilde{q}(v, u)/\widetilde{q}(v, v)$ and $v^\perp = u - (\widetilde{q}(v, u)/\widetilde{q}(v, v)) \cdot v$. Now, repeat these arguments to $L^\perp$ to find the next base vector, and so on. □

<u>Corollary 2.2</u>. *Over an algebraically closed field any quadric may be written as $\sum x_i^2 = 0$ in some coordinate system; in particular all non singular quadrics are projectively equivalent.*

Proof. Diagonal elements of the Gram matrix turns into units after the base change $e_i \longmapsto e_i/\sqrt{q(e_i)}$. □

<u>Example 2.3:</u> quadric on $\mathbb{P}_1$, by the Theorem 2.1, can be written in an appropriate coordinates either as $a\,x_0^2 + b\,x_1^2 = 0$ or as $a\,x_0^2 = 0$. The second is singular (i. e. has $\det q = 0$) and is called *a double point*. It consists of just one point $(0 : 1)$, which has «a multiplicity 2» in any reasonable sense. The first quadric is non singular and either is empty or consists of two distinct points. More precisely, $Q = \varnothing$ iff $-\det q \equiv -b/a \pmod{k^2}$ is not a square in k; in particular, this is impossible over an algebraically closed field. If $-\det q = \delta^2$ is a square, then $Q = \{\, (-\delta/a : 1)\,,\, (\delta/a : 1) \,\}$.

<u>QUADRIC AND LINE.</u> By the above example, for any quadric Q and a line ℓ there are exactly 4 possibilities: either $\ell \subset Q$, or $\ell \cap Q$ consists of 2 distinct points, or $\ell \cap Q$ is a double point, or $\ell \cap Q = \varnothing$. Moreover, the last case is impossible, if k is algebraically closed. A line is called *a tangent line*, if it lays on Q or crosses Q via a double point.

<u>CORRELATIONS.</u> Any quadratic form q on V induces the linear map $V \xrightarrow{\widehat{q}} V^*$ that sends a vector $v \in V$ to the linear form

$$\widehat{q}(v)\ :\ w \longmapsto \widetilde{q}(w, v)$$

The map $\widehat{q}$ is called *a correlation* (or *a polarity*) of q. Its matrix written in dual bases $\{e_i\} \subset V$, $\{x_i\} \subset V^*$ coincides with the Gram matrix A. In particular, q is non singular iff $\widehat{q}$ is an isomorphism. The space

$$\ker(q) \stackrel{\text{def}}{=} \ker \widehat{q} = \{\, v \in V \mid \widetilde{q}(w, v) = 0\ \ \forall\, w \in V \,\}$$

[1] WARNING: in this lecture we always suppose that $\operatorname{char} k \neq 2$

[2] It is known as the *Lagrange theorem*

is called *a kernel* of q. Its projectivization $Sing\,Q \stackrel{\text{def}}{=} \mathbb{P}(\ker q) \subset \mathbb{P}(V)$ is called *a vertex space* of $Q = (q)_0$ and $\operatorname{codim}_{\mathbb{P}(V)} Sing\,Q$ is called *a corank* of Q.

Theorem 2.4. *The intersection $Q' = L \cap Q$ is non singular for any projective subspace $L \subset \mathbb{P}(V)$ complementary to $Sing\,Q$; moreover, Q is the cone over Q' with the vertex space $Sing\,Q$, i. e. Q is the union of all lines crossing both Q' and $Sing\,Q$.*

Proof. Take any direct decomposition $V = \ker q \oplus U$ and let $L = \mathbb{P}(U)$. If $u \in U$ satisfy $\widetilde{q}(u,u') = 0 \quad \forall\, u' \in U$, then automatically $\widetilde{q}(u,v) = 0 \quad \forall\, v \in V$ and $u = 0$, because of $\ker q \cap U = 0$. Since $Q' = Q \cap L$ is given by the restriction $q|_U$, it is non singular. Further, for any line $\ell = \mathbb{P}(W)$ such that $\dim W \cap \ker Q = 1$ we have $\dim W \cap U = 1$ and $\operatorname{cork} q|_U \geqslant 1$. So, if $\ell \cap Sing\,Q = \{p\}$ is just one point, then $\ell \cap L \neq \varnothing$ and either $\ell \subset Q$ or $\ell \cap Q\{p\}$. That's all we need. □

Corollary 2.5. *A quadric $Q \subset \mathbb{P}_n$ over an algebraically closed field is uniquely up to an isomorphism defined by its corank; it may be equal to* $0,\,1,\,\dots\,,n$.

Proof. Corank is the number of diagonal zeros in the diagonal Gram matrix. □

Tangent space T_pQ to a quadric Q at a point $p \in Q$ is defined as the union of all tangent lines passing through p.

Lemma 2.6. *Let p and p' be distinct points and $p \in Q = (q)_0$. The line $\ell = (p\,p')$ is tangent to Q iff $\widetilde{q}(p,p') = 0$, i. e. iff p and p' are orthogonal with respect to polarization of q.*

Proof. Take some vectors u, u' representing p and p'. Then $\ell = \mathbb{P}(U)$. The restriction $q|_U$ has the Gram matrix

$$\begin{pmatrix} \widetilde{q}(u,u) & \widetilde{q}(u,u') \\ \widetilde{q}(u',u) & \widetilde{q}(u',u') \end{pmatrix} .$$

It is singular iff $\widetilde{q}(u,u') = 0$, because of $\widetilde{q}(u,u) = 0$ by the lemma assumption. □

Corollary 2.7. $p \in Sing\,Q \iff T_pQ$ *is the whole space* $\iff \dfrac{\partial q}{\partial x_i}(p) = 0\ \forall i$. □

Corollary 2.8. *If $p \in (Q \setminus Sing\,Q)$, then $T_pQ = \{x \in \mathbb{P}_n \mid \widetilde{q}(p,x) = 0\}$ is a hyperplane of codimension 1.* □

Corollary 2.9. *Let $p \notin Q$ and a hyperplane $C \subset \mathbb{P}_n$ be given by the equation $\widetilde{q}(p,x) = 0$ in x. Then $Q \cap L$ consists of all points where Q is touched by the tangent lines coming from p.* □

Polar mappings. The spaces $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ are called *dual* and denoted by $\mathbb{P}_n$ and $\mathbb{P}_n^{\times}$ when V is not essential. Since any codimension 1 subspace $U \subset V$ is defined by linear form $\xi \in V^*$, which is unique up to proportionality, $\mathbb{P}_n^{\times}$ is just *the space of hyperplanes* in $\mathbb{P}_n$ and vice versa. If $Q = (q)_0 \subset \mathbb{P}(V)$ is non singular, then the linear isomorphism $\mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V^*)$ induced by the correlation $\widehat{q}$ is called *a polarity* of Q. It sends a point $p \in \mathbb{P}_n$ to the hyperplane $L \subset \mathbb{P}_n$ given by the equation $\widetilde{q}(p,x) = 0$ like in the previous Corollary. L is called *a polar* of p and p is called *a pole* of L with respect to q. So, Q is just the set of all points laying on their own polars. Note that some non singular quadratic forms q can produce *empty* quadrics Q over an non closed field; but their polar mappings will be always visible!

Exercice 2.10. Show that p lays on the polar of q iff q lays on the polar of p (for any pair of distinct points and any polarity).

Exercice 2.11. Consider a circle in the real Euclidean affine plane $\mathbb{R}^2$. How to draw the polar of a point that lays: a) outside b) inside this circle? Describe geometrically the polarity defined by the «imaginary» circle «given» in $\mathbb{R}^2$ by the equation $x^2 + y^2 = -1$.

Proposition 2.12. *Two polarities coincide iff the corresponding quadratic forms are proportional.*

Proof. This follows from Lemma 1.11 □

Corollary 2.13. *Over an algebraically closed field two quadrics coincide iff their quadratic equations are proportional.*

Proof. Let $Q = Q'$. We can suppose that the quadrics are non singular, because of their equations are not changed under direct summation with the kernel $\ker q = \ker q'$. Non singular case is covered by the above Proposition. □

The space of quadrics. All the polarities on $\mathbb{P}_n = \mathbb{P}(V)$ are one-to-one parameterized by the points of the projective space

$$\mathbb{P}_{\frac{n(n+3)}{2}} = \mathbb{P}(S^2V^*) ,$$

which will be referred as *a space of quadrics*.

Complex projective conics. A quadric on the projective plane is called *a projective conic*. A projective conic over $\mathbb{C}$, up to an isomorphism, coincides either with *a double line* $x_0 = 0$, which has corank 2, or with *a reducible conic*[3] $x_0^2 + x_1^2 = 0$, which has corank 1, or with the non singular conic $x_0^2 + x_1^2 + x_2^2 = 0$. The space of all conics in $\mathbb{P}_2 = \mathbb{P}(V)$ is $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$.

[3] i. e. a pair of crossing lines

Example 2.14: standard model for non singular conic. Let U be 2-dimensional vector space. The quadratic *Veronese map* (Example 5.14), which sends a linear form ξ to its square ξ^2, gives the embedding $\mathbb{P}(U^*) = \mathbb{P}_1 \overset{v}{\hookrightarrow} \mathbb{P}_2 = \mathbb{P}(S^2U^*)$. Let us identify $\mathbb{P}(S^2U^*)$ with the space of quadrics on $\mathbb{P}_1 = \mathbb{P}(U)$. Then the Veronese embedding is a bijection between the points of $\mathbb{P}_1$ and the singular quadrics on $\mathbb{P}_1$, that is with the double points. So, the Veronese conic $Q_V = \{\, q \in S^2U^* \mid \det q = 0 \,\}$ is the quadric; it is called *a Veronese quadric*. Fixing a base (x_0, x_1) for U^* and induced base $\{x_0^2,\, 2\,x_0x_1,\, x_1^2\}$ for S^2U^*, we can coordinate a quadric $q(x) = q_0\,x_0^2 + 2\,q_1\,x_0x_1 + q_2\,x_1^2 \in S^2U^*$ by $(q_0 : q_1 : q_2)$ and a linear form $\xi(x) = t_0x_0 + t_1x_1 \in U^*$ by $(t_0 : t_1)$. Then Veronese sends $(t_0x_0 + t_1x_1) \longmapsto (t_0x_0 + t_1x_1)^2 = t_0^2\,x_0^2 + 2\,t_0t_1\,x_0x_1 + t_1^2\,x_1^2$. So, Q_V has the equation $\det(q) = q_0q_2 - q_1^2 = 0$ and admits the quadratic parameterization $(q_0 : q_1 : q_2) = (t_0^2 : t_0t_1 : t_1^2)$. Note that any non singular conic $Q \subset \mathbb{P}_2$ can be identified with Q_V by an appropriate base choice, in particular, any smooth Q admits a quadratic parameterization by the points of $\mathbb{P}_1$.

Proposition 2.15. *Two distinct non singular conics cann't have more then 4 common points.*

Proof. Parameterize the first conic by $x = v(t_0, t_1)$ via Veroneze. If the second conic is given by the quadratic equation $q(x) = 0$, then the t-coordinates of the intersection points are given by the 4-th degree equation $q(v(t)) = 0$. □

Proposition 2.16. *Any 5 points in $\mathbb{P}_2$ lay on some conic. If no 3 of the points are collinear, then this conic is unique and non singular.*

Proof. Since the equation $q(p) = 0$ is linear in q for a given p, the conics passing through $p \in \mathbb{P}_2$ form the hyperplane in $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$. Any 5 hyperplanes in $\mathbb{P}_5$ have non zero intersection, so, some conic exists. If it is singular, it consists of one or two lines. Hence at least 3 of 5 points are collinear. By the previous proposition, non singular conic is unique. □

COMPLEX PROJECTIVE QUADRICS ON $\mathbb{P}_3$ up to an isomorphism are: *a double plane* $x_0^2 = 0$; *a reducible quadric* $x_0^2 + x_1^2 = 0$, which is a pair of crossing planes (or a cone with a line vertex over a pair of distinct points on an complementary line); *a simple cone* $x_0^2 + x_1^2 + x_2^2 = 0$, which is a cone with one point vertex over a non singular plane conic; and a non singular quadric $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$. The projective space of all quadrics on $\mathbb{P}_3$ has dimension $3 \cdot 6 : 2 = 9$. In particular *any 9 points in $\mathbb{P}_3$ lay on some quadric.*

Claim 2.17. *Any 3 lines on $\mathbb{P}_3$ lay on some quadric.*

Proof. Take 3 distinct points on each line and draw a quadric through these 9 points. □

Note that if 3 lines are pairwise skew, then a quadric containing these lines must be non singular. If 2 of the lines are skew, then such a quadric is ether reducible or non singular (but nether a cone nor a double plane).

SEGRE QUADRIC IN $\mathbb{P}_3$. Let U_-, U_+ be 2-dimensional spaces and $W = \mathrm{Hom}(U_-, U_+)$ be the space of all linear maps. Then $\mathbb{P}_3 = \mathbb{P}(W)$ consists of non zero linear maps considered up to proportionality. Non zero but degenerate maps form a quadric $Q_S \subset \mathbb{P}_3$, which is called *a Segre quadric*. In the matrix notations, Q_S is presented by the determinantal equation

$$\det \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} = \alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10} = 0\,. \tag{4}$$

This quadric is bijectively parameterized by the points of $\mathbb{P}_1 \times \mathbb{P}_1 = \mathbb{P}(U_-^*) \times \mathbb{P}(U_+)$ as follows. *A Segre embedding*

$$\mathbb{P}(U_-^*) \times \mathbb{P}(U_+) \overset{s}{\hookrightarrow} \mathbb{P}(\mathrm{Hom}(U_-, U_+))$$

sends a pair $(\xi, v) \in U_-^* \times U_+$ to rank 1 operator $U_- \xrightarrow{\xi \otimes v} U_+$ defined by the rule $U_- \ni u \longmapsto \xi(u)\,v \in U_+$. So, $\ker(\xi \otimes v)$ coincides with the zero subspace of the linear form ξ and $\mathrm{im}\,(\xi \otimes v)$ is spanned by v. Since any non zero degenerate operator $U_- \xrightarrow{\varphi} U_+$ has 1-dimensional kernel and 1-dimensional image, it may be written as $\varphi = \xi \otimes v$ for some $\xi \in U_-^*$, $v \in U_+$, which are up to proportionality uniquely defined by φ. So, the Segre embedding maps $\mathbb{P}_1 \times \mathbb{P}_1$ bijectively onto Q_S.

In terms of coordinates, if $\xi = (\xi_0 : \xi_1) \in U_-^*$ and $v = (t_0 : t_1) \in U_+$, then the operator $\xi \otimes v$ sends the first basic vector $(1, 0) \in U_-$ to $\xi_0 \cdot v$ and the second basic vector $(0, 1) \in U_-$ to $\xi_1 \cdot v$. Hence, $\xi \otimes v$ has a the matrix $\begin{pmatrix} \xi_0 t_0 & \xi_1 t_0 \\ \xi_0 t_1 & \xi_1 t_1 \end{pmatrix}$. So, the Segre embedding gives a parametrization for the quadric (4) by the formulas

$$\alpha_{00} = \xi_0 t_0\,, \quad \alpha_{01} = \xi_1 t_0\,, \quad \alpha_{10} = \xi_0 t_1\,, \quad \alpha_{11} = \xi_1 t_1\,.$$

where $\big(\,(\xi_0 : \xi_1)\,,\,(t_0 : t_1)\,\big) \in \mathbb{P}_1 \times \mathbb{P}_1$. In other words, any rank one 2×2 - matrix (a_{ij}) has proportional rows and proportional columns and their ratios

$$([\text{row } 1)] : [\text{row } 2]) \overset{\text{def}}{=} (t_0 : t_1) \qquad \text{and} \qquad ([\text{col } 1)] : [\text{col } 2]) \overset{\text{def}}{=} (\xi_0 : \xi_1)\,.$$

define the matrix up to a scalar factor.

<u>Two rulings for non singular quadric on $\mathbb{P}_3$.</u> Note that $\mathbb{P}_1 \times \mathbb{P}_1$ is ruled by two families of «coordinate» lines $\xi \times \mathbb{P}_1$ and $\mathbb{P}_1 \times t$. They are called *a first* and *a second* line families.

<u>Lemma 2.18</u>. *The Segre embedding sends each coordinate line family to the ruling of Q_s by a family of pairwise skew lines. These two line families exhaust all the lines on Q_s. Any two lines from different families are intersecting and each point of Q_s is an intersection point of two lines from different families.*

Proof. The line $\xi \times \mathbb{P}_1$, where $\xi = (\xi_0 : \xi_1) \in \mathbb{P}(U_-^*)$, goes to the set of all degenerated matrices with the fixed column ratio, i. e. to the line given in $\mathbb{P}_3$ by two linear equations $a_{00} : a_{01} = a_{10} : a_{11} = \xi_0 : \xi_1$ Analogously, $s(\mathbb{P}_1 \times t)$, where $t = (t_0 : t_1) \in \mathbb{P}(U_+)$, goes to the line given in $\mathbb{P}_3$ by $a_{00} : a_{10} = a_{01} : a_{11} = t_0 : t_1$. Since the Segre embedding is bijective, each line family consists of pairwise skew lines, any two lines from the different families are intersecting, and for any $x \in Q_s$ there is a pair of the different type lines passing through x. So, $Q_s \cap T_x Q_s$ is the reducible conic, which splits into the last pair of lines. In particular, there are no other lines on Q_s. □

<u>Corollary 2.19</u>. *There exists a unique quadric passing through 3 given pairwise skew lines in $\mathbb{P}_3$. It is ruled by all lines intersecting each of 3 given lines.*

Proof. Let a quadric Q contain 3 given lines. Since Q is non singular, it can be ruled by two line families via Segre. 3 given lines must belong to the same, say, to the first family. Any line from the second family intersect all the lines of the first family (in particular, 3 given lines). Moreover, any line on $\mathbb{P}_3$ which intersect 3 given lines must lay on Q and belong to the second family. □

Exercice 2.20. How many lines intersect 4 given pairwise skew lines in $\mathbb{P}_3$?

Hint. Draw all lines intersecting some 3 of given 4 and look at the interaction of this figure with the remaining 4-th line.

Exercice 2.21*. Do the previous exercise for $\mathbb{A}^3$ instead of $\mathbb{P}_3$ and also over $\mathbb{R}$ instead of $\mathbb{C}$. There will be several answers depending on the line configuration. Find all possible answers and indicate which of them are stable under small perturbations.

<u>Linear subspaces laying on a non singular quadric</u> . Lemma 2.18 can be extended in higher dimensions. Let $Q_n \subset \mathbb{P}_n = \mathbb{P}(V)$ be non singular quadric and $L = \mathbb{P}(W) \subset Q_n$.

<u>Theorem 2.22</u>. $\dim L \leqslant \left[\dfrac{n-1}{2}\right]$, *where $[\,*\,]$ means the integer part.*

Proof. Let Q_n be given by a quadratic form q with the polarization $\widetilde{q}$. Then

$$L \subset Q_n \iff \widetilde{q}(w_1, w_2) = 0 \;\forall\, w_1, w_2 \in W \iff \widehat{q}(W) \subset \mathrm{Ann}\,(W) = \{\, \xi \in V^* \mid \xi(w) = 0 \quad \forall\, w \in W \,\}\,,$$

where $\widehat{q} : v \longmapsto \widetilde{q}(v, *)$ is the correlation associated with Q_n. This correlation is injective, because of Q_n is non singular. So, $\dim W \leqslant \dim \mathrm{Ann}\, W = \dim V - \dim W$. Hence, $\dim L = \dim W - 1 \leqslant (\dim V)/2 - 1 = [(n-1)/2]$. □

<u>Lemma 2.23</u>. *$\mathrm{cork}\,(H \cap Q_n) \leqslant 1$ for any codimension 1 hyperplane $H \subset \mathbb{P}_n$.*

Proof. If $H = \mathbb{P}(W)$, then $\dim \ker (q|_W) \leqslant \dim\big(W \cap \widehat{q}^{-1}(\mathrm{Ann}\, W) \big) \leqslant \dim \widehat{q}^{-1}(\mathrm{Ann}\, W) = \dim \mathrm{Ann}\, W = 1$. □

<u>Lemma 2.24</u>. *For any $x \in Q_n$, $Q_n \cap T_x Q_n$ is a simple cone with the vertex x over a non singular quadric Q_{n-2} in a $(n-2)$-dimensional projective subspace in $T_x Q_n \setminus x$.*

Proof. Since $T_x Q = \mathbb{P}(\ker \widetilde{q}(x, *))$ and $\widetilde{q}(x, x) = q(x) = 0$, the restriction of q onto $T_x Q$ has at least 1-dimensional kernel presented by x itself. By the previous lemma this kernel is spaned by x. □

<u>Theorem 2.25</u>. *Let $d_n = \left[\dfrac{n-1}{2}\right]$ be the upper bound from Theorem 2.22. The d_n-dimensional subspaces $L \subset Q_n$ passing through a given point on Q_n are parameterized bijectively by the all $(d_n - 1)$-dimensional subspaces laying on Q_{n-2}.*

Proof. Fix some $(n-1)$-dimensional projective subspace $H \subset T_x Q \setminus x$ and present $Q_n \cap T_x Q_n$ as a simple cone ruled by lines passing through x and some $Q_{n-2} \subset H$. Since any $L \subset Q_n$ which pass through x is contained inside $Q_n \cap T_x Q_n$, it has to be the linear span of x and some $(d_n - 1)$-dimensional subspace $L' \subset Q_{n-2}$. □

For example, there are only 0-dimensional subspaces on Q_1 and Q_2. Next two quadrics, Q_3 and Q_4, do not contain planes. But any point $x \in Q_3$ lies on two lines passing through x and 2 points of $Q_1 \subset T_x Q_3 \setminus \{x\}$ and any point of Q_4 belongs to 1-dimensional family of lines parameterized by the points of a non singular conic $Q_2 \subset T_x Q_4 \setminus \{x\}$. Further, non singular quadric $Q_5 \subset \mathbb{P}_5$ does not contain 3-dimensional subspaces but for any point $x \in Q_5$ there are two 1-dimensional families of planes passing through x. Each family is parameterized by the corresponding family of lines on $Q_3 \subset T_x Q_5 \setminus \{x\}$, i. e. by $\mathbb{P}_1$ actually.