Middle term test

The problems may be solved in any order. The complete solution of each problem is worth 10 points. Only an answer without its explanation gives 0 points irrespective of its correctness. To get the 100% result it is enough to collect 40 points in sum, e.g., solve completely any 4 of the 6 problems.

Problem 1 (10 points). Over an algebraically closed field of characteristic $\operatorname{char}\Bbbk \neq 2$, does there exist a pencil of plane conics that contains exactly one singular conic but does not contain a double line? If the answer is yes, give an explicit example of such pencil. If not, explain why.

Problem 2 (10 points). For 6 points $p_1, p_2, \dots, p_6 \in \mathbb{P}_2 = \mathbb{P}(V)$ such that no 3 of them are collinear, compute the dimension of the vector space[1] $W = \{f \in S^3V^* \mid f(p_i) = 0 \text{ for all } i = 1, 2, \dots, 6\}$ in the following two cases: when there is a conic passing through all the p_i, and when there is no such a conic.

Problem 3. Is there a 2×4 matrix whose 2×2 minors (written in some random order) are

a) (10 points) $\{2, 3, 4, 5, 6, 7\}$

b) (10 points) $\{3, 4, 5, 6, 7, 8\}$?

If the answer is yes, give an explicit example of such matrix. If not, explain why.

Problem 4. Consider two vector spaces U, W of arbitrary finite dimensions and write

$$S \subset \mathbb{P}\big(\operatorname{Hom}(U, W)\big) = \mathbb{P}(U^* \otimes W)$$

for the Segre variety formed by rank 1 linear maps $\xi \otimes w : U \to W$, $u \mapsto \xi(u) \cdot w$ considered up to proportionality. Is it true or not that

a) (10 points) every line laying on S is contained in a subspace of the form either $\mathbb{P}(U^*) \times w$ or $\xi \times \mathbb{P}(W)$ for some $w \in \mathbb{P}(W)$, $\xi \in \mathbb{P}(U^*)$?

b) (10 points) the tangent space $T_{\xi \otimes w}S$ to S at $\xi \otimes w \in S$ consists of all linear maps $f : U \to W$ such that $f(\operatorname{Ann}\xi) \subset \Bbbk \cdot w$?

[1] Geometrically, the projectivization $\mathbb{P}(W)$ consists of all cubic curves passing through $p_1, p_2, \dots, p_6$.