Affine and projective spaces, polynomials, projections, and projective linear isomorphisms.

<u>Polynomials.</u> Let V be n-dimensional vector space over an arbitrary field k, $(e_1, e_2, \dots, e_n)$ be a basis in V. The *dual space* V^* consists of all k-linear functions $V \longrightarrow k$; its *dual base* $(x_1, x_2, \dots, x_n)$ is formed by the coordinate functions in V w.r.t. $\{e_i\}$, that is $x_i(e_j) = 1$, if $i = j$, and $x_i(e_j) = 0$ otherwise. We write $S^dV^* \subset k[x_1, x_2, \dots, x_n]$ for a subspace of all homogeneous polynomials of degree d. So, $k[x_1, x_2, \dots, x_n] = \bigoplus\limits_{d \geqslant 0} S^dV^*$ and $S^rV^* \cdot S^sV^* \subset S^dV^{r+s}$, i.e. $k[x_1, x_2, \dots, x_n]$ is a *graded* k-algebra with the graded components S^dV^*. There is *the evaluation map* $k[x_1, x_2, \dots, x_n] \xrightarrow{\text{ev}} \{k\text{-valued functions on } V\}$, which sends a polynomial $f(x_1, x_2, \dots, x_n)$ to the function $p = (p_1, p_2, \dots, p_n) \longmapsto f(p) = f(p_1, p_2, \dots, p_n)$. The image of ev is called *a space of algebraic functions* on V.

Exercice 1.1. Show that this subspace and all the subspaces $\text{ev}(S^dV^*)$ do not depend on a base choice in V.

<u>Lemma 1.2</u>. *The evaluation map is injective*[1] *iff k is infinite.*

Proof. If k is finite, say consists of q elements, then the space of k-valued functions on V consists of q^{q^n} elements and is finite as well. Since the polynomial algebra is infinite, $\ker \text{ev} \neq 0$. The inverse argumentation uses induction on n. Non zero polynomial $f(x)$ in one variable can't have more than $\deg f$ roots. So, $f(x) \equiv 0$ as soon as $f(p) = 0$ for infinitely many $p \in k$. A polynomial in n variables $f(x_1, x_2, \dots, x_n) = \sum\limits_{\nu=0}^{d} \varphi_\nu(x_1, x_2, \dots, x_{n-1}) \cdot x_n^{d-\nu}$ may be considered as a polynomial in one variable x_n with coefficients from $k[x_1, x_2, \dots, x_{n-1}]$. Evaluating all φ_ν at any point $(p_1, p_2, \dots, p_{n-1}) \in k^{n-1}$, we get a polynomial in x_n with constant coefficients. It equals zero by the above reason. So, all φ_ν are identically zero functions. By the the inductive assumption they should be the zero polynomials as well. □

<u>Affine space $\mathbb{A}(V)$</u> is just a set of all vectors in V. A point presenting the zero vector is called *an origin*; we denote it by O. All other points may be pictured as the ends of non zero vectors drawn from the origin. We write $\mathbb{A}^n$ instead of $\mathbb{A}(V)$ for n-dimensional space whose feeling is not essential. A set $X \subset \mathbb{A}(V)$ is called *a closed affine algebraic set* if it coincides with a set of all solutions of some (possibly, infinite) system of polynomial equations on V.

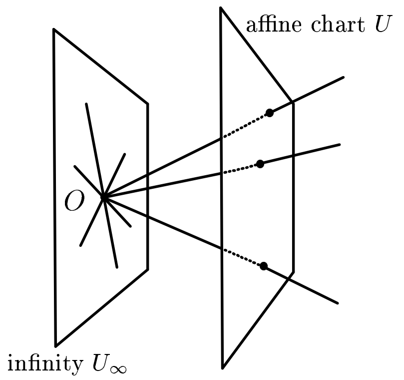

Fig. 1.1. The projective world.

<u>Projective space[2] $\mathbb{P}(V)$</u> is a set of all 1-dimensional vector subspaces in V, that is, a set of all lines through the origin in $\mathbb{A}(V)$. If $\dim V = (n+1)$, we write $\mathbb{P}_n$ instead of $\mathbb{P}(V)$ and call it *n-dimensional projective space*. The rays going from the origin produce "the usual" points as soon as they meet some screen inside $\mathbb{A}(V)$, that is, a codimension 1 affine hyperplane $U \subset \mathbb{A}(V)$ that does not contain the origin — like ion fig. 1.1. Such a hyperplane is called *an affine chart* on $\mathbb{P}(V)$. Of course, no such a chart does cover the whole of $\mathbb{P}_n$: the difference $U_\infty \overset{\text{def}}{=} \mathbb{P}_n \setminus U$ consists of all lines laying in a parallel copy of U passing through the origin. It may be identified with $\mathbb{P}_{n-1} \subset \mathbb{P}(V)$ and is called *an infinity hyperplane* for the chart U. So, $\mathbb{P}_n$ is decomposed into disjoint union of "affine cells": $\mathbb{P}_n = U \sqcup U_\infty = \mathbb{A}^n \sqcup \mathbb{P}_{n-1} = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{P}_{n-2} = \dots = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \dots \sqcup \mathbb{A}^0$

Exercice 1.3*. If you have some experience in CW-topology, try to calculate integer homology groups of the real and complex projective spaces using the above cell decomposition.

<u>Global homogeneous coordinates.</u> Let $\dim V = (n+1)$ and $v = (x_0, x_1, \dots, x_n)$, $w = (y_0, y_1, \dots, y_n)$ be two vectors presented by their coordinates w.r.t. a given fixed basis $\{e_0, e_1, \dots, e_n\} \subset V$. Then v and w give the same point $p \in \mathbb{P}_n$ iff they are proportional, i.e. $x_\nu = \lambda y_\nu \ \forall \nu$ for some $\lambda \neq 0$. So, only the ratios $x_\mu : x_\nu$ are well defined for the point $p \in \mathbb{P}_n$. These ratios are called *homogeneous coordinates* on $\mathbb{P}_n$ (w.r.t. the base $\{e_0, e_1, \dots, e_n\} \subset V$). We write $p = (x_0 : x_1 : \dots : x_n)$ for a point p presented (up to proportionality) by $v = (x_0, x_1, \dots, x_n)$. For any homogeneous polynomial $f \in S^d(V^*)$, the zero set $(f)_0 \overset{\text{def}}{=} \{ v \in V \mid f(v) = 0 \}$

[1] i. e. different polynomials always give the different polynomial functions on V

[2] in this section we always take $\dim V = n + 1$

is well defined subset in $\mathbb{P}(V)$, because of $f(v)=0 \iff f(\lambda v)=\lambda^d f(v)=0$. Such the subsets and their intersections are called *closed projective algebraic sets*. For example, the equation

$$x_0^2+x_1^2=x_2^2 \tag{1}$$

presents the curve $C\subset\mathbb{P}_2$; if char $(k)\neq 2$, it is known as *non degenerate plane conic*.

LOCAL AFFINE COORDINATES. Let $U\subset\mathbb{A}(V)$ be an affine chart on $\mathbb{P}_n=\mathbb{P}(V)$, say, given by the equation $\alpha(x)=1$, where $\alpha\in V^*$. Take any n linear forms $\xi_1,\xi_2,\ldots,\xi_n\in V^*$ whose restrictions $\xi_i|_U$ are linearly independent[3]. Then for any $v\in V$ we have $v/\alpha(v)\in U$ and can use n functions $x_i(p)=\xi_i(v/\alpha(v))$, $1\leqslant i\leqslant n$ as *local* affine coordinates inside $U\subset\mathbb{P}_n$ for the point $p\in U$ corresponding to v. These coordinates depend on both U and a choice of ξ_i's. Note, that the sentences: «$\alpha(p)\to 0$», «$x_i(p)\to\infty$ for some i», and «$p\to U_\infty=\mathbb{P}_n\setminus U$» do really mean the same: «p is running away from U to infinity».

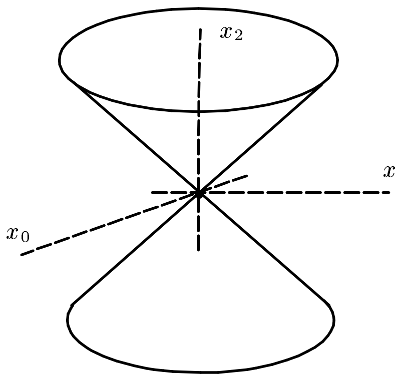

Fig. 1.2. The cone.

Example 1.4: affine conics. Consider the non degenerate conic (1) over $k=\mathbb{R}$ in some affine charts. Let U_0 be given by the equation $\{x_0=1\}$. Then $t_1=x_1|_{U_0}=x_1/x_0$ and $t_2=x_2|_{U_0}=x_2/x_0$ can be taken as local affine coordinates on U_0. Dividing the both sides of (1) by x_2^2, we get for $C\cap U_0$ the equation $t_2^2-t_1^2=1$, that is, $C\cap U_0$ is a hyperbola. Similarly, in a chart $U_2=\{x_0=1\}$ with local affine coordinates $t_0=x_0/x_2$, $t_1=x_1/x_2$ we get the equation $t_0^2+t_1^2=1$, which shows that $C\cap U_2$ is a circle. Finally, consider a chart U given by $x_2-x_1=1$ with local affine coordinates $t=x_0|_U=\dfrac{x_0}{x_2-x_1}$ and $u=(x_2+x_1)|_U=\dfrac{x_2+x_1}{x_2-x_1}$. After dividing by $(x_2-x_1)^2$ and some eliminations, we see that $C\cap U$ is the parabola $u=t^2$.

Exercice 1.5. The affine cone $x_0^2+x_1^2=x_2^2$ in $\mathbb{A}^3$ is drawn on fig. 1.2. Picture there each of 3 previous affine charts and outline their intersections with the cone.

HINT. A good idea is to draw 3 separated pictures.

PROJECTIVE CLOSURE. Any closed affine algebraic set $X\subset\mathbb{A}^n$ is always an affine piece of appropriate closed projective algebraic set $\widetilde{X}\subset\mathbb{P}_n$ called *a projective closure of* X. If X is given by polynomial equations $\{\,f_\nu(t_1,t_2,\ldots,t_n)=0\,\}$, we substitute $t_i=x_i/x_0$ and multiply the ν-th equation by $x_0^{\deg f_\nu}$; then the resulting equations $\widetilde{f}_\nu(x_0,x_1,\ldots,x_n)=0$ are homogeneous and define a closed projective algebraic set $\widetilde{X}\subset\mathbb{P}_n$ such that $\widetilde{X}\cap U_0=X$, where U_0 is the affine chart given by $x_0=1$ (it is identified with original $\mathbb{A}_n\supset X$ by taking $t_i=x_i|_{U_0}$).

STANDARD AFFINE COVERING AND GLUING RULES. $\mathbb{P}_n$ is covered by $(n+1)$ affine charts $U_\nu\subset\mathbb{A}^{n+1}$ given by $\{x_\nu=1\}$. This covering is called *standard*. For each $\nu=0,1,\ldots,n$ we take n forms

$$t_i^{(\nu)}=x_i|_{U_\nu}=\frac{x_i}{x_\nu}\,,\quad \text{where } 0\leqslant i\leqslant n\,,\ i\neq\nu$$

as standard local affine coordinates on U_ν. Topologically, $\mathbb{P}_n$ is constructed from $(n+1)$ distinct copies of $\mathbb{A}_n$ denoted as $U_0,U_1,\ldots,U_n$ by gluing them together along the actual intersections $U_\mu\cap U_\nu\subset\mathbb{P}_n$. Namely, we identify a point of U_ν with a point of U_μ iff they correspond to the same point of $\mathbb{P}_n$. In homogeneous coordinates, the intersection $U_\mu\cap U_\nu$ consists of all x such that both x_μ and x_ν are non zero. This locus is presented inside U_μ and U_ν by inequalities $t_\nu^{(\mu)}\neq 0$ and $t_\mu^{(\nu)}\neq 0$ respectively and a point $t^{(\mu)}\in U_\mu$ is glued with a point $t^{(\nu)}\in U_\nu$ iff

$$t_\nu^{(\mu)}=1/t_\mu^{(\nu)}\qquad\text{and}\qquad t_i^{(\mu)}=t_i^{(\nu)}/t_\mu^{(\nu)}\quad\text{for}\quad i\neq\mu,\nu$$

Right hand sides of these equations are called *the transition functions* from $t^{(\nu)}$ to $t^{(\mu)}$ over $U_\mu\cap U_\nu$. For example, $\mathbb{P}_1$ can be produced from two copies of $\mathbb{A}^1$ by identifying for all $t\neq 0$ the point t in one of them with the point $1/t$ in the other.

Exercice 1.6*. If you have some experience in smooth topology, prove that real and complex projective lines are analytic manifolds isomorphic to the circle S^1 (in real case) and to the Riemann sphere S^2 (in complex case).

PROJECTIVE SUBSPACES. A closed projective algebraic subset is called *a projective subspace* if it can be given by a system of *linear* homogeneous equations. Any projective subspace $L\subset\mathbb{P}(V)$ has a form $L=\mathbb{P}(W)$, where

[3] this means that $n+1$ forms $\alpha,\xi_1,\xi_2,\ldots,\xi_n$ form a base of V^*

$W \subset V$ is a vector subspace. Note that 0-dimensional projective subspaces[4] coincide with the points. Since $\operatorname{codim}_{\mathbb{P}(V)}\mathbb{P}(W) = \operatorname{codim}_V W$, we have $L_1 \cap L_2 \neq \varnothing$ for any two projective subspaces L_1 and L_2 such that $\operatorname{codim} L_1 + \operatorname{codim} L_2 \leqslant n$. For example, any two lines on $\mathbb{P}_2$ have non empty intersection[5].

Two projective subspaces L_1 and L_2 in $\mathbb{P}_n$ are called *complementary* to each other, if

$$L_1 \cap L_2 = \varnothing \qquad \text{and} \qquad \dim L_1 + \dim L_2 = n - 1\,.$$

For example, any two skew lines in 3-dimensional space are complementary.

Exercice 1.7. Show that $\mathbb{P}(U)$ and $\mathbb{P}(W)$ are complementary in $\mathbb{P}(V)$ iff $V = U \oplus W$.

Lemma 1.8. *If $L_1, L_2 \subset \mathbb{P}(V)$ are two complementary linear subspaces, then any point $p \in \mathbb{P}(V) \setminus (L_1 \cup L_2)$ lies on a unique line crossing the both subspaces.*

Proof. We have $V = U_1 \oplus U_2$, where $\mathbb{P}(U_i) = L_i$. So, any $v \in V$ has a unique decomposition $v = u_1 + u_2$ with $u_i \in U_i$. If $v \notin U_1 \cup U_2$, then both u_i, u_2 are non zero and there is a unique 2-dimensional subspace that contains v and has non zero intersections with both U_i: it coincides with a linear span of u_1, u_2. □

Projections. For any two complementary projective subspaces $L_1, L_2 \subset \mathbb{P}_n$, a *projection* from L_1 onto L_2 is a map $\pi_{L_2}^{L_1} : (\mathbb{P}_n \setminus L_1) \longrightarrow L_2$ that sends any point $q \in L_2$ to itself and any point $p \in \mathbb{P}_n \setminus (L_1 \sqcup L_2)$ to $\ell \cap L_2$, where ℓ is the unique line passing through p and crossing both L_1 and L_2 (like in Lemma 1.8). In homogeneous coordinates $(x_0 : x_1 : \ \dots \ : x_n)$ such that L_1 is coordinated by $(x_0 : x_1 : \ \dots \ : x_m)$ and L_2 is coordinated by $(x_{m+1} : x_{m+2} : \ \dots \ : x_n)$, the projection $\pi_{L_2}^{L_1}$ is nothing but taking $x_\nu = 0$ for $0 \leqslant \nu \leqslant (m+1)$.

Example 1.9: projecting a conic onto a line. Let $\pi_\ell^p : Q \longrightarrow \ell$ be a projection of the conic $Q = \{x_0^2 + x_1^2 = x_2^2\} \subset \mathbb{P}_2$ onto the line $\ell = \{x_0 = 0\}$ from the point $p = (1 : 0 : 1) \in Q$. Inside the standard affine chart U_2, where $x_2 = 1$, it looks like on fig. 1.3. It is bijective, because the pencil of all lines passing through p is bijectively parameterized by the points $t \in \ell$ and any such a line (pt) intersects Q exactly in one more point $q = q(t)$ in addition to p except for the the tangent line at p, which is given by $x_0 = x_2$ and crosses ℓ at the point[6] $t = (0 : 1 : 0)$ that corresponds to $q(t) = p$ itself. Moreover, this bijection is *birational*, i. e. the corresponding $(q_0 : q_1 : q_2) \in Q$ and $(0 : t_1 : t_2) \in L$ are *rational* algebraic functions of each other. Namely, $(t_1 : t_2) = (\, q_1 : (q_2 - q_0)\,)$ and $(q_0 : q_1 : q_2) = (\,(t_1^2 - t_2^2) : 2\,t_1 t_2 : (t_1^2 + t_2^2)\,)$.

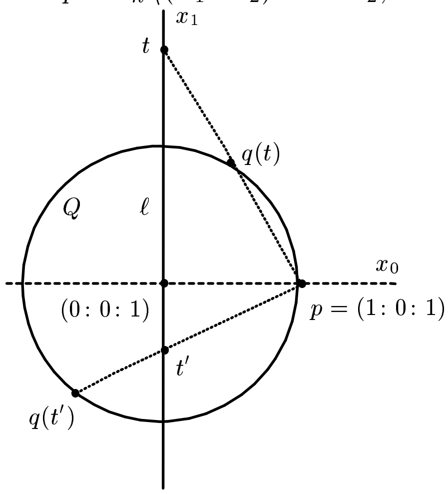

Fig. 1.3. Projecting a conic.

Exercice 1.10. Check these formulas and note that the second of them gives the full list of the Pythagorean triples[7] $(q_0 : q_1 : q_2)$ when (t_1, t_2) runs through $\mathbb{N} \times \mathbb{N}$.

Matrix notations for linear maps. Let $\operatorname{Hom}(U, W)$ be the space of all linear maps from n-dimensional vector space U to m-dimensional vector space W. Denote by $Mat_{m\times n}(k)$ the space of matrices with m rows, n columns, and entries in the field k. Any pair of basises $\{u_1, u_2, \dots, u_n\} \subset U$ and $\{w_1, w_2, \dots, w_m\} \subset W$ presents an isomorphism $\operatorname{Hom}(U, W) \xrightarrow{\sim} Mat_{m\times n}(k)$ that sends an operator $U \xrightarrow{\alpha} W$ to a matrix $A = (a_{ij})$ whoose j-th column consists of m coordinates of the vector $\alpha(u_j) \in W$ w.r.t. the basis $\{w_1, w_2, \dots, w_m\}$, i.e.

$$\alpha(u_j) = \sum_{\nu=1}^{m} a_{\nu j}\, w_\nu$$

or shortly $(\alpha(u_1), \alpha(u_2), \dots, \alpha(u_n)) = (w_1, w_2, \dots, w_m) \cdot A$. Let ${}^t z$ and ${}^t y$ be the transposed[8] coordinate rows for $u = (z_1, z_2, \dots, z_m) \in U$ and $w = \alpha(u) = (y_1, y_2, \dots, y_n) \in W$. Then ${}^t z = A\, {}^t y$, because of

$$\begin{aligned}(w_1, w_2, \dots, w_m) \cdot {}^t z = w = \alpha(u) = \alpha((u_1, u_2, \dots, u_n) \cdot {}^t y) = \\ = (\alpha(u_1), \alpha(u_2), \dots, \alpha(u_n)) \cdot {}^t y = (w_1, w_2, \dots, w_m) \cdot (\, A\, {}^t y\,)\end{aligned}$$

Linear projective transformations. If $\dim U = \dim W = (n+1)$, then any linear isomorphism $U \xrightarrow{\alpha} W$ induces the bijection $\mathbb{P}(U) \xrightarrow{\overline{\alpha}} \mathbb{P}(W)$, which is called *the projective linear transformation* or *the linear isomorphism*. A point set $\{p_1, p_2, \dots, p_m\} \subset \mathbb{P}_n$ is called *linearly general*, if any $(n+1)$ of p_i don't lay together in

[4] that is, $\mathbb{P}_0 = \mathbb{P}(k^1)$
[5] in terms of $\mathbb{A}^3$ this means that any two planes containing the origin are intersected along a line
[6] this intersection point lays at the infinity on fig. 1.3
[7] i. e. the right triangles with integer side lengths
[8] i. e. arranged in columns

any hyperplane $\mathbb{P}_{n-1} \subset \mathbb{P}_n$. Equivalently, the points $\{p_i\}$ are linearly general in $\mathbb{P}_n = \mathbb{P}(V)$ iff $(n+1)$ vectors representing any $(n+1)$ of them always form a basis of V.

Lemma 1.11. *For any two linearly general collections of $(n+2)$ points $\{p_0, p_1, \dots, p_{n+1}\} \in \mathbb{P}(U)$ and $\{q_0, q_1, \dots, q_{n+1}\} \in \mathbb{P}(V)$, where $\dim U = \dim V = (n+1)$, there exists a unique up to proportionality linear isomorphism $V \xrightarrow{\alpha} W$ such that $\overline{\alpha}(p_i) = q_i$ $\forall i$. In particular, two matrices give the same projective linear transformation iff they are proportional.*

Proof. Fix some vectors u_i and w_i representing the points p_i and q_i. By the linear generality, we can take $\{u_1, u_2, \dots, u_{n+1}\}$ and $\{w_1, w_2, \dots, w_{n+1}\}$ as the basises in U and W and identify a map $\alpha \in \mathrm{Hom}(U, W)$ by the square matrix in these basises. Then, $\overline{\alpha}(p_i) = q_i$ for $1 \leqslant i \leqslant (n+1)$ iff the matrix A of α is diagonal, say with $(d_1, d_2, \dots, d_{n+1})$ on the main diagonal. Now consider the first vectors u_0 and w_0. Again by the linear generality, all the coordinates of $u_0 = (x_1, x_2, \dots, x_{n+1})$ w. r. t. the basis $\{u_i\}_{1 \leqslant i \leqslant (n+1)}$ and all the coordinates of $w_0 = (y_1, y_2, \dots, y_{n+1})$ w. r. t. the basis $\{w_i\}_{1 \leqslant i \leqslant (n+1)}$ are non zero. Since the coordinates of $\alpha(u_0)$ are proportional to the ones of w_0, we have $y_i : y_j = (d_i x_i) : (d_j x_j)$ $\forall i, j$. Hence, all d_i are uniquely recovered from just one of them, say d_1, as $d_i = d_1 \cdot (y_i x_1) : (x_i y_1)$. □

Exercice 1.12. Let ℓ_1 and ℓ_2 be two lines on $\mathbb{P}_2$. Fix any point p outside $\ell_1 \cup \ell_2$ and consider the a $\ell_1 \xrightarrow{\gamma_p} \ell_2$ that sends $t \in \ell_1$ to the crossing point of ℓ_2 and the line (tp). Check that γ_p is the linear isomorphism.

Linear projective group. All linear isomorphisms $V \longrightarrow V$ form a group called *the general linear group* of V and denoted by $GL(V)$. This group acts on $\mathbb{P}(V)$. By Lemma 1.11, the kernel of this action coincides with the subgroup of all scalar dilatations $H \subset GL(V)$. Hence, the group of all projective linear automorphisms of $\mathbb{P}(V)$ equals to the factor group $GL(V)/H$, which is denoted by $PGL(V)$ and called *the projective linear group*. As soon as some basis $\{e_0, e_1, \dots, e_n\} \subset V$ is fixed, we can identify $GL(V)$ with the group $GL_{n+1}(k) \subset Mat_{n+1}(k)$ of all non degenerated square matrices. Under this identification the dilatations go to the scalar matrices and $PGL(V)$ turns into the group $PGL_{n+1}(k) \overset{\text{def}}{=} GL_{n+1}(k)/\{\text{scalar diagonal matrices } \lambda E\}$ of all non degenerate square matrices considered up to proportionality.

Example 1.13: linear-fractional group. The group $PGL_2(k)$ consists of all square matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which have $ad - bc \neq 0$ and are considered up to proportionality. It acts on $\mathbb{P}_1$ via $(x_0 : x_1) \overset{A}{\longmapsto} ((ax_0 + bx_1) : (cx_0 + dx_1))$. In terms of the standard affine chart $U_0 \simeq \mathbb{A}_1$ with the affine coordinate $t = x_1/x_0$, this action takes $t \longmapsto (ct + d)/(at + b)$

Exercice 1.14. Check that $(AB)(t) = A(B(t))$.

Exercice 1.15. For any 3 different points $p, q, r \in U_0$ write down the unique linear-fractional transformation α such that $\alpha(p) = 0$, $\alpha(q) = 1$, and $\alpha(r) = \infty$.

Hint. It has to be of the form $t \overset{\alpha}{\longmapsto} \vartheta \cdot (t-p)/(t-r)$, where $\vartheta \in k$.

Proposition 1.16. *Let $\mathbb{P}_1 \xrightarrow{\varphi} \mathbb{P}_1$ be a bijection over an algebraically closed field. If everywhere except for a finite number of points φ is given by the formula $\varphi(x_0 : x_1) = (f_0(x_0, x_1) : f_1(x_0, x_1))$, where f_i are rational algebraic functions, then $\varphi \in PGL_2(k)$.*

Proof. To give a well defined map, the functions f_i have to be homogeneous of the same degree. Moreover, multiplying by denominators and eliminating common factors we can suppose that f_i are *coprime polynomials*. Since φ is bijective, each $\vartheta = (\vartheta_0 : \vartheta_1) \in \mathbb{P}_1$ (except for a finite set) has precisely one preimage, that is the homogeneous equation

$$\vartheta_1 \cdot f_0(x_0, x_1) - \vartheta_0 \cdot f_1(x_0, x_1) = 0 \tag{2}$$

has just one root up to proportionality, i. e. is pure d-th power of some linear form in $(x_0 : x_1)$, where $d = \deg f_1 = \deg f_2$. Now note, that all degree d homogeneous polynomials in $(x_0 : x_1)$ considered up to a scalar factor form the projective space $\mathbb{P}_d = \mathbb{P}(S^d U^*)$, where U is 2-dimensional vector subspace underlying $\mathbb{P}_1$ in question. When ϑ varies through $\mathbb{P}_1$, the equations (2) draw a line in this $\mathbb{P}_d$, whereas pure d-th powers of linear forms form a curve called the *Veronese curve* of degree d. It is easy to see (see the Exercice 1.17 below) that over an infinite field the Veronese curve of degree $d \geqslant 2$ is *twisted*, that is any $(k+1)$ its distinct points do not lay in a common $(k-1)$-dimensional subspace for all $k = 2, 3, \dots, d$. In particular, the line (2) can intersect the Veronese curve in at most two points, if $d \geqslant 2$. So, $d = 1$, i. e. φ is a projective linear isomorphism. □

Exercice 1.17. The Veronese curve of degree d is defined as an image of the Veronese map

$$\mathbb{P}_1 = \mathbb{P}(U^*) \xrightarrow{v_d} \mathbb{P}_d = \mathbb{P}\left(S^d(U^*)\right) \tag{3}$$

which takes a linear form $\varphi \in U^*$ to its d-th power $\psi^d \in S^d(U^*)$. Show that if the ground field contains more than d points, then any $d+1$ distinct points of the Veronese curve are linearly general.

Hint. Write $\varphi \in U^*$ as $\alpha_0 x_0 + \alpha_1 x_1$ and $f \in S^d(U^*)$ as $\sum\limits_{\nu=0}^{d} a_\nu \cdot \binom{d}{\nu} x_0^{d-\nu} x_1^\nu$ and use $(\alpha_0 : \alpha_1)$, $(a_0 : a_1 : \dots : a_d)$ as homogeneous coordinates on $\mathbb{P}_1$ and $\mathbb{P}_d$ in question. Then an intersection of the Veronese curve with $(d-1)$-dimensional hyperplane given by linear equation $\sum A_\nu a_\nu = 0$ consists of $(\alpha_0 : \alpha_1)$ satisfying degree d equation $\sum A_\nu \cdot \alpha_0^{d-\nu} \alpha_1^\nu = 0$.