§9 Dimension

Everywhere in §8 we assume on default that the ground field $\Bbbk$ is algebraically closed.

9.1 Basic properties of the dimension. Given an algebraic manifold X and a point $x \in X$, the maximal $n \in \mathbb{N}$ such that there exists an increasing chain of closed irreducible submanifolds

$$\{x\} = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_{n-1} \subsetneq X_n \subset X \tag{9-1}$$

is called the *dimension* of X at x and denoted by $\dim_x X$. For an irreducible X, the maximality of a chain (9-1) forces $X_n = X$. Thus, if the point x belongs to several irreducible components of X, then $\dim_x X$ equals the maximal dimension among the dimensions of those components.

EXERCISE 9.1. Check that $\dim_x X = \dim_x U$ for every affine chart $U \ni x$.

LEMMA 9.1
Given a finite morphism of irreducible algebraic varieties $\varphi : X \to Y$, then $\dim_x X \leqslant \dim_{\varphi(x)} Y$ for all $x \in X$. If φ is not surjective, then the inequality is strict.

PROOF. Replacing Y by an affine neighborhood of $\varphi(x)$ and X by the preimage of this neighborhood allows us to assume, by Exercise 9.1, that both X, Y are affine. It follows from Proposition 7.12 on p. 94 that every chain (9-1) in X is mapped to the strictly increasing chain of closed irreducible subvarieties $\varphi(X_i)$ in Y. This leads to the required inequality. If $\varphi(X) \neq Y$, then the last subvariety of the chain is proper in Y, and therefore the chain can be enlarged at least by Y. □

PROPOSITION 9.1
$\dim_x \mathbb{A}^n = n$ for all $x \in \mathbb{A}^n$.

PROOF. Since for every $x \in \mathbb{A}^n$ there is a chain (9-1) of strictly increasing affine subspaces $X_i = \mathbb{A}^i$ passing through x, the inequality $\dim_x \mathbb{A}^n \geqslant n$ holds. The opposite inequality is established by induction in n. It is obvious for $\mathbb{A}^0$. Let $\dim_x \mathbb{A}^n = m$. Then the last element in every maximal chain (9-1) for $X = \mathbb{A}^n$ is $X_m = \mathbb{A}^n$. The next to last element $X_{m-1} \subsetneq X_m$ is a proper subvariety in $\mathbb{A}^n$. By Corollary 8.4 on p. 105, it admits a finite map to some proper affine subspace $\mathbb{A}^k \subsetneq \mathbb{A}^n$. By Lemma 9.1 and the inductive assumption applied for k, $\dim X_{m-1} \leqslant \dim \mathbb{A}^k \leqslant k < n$. Hence, $m - 1 \leqslant n - 1$ as required. □

PROPOSITION 9.2
Let X be an irreducible algebraic manifold. Then $\dim_x X$ does not depend on $x \in X$. If X is affine, then $\dim X = \operatorname{tr}\deg \Bbbk[X]$.

PROOF. Replacing X by an affine neighborhood of $x \in X$ allows us to assume that X is affine. By the Corollary 8.4 on p. 105, there exists a finite regular surjection $\pi : X \twoheadrightarrow \mathbb{A}^n$. Its pullback

$$\pi^* : \Bbbk[x_1, x_2, \dots, x_n] \hookrightarrow \Bbbk[X]$$

realizes $\Bbbk[X]$ as an algebraic extension of $\Bbbk[x_1, x_2, \dots, x_n]$. Therefore, $\operatorname{tr}\deg \Bbbk[X] = n$. By the Proposition 9.1 and Lemma 9.1, $\dim_x X \leqslant \dim \mathbb{A}^n = n$ for all $x \in X$. It remains to prove the opposite inequality. Consider a maximal chain of increasing irreducible subvarieties in $\mathbb{A}^n$

$$\{\pi(x)\} = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_{n-1} \subsetneq Y_n = \mathbb{A}^n .$$

By Proposition 7.13, every irreducible component of $\pi^{-1}(Y_i)$ is surjectively mapped onto Y_i for all i. Hence, there exists a strictly increasing chain $\{x\} = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_{n-1} \subsetneq X_n = X$ in which every X_i is an irreducible component of $\pi^{-1}(Y_i)$ that contains X_{i-1} and is surjectively mapped onto Y_i. This forces $\dim_x X \geqslant n$. □

COROLLARY 9.1
For every irreducible affine variety X and finite regular surjection $X \twoheadrightarrow \mathbb{A}^n$, the equality $n = \dim X$ holds. □

COROLLARY 9.2
The inequality $\dim X \leqslant \dim Y$ for a regular finite map $\varphi : X \to Y$ of irreducible manifolds[1] becomes the equality if and only if φ is surjective.

PROOF. For nonsurjectife φ the inequality is strong by Lemma 9.1. For surjective φ, the algebra $\Bbbk[X]$ is an algebraic extension of $\Bbbk[Y]$, and therefore $\operatorname{tr}\deg \Bbbk[X] = \operatorname{tr}\deg \Bbbk[Y]$. □

COROLLARY 9.3
$\dim(X \times Y) = \dim X + \dim Y$ for irreducible varieties X, Y.

PROOF. We can assume that X, Y both are affine, and $\dim X = n$, $\dim Y = m$. Then there exist finite surjections $\pi_X : X \twoheadrightarrow \mathbb{A}^n$, $\pi_Y : Y \twoheadrightarrow \mathbb{A}^m$. Their direct product $\pi_X \times \pi_Y : X \times Y \twoheadrightarrow \mathbb{A}^{n+m}$ is obviously regular and surjective. It is finite, because if some finite collections of elements f_i and g_j span, respectively, $\Bbbk[X]$ as a $\Bbbk[x_1, x_2, \dots, x_n]$-module and $\Bbbk[Y]$ as a $\Bbbk[y_1, y_2, \dots, y_m]$-module, then the products $f_i \otimes g_j$ span $\Bbbk[X] \otimes \Bbbk[Y]$ as a module over $\Bbbk[x_1, \dots, x_n, y_1, \dots, y_m]$. □

EXERCISE 9.2. Verify the latter statement.

9.2 Dimensions of subvarieties. If an algebraic manifold X is reducible and a regular nonzero function $f : X \to \Bbbk$ vanishes identically along an irreducible component $X' \subset X$ such that $\dim X' = \dim X$, then for every point $x \in X'$, the hypersurface $V(f) \subset X$ has $\dim_x V(f) = \dim_x X$, though $V(f) \neq X$ globally. For an irreducible X, such phenomenon never happens.

PROPOSITION 9.3
Let X be an irreducible affine algebraic variety and $f \in \Bbbk[X]$. Then $\dim_p V(f) = \dim_p(X) - 1$ for all $p \in V(f)$.

PROOF. If $V(f) = \varnothing$, there is nothing to prove. Assume that $V(f) \neq \varnothing$ and therefore, $f \neq \text{const}$. Then, for $X = \mathbb{A}^n$, the statement follows from the Example 8.7 on p. 106 and the Corollary 9.1. The general case is reduced to $X = \mathbb{A}^n$ by the same geometric construction as in the proof of Proposition 7.13 on p. 95. Namely, fix a finite surjection $\pi : X \twoheadrightarrow \mathbb{A}^m$ and consider the map

$$\varphi = \pi \times f : X \to \mathbb{A}^m \times \mathbb{A}^1, \quad x \mapsto \big(\pi(x), f(x)\big).$$

As we have seen in the proof of Proposition 7.13, the map φ provides X with the finite surjection onto the hypersurface $V(\mu_f) \subset \mathbb{A}^m \times \mathbb{A}^1$, the zero set of the minimal polynomial

$$\mu_f(u, t) = t^n + \alpha_1(u) t^{n-1} + \cdots + \alpha_n(u) \in \Bbbk[u_1, u_2, \dots, u_m][t]$$

[1]See Lemma 9.1 on p. 107.

for f over $\Bbbk(\mathbb{A}^m)$. The hypersurface $V(f) \subset X$ is surjectively mapped by φ onto the intersection $V(\mu_f) \cap V(t)$. Within the affine space $\mathbb{A}^m = V(t)$ the intersection $V(\mu_f) \cap V(t)$ is given by the equation $a_n = 0$, and therefore $\dim V(\mu_f) \cap V(t) = \dim V(a_n) = m - 1$ at every point of this intersection. By the Corollary 9.2, $\dim V(f) = V(\mu_f) \cap V(t) = \dim X - 1$. □

COROLLARY 9.4
Let X be an affine algebraic variety and $f_1, f_2, \dots, f_m \in \Bbbk[X]$. Then

$$\dim_p V(f_1, f_2, \dots, f_m) \geqslant \dim_p(X) - m \tag{9-2}$$

for all $p \in V(f_1, f_2, \dots, f_m)$. If the class of f_i in the quotient $\Bbbk[X]/(f_1, f_2, \dots, f_{i-1})$ does not divide zero for every[1] $i = 1, 2, \dots, m$, then the inequality (9-2) becomes an equality.

PROOF. Induction in m. Let $Y = V(f_1, f_2, \dots, f_{i-1})$, $p \in Y$, and Z be an irreducible component of Y passing through p. The function f_i ether vanishes identically on Z or is restricted to nonzero element of $\Bbbk[Z]$. The first means that f_i divides zero in $\Bbbk[Y] = \Bbbk[X]/(f_1, f_2, \dots, f_{i-1})$, and forces $\dim_p(Z \cap V(f_1, f_2, \dots, f_i)) = \dim_p Z$. In the second case, $\dim_p(Z \cap V(f_1, f_2, \dots, f_i)) = \dim_p Z - 1$ by Proposition 9.3. □

CAUTION 9.1. Note that Proposition 9.3 and Corollary 9.4 do *not* assert that $V(f_1, f_2, \dots, f_m) \neq \varnothing$. Since the empty set contains no points p, for $V(f_1, f_2, \dots, f_m) = \varnothing$, the Corollary 9.4 remains formally true but becomes empty. The weak Nullstellensatz implies that $V(f_1, f_2, \dots, f_m) = \varnothing$ if and only if the class of f_i in $\Bbbk[X]/(f_1, f_2, \dots, f_{i-1})$ is invertible for some i, and this may routinely happen. For example, consider $X = \mathbb{A}^3 = \operatorname{Spec}_{\mathrm{m}} \Bbbk[x, y, z]$, $f_1 = x$, $f_2 = x + 1$. Obviously, $V(x, x+1) = \varnothing$. The same warning applies to the next corollary as well.

COROLLARY 9.5
For affine algebraic varieties $X_1, X_2 \subset \mathbb{A}^n$ and every point $x \in X_1 \cap X_2$,

$$\dim_x(X_1 \cap X_2) \geqslant \dim_x X_1 + \dim_x X_2 - n\,.$$

PROOF. Let $\varphi_i : X_i \hookrightarrow \mathbb{A}^n$, $i = 1, 2$, be the closed immersions corresponding to the quotient maps $\varphi_i^* : \Bbbk[x_1, x_2, \dots, x_n] \twoheadrightarrow \Bbbk[X_i]$. Then $X_1 \cap X_2$ is isomorphic to the preimage of the diagonal $\Delta_{\mathbb{A}^n} \subset \mathbb{A}^n \times \mathbb{A}^n$ under the map $\varphi_1 \times \varphi_2 : X_1 \times X_2 \hookrightarrow \mathbb{A}^n \times \mathbb{A}^n$. Within $X_1 \times X_2$, this preimage is determined by the n equations $\varphi_1^* \times \varphi_2^*(x_i) = \varphi_1^* \times \varphi_2^*(y_i)$, the pullbacks of equations $x_i = y_i$ for $\Delta_{\mathbb{A}^n}$ in $\mathbb{A}^n \times \mathbb{A}^n$. It remains to apply Corollary 9.4. □

PROPOSITION 9.4
For any irreducible projective varieties $X_1, X_2 \subset \mathbb{P}_n$, the inequality $\dim X_1 + \dim X_2 \geqslant n$ forces $X_1 \cap X_2 \neq \varnothing$.

PROOF. Let $\mathbb{P}_n = \mathbb{P}(V)$ and $\mathbb{A}^{n+1} = \mathbb{A}(V)$. Given a nonempty irreducible projective variety $Z \subset \mathbb{P}_n$, write $Z' \subset \mathbb{A}^{n+1}$ for the affine cone over Z provided by the same homogeneous equations on the coordinates. Then the origin $O \in \mathbb{A}^{n+1}$ belongs to Z' and $\dim_O Z' \geqslant \dim Z + 1$, because every chain

[1] For $i = 1$ this means that f_1 is not a zero divisor in $\Bbbk[X]$. A sequence of functions possessing this property is called a *a regular sequence*, and the corresponding subvariety $V(f_1, f_2, \dots, f_m) \subset X$ ia called a *complete intersection*.

$\{z\} \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_m = Z$ produces the chain of cones $\{O\} \subsetneq (O,z) \subsetneq Z_1' \subsetneq \cdots \subsetneq Z_m' = Z'$ starting with the point O and the line (O,z). Therefore, by Corollary 9.5

$$\dim_O(X_1' \cap X_2'') \geqslant \dim_O(X_1) + 1 + \dim_O(X_2) + 1 - n - 1 \geqslant 1\,.$$

Thus, $X_1' \cap X_2''$ is not exhausted by O. □

9.2.1 Dimensions of fibers of regular maps. In a contrast to differential geometry and topology, the dimensions of nonempty fibers of regular maps are controlled in algebraic geometry almost as strictly as in linear algebra.

THEOREM 9.1

Let $\varphi : X \to Y$ be a dominant regular map of irreducible algebraic varieties. Then for all $x \in X$,

$$\dim_x \varphi^{-1}(\varphi(x)) \geqslant \dim X - \dim Y\,. \tag{9-3}$$

Moreover, there exists a dense Zariski open set $U \subset Y$ such that for all $y \in U$ and all $x \in \varphi^{-1}(y)$,

$$\dim_x \varphi^{-1}(y) = \dim_x X - \dim_y Y\,. \tag{9-4}$$

PROOF. Replacing Y by an affine chart $U \ni \varphi(x)$ and X by an affine neighborhood of x in $\varphi^{-1}(U)$ allows us to assume that X, Y are affine. Composing φ with a finite surjection $Y \twoheadrightarrow \mathbb{A}^m$, we may assume that $Y = \mathbb{A}^m = \operatorname{Spec}_{\mathrm{m}} \Bbbk[y_1, y_2, \dots, y_m]$ and $\varphi(x) = 0$. Then $\varphi^{-1}(0) \subset X$ is given by the m equations $\varphi^*(y_i) = 0$, the pullbacks of the equations $y_i = 0$, which describe the origin within $\mathbb{A}^m$. Thus, Corollary 9.4 implies inequality (9-3).

To prove the second statement, let us factorize φ into a closed immersion $X \hookrightarrow Y \times \mathbb{A}^m$ followed by the projection $\pi : Y \times \mathbb{A}^m \twoheadrightarrow Y$, as in formula (7-7) on p. 94, and apply Corollary 8.3 on p. 105 to the fibers of π. Consider the projective closure $\overline{X} \subset Y \times \mathbb{P}_m$, fix a projective hyperplane $H \subset \mathbb{P}_m$ and a point $p \in \mathbb{P}_m \smallsetminus H$ such that the section $Y \times \{p\} \subset Y \times \mathbb{P}_m$ is not contained in $\overline{X}$. Then the fiberwise projection from p to H satisfies the conditions of Proposition 8.1 in the fibers over all

$$y \in Y \smallsetminus \overline{\pi}\Big(\big(Y \times \{p\}\big) \cap \overline{X}\Big)\,,$$

where $\overline{\pi} : Y \times \mathbb{P}_m \twoheadrightarrow Y$ is the projection along $\mathbb{P}_m$. Since the latter is a closed map, the inadmissible points y form a proper Zariski closed subset in Y. Therefore, there exists a nonempty principal open set $U \subset Y$ such that Proposition 8.1 can be applied fiberwise over all points $y \in U$. Since U is an affine algebraic variety as well, we can replace Y by U and X by $X \cap \pi^{-1}(U)$. After that, Corollary 8.3 gives a finite parallel fiberwise projection of X in the direction p to affine hyperplane $Y \times \mathbb{A}^{m-1} = (Y \times H) \cap (Y \times \mathbb{A}^n)$. If it is not surjective, we repeat the procedure until we get a finite surjection $\psi : X \twoheadrightarrow Y \times \mathbb{A}^n$ whose composition with the projection onto Y equals φ. This forces $\dim X = n + \dim Y$. Since the fiber $\varphi^{-1}(y)$ is surjectively and finitely mapped onto $\{y\} \times \mathbb{A}^n$ for all $y \in Y$, we conclude from Lemma 9.1 that $\dim_x \varphi^{-1}(y) = n = \dim X - \dim Y$ for all $x \in \varphi^{-1}(y)$. □

COROLLARY 9.6 (SEMICONTINUITY THEOREM)

For every regular map of algebraic manifolds $\varphi : X \to Y$, the sets

$$X_k \stackrel{\text{def}}{=} \{x \in X \mid \dim_x \varphi^{-1}(\varphi(x)) \geqslant k\}$$

are closed in X for all $k \in \mathbb{Z}$.

PROOF. If $\dim Y = 0$, then this is trivially true for all X and k. For $\dim Y = m > 0$ we can assume by induction that the statement holds for all X, k, and all Y with $\dim Y < m$. Replacing Y and X by some irreducible components of maximal dimension passing through $\varphi(x)$ and x respectively allows us to assume that both X and Y are irreducible. Since $X_k = X$ for $k \leqslant \dim(X) - \dim(Y)$ by Theorem 9.1, the statement holds for all such k. For $k > \dim(X) - \dim(Y)$, we can replace Y and X by $Y' = Y \smallsetminus U$ and $X' = \varphi^{-1}(Y')$, where $U \subset Y$ is that from Theorem 9.1, and apply the inductive assumption, because $X_k \subset X'$ and $\dim Y' < \dim Y$. □

COROLLARY 9.7
Let $\varphi : X \to Y$ be a closed regular morphism of algebraic manifolds. Then the sets

$$Y_k \stackrel{\text{def}}{=} \{y \in Y \mid \dim \varphi^{-1}(y) \geqslant k\}$$

are closed in Y for all $k \in \mathbb{Z}$. □

THEOREM 9.2 (DIMENSION CRITERION OF IRREDUCIBILITY)
Assume that a closed regular surjection of algebraic manifolds $\varphi : X \twoheadrightarrow Y$ has irreducible fibers of the same constant dimension. Then X is irreducible if Y is.

PROOF. Let $X = X_1 \cup X_2$ be reducible. Since every fiber of φ is irreducible, it is entirely contained in X_1 or in X_2. Put $Y_i \stackrel{\text{def}}{=} \{y \in Y \mid \varphi^{-1}(y) \subset X_i\}$ for $i = 1, 2$. Then $Y = Y_1 \cup Y_2$, and the subsets $Y_1, Y_2 \subsetneq Y$ are proper if $X_1, X_2 \subsetneq X$ are proper. Since Y_i coincides with the locus of points in Y over which the fibers of the restricted map $\varphi|_{X_i} : X_i \to Y$ achieve their maximal value, we conclude from Corollary 9.7 that Y_i is closed in Y for both $i = 1, 2$. Thus, reducibility of X forces Y to be reducible. □

9.3 Dimensions of projective varieties. It follows from Proposition 9.4 on p. 109 that every irreducible projective manifold $X \subset \mathbb{P}_n = \mathbb{P}(V)$ of dimension $\dim X = d$ intersects all projective subspaces $H \subset \mathbb{P}_n$ of dimension $\dim H \geqslant n - d$. We are going to show that a generic projective subspace H of dimension $\dim H < n - d$ does not intersect X, and therefore, the dimension $\dim X$ is characterized as the maximal d such that X intersects all projective subspaces of codimension d. We know from n° 4.6.4 on p. 58 that all projective subspaces of codimension $d + 1$ in $\mathbb{P}_n = \mathbb{P}(V)$ form the Grassmannian $\mathrm{Gr}(n-d, n+1) = \mathrm{Gr}(n-d, V)$, which is an irreducible projective manifold. Consider the *incidence variety*

$$\Gamma \stackrel{\text{def}}{=} \{(x, H) \in X \times \mathrm{Gr}(n-d, V) \mid x \in H\} \tag{9-5}$$

and write $\pi_1 : \Gamma \twoheadrightarrow X$ and $\pi_2 : \Gamma \to \mathrm{Gr}(n-d, V)$ for the canonical projections.

EXERCISE 9.3. Convince yourself that Γ is a projective algebraic variety.

The fiber of the first projection $\pi_1 : \Gamma \twoheadrightarrow X$ over an arbitrary point $x \in X$ consists of all projective subspaces passing trough x. It is naturally identified with the Grassmannian $\mathrm{Gr}(n-d-1, n) = \mathrm{Gr}(n-d-1, V/\Bbbk \cdot x)$ of all $(n-d-1)$-dimensional vector subspaces in the quotient space $V/\Bbbk x$. Thus, π_1 is a closed surjective morphism with irreducible fibers of the same constant dimension $(n-d-1)(d+1)$. By Theorem 9.2, the incidence variety Γ is irreducible, and

$$\dim \Gamma = d + (n-d-1)(d+1) = (n-d)(d+1) - 1\,.$$

This forces the image of the second projection $\pi_2(\Gamma) \subset \mathrm{Gr}(n-d, V)$, which consists of all $(n-d-1)$-dimensional projective subspaces intersecting X, to be a closed irreducible subvariety of dimension at most $\dim \Gamma$ in the grassmannian $\mathrm{Gr}(n-d,V)$ of dimension $(n-d)(d+1) > \dim \Gamma$. Therefore, the codimension $(d+1)$ projective subspaces H not intersecting X form a dense Zariski open subset in the Grassmannian $\mathrm{Gr}(n-d,V)$.

In fact, dimensional arguments allow us to say much more about the interaction of X with the projective subspaces in $\mathbb{P}_n$. If we repeat the previous construction for the Grassmannian $\mathrm{Gr}(n-d+1,V)$ of codimension-d subspaces $H' \subset \mathbb{P}(V)$ and the incidence variety

$$\Gamma' \overset{\text{def}}{=} \{(x,H') \in X \times \mathrm{Gr}(n-d+1,V) \mid x \in H\},$$

which is an irreducible projective manifold of dimension

$$\dim X + \dim \mathrm{Gr}(n-d,n) = d + d(n-d) = d(n-d+1)$$

for the same reasons as above, we get a surjective projection $\pi_2 : \Gamma' \twoheadrightarrow \mathrm{Gr}(n-d+1,V)$, because $X \cap H' \neq \varnothing$ for all $H' \subset \mathbb{P}(V)$. Theorem 9.1 forces the fibers of π_2 to achieve their minimal possible dimension $\dim \Gamma - \dim \mathrm{Gr}(n-d+1,n+1) = d(n-d+1) - (n-d+1)d = 0$ over all points of some open dense subset in the Grassmannian. This means that a generic projective subspace of codimension d intersects X in a *finite number* of points. Let us fix such a subspace H' and draw an $(n-d-1)$-dimensional subspace $H \subset H'$ through some intersection point $p \in X \cap H'$. Then $H \cap X$ is a nonempty finite set. Therefore, the second projection of the incidence variety (9-5)

$$\pi_2 : \Gamma \to \mathrm{Gr}(n-d,V)$$

has a zero-dimensional fiber. This forces the minimal dimension of nonempty fibers to be zero. It follows from Theorem 9.1 that $\dim \pi_2(\Gamma) = \dim \Gamma = \dim \mathrm{Gr}(n-d,V) - 1$. In other words, the codimension $(d+1)$ projective subspaces $H \subset \mathbb{P}(V)$ intersecting an irreducible variety $X \subset \mathbb{P}(V)$ of dimension d form an irreducible hypersurface in the Grassmannian $\mathrm{Gr}(n-d,V)$ of all codimension-$(d+1)$ projective subspaces in $\mathbb{P}_n = \mathbb{P}(V)$.

Exercise 9.4. Deduce from this that for every irreducible projective variety $X \subset \mathbb{P}_n$ of dimension d, there exists a unique, up to a scalar factor, irreducible homogeneous polynomial in the Plücker coordinates of a codimension-d subspace $H \subset \mathbb{P}_n$ that vanishes at a given H if and only if $H \cap X \neq \varnothing$.

The above analysis illustrates a method commonly used in geometry for calculating the dimensions of projective manifolds by means of auxiliary incidence varieties. Below are two more examples.

Example 9.1 (resultant)
Given collection of positive integers $d_0, d_1, \dots, d_n \in \mathbb{N}$, write $\mathbb{P}_{N_i} = \mathbb{P}\left(S^{d_i}V^*\right)$ for the space of degree-d_i hypersurfaces in $\mathbb{P}_n = \mathbb{P}(V)$. We are going to show that the resultant variety[1]

$$\mathcal{R} = \{(S_0, S_1, \dots, S_n) \in \mathbb{P}_{N_0} \times \mathbb{P}_{N_1} \times \cdots \times \mathbb{P}_{N_n} \mid \cap S_i \neq \varnothing\}$$

of a system of $(n+1)$ homogeneous polynomial equations of given degrees in $n+1$ unknowns is an irreducible hypersurface, i.e., there exists a unique, up to proportionality, irreducible polynomial R in the coefficients of the equations, homogeneous in the coefficients of each equation, such that R vanishes at a given collection of polynomials $f_0, f_1, \dots, f_n$ if and only if the equations

[1]See n° 6.8 on p. 79.

$f_i(x_0, x_1, \dots, x_n) = 0$, $0 \leqslant i \leqslant n$, have a nonzero solution. The polynomial R is called the *resultant* of $n+1$ homogeneous polynomials of degrees $d_1, d_2, \dots, d_n$.

Consider the incidence variety $\Gamma \overset{\text{def}}{=} \{(S_1, S_2, \dots, S_n, p) \in \mathbb{P}_{N_0} \times \cdots \times \mathbb{P}_{N_n} \times \mathbb{P}_n \mid p \in \cap S_i\}$.

EXERCISE 9.5. Convince yourself that Γ is an algebraic projective variety.

Since the equation $f(p) = 0$ is linear in f, all degree-d_i hypersurfaces in $\mathbb{P}_n$ passing through a given point $p \in \mathbb{P}_n$ form a hyperplane in $\mathbb{P}_{N_i}$. Therefore, the projection $\pi_2 : \Gamma \twoheadrightarrow \mathbb{P}_n$ is surjective, and all its fibers, which are the products of projective hyperplanes in the spaces $\mathbb{P}_{N_i}$, are irreducible and have the same constant dimension $\sum(N_i - 1) = \sum N_i - n - 1$. Thus, Γ is an irreducible projective variety of dimension $\sum N_i - 1$.

EXERCISE 9.6. Write $n+1$ hypersurfaces $V(f_i) \subset \mathbb{P}_n$ of prescribed degrees $d_i = \deg f_i$ such that $V(f_0, f_1, \dots, f_n)$ is just one point.

The exercise shows that the projection $\pi_1 : \Gamma \to \mathbb{P}_{N_0} \times \mathbb{P}_{N_1} \times \cdots \times \mathbb{P}_{N_n}$ has a nonempty fiber of dimension zero. This forces a generic nonempty fiber to be of dimension zero, and implies the equality $\dim \pi_1(\Gamma) = \dim \Gamma$. Therefore, $\pi_1(\Gamma)$ is an irreducible submanifold of codimension 1 in $\mathbb{P}_{N_0} \times \cdots \times \mathbb{P}_{N_n}$.

EXERCISE 9.7. Show that every irreducible submanifold of codimension 1 in a product of projective spaces is the zero set of an irreducible polynomial in the homogeneous coordinates on the spaces, homogeneous in the coordinates of each space.

EXAMPLE 9.2 (LINES ON SURFACES)

Algebraic surfaces of degree d in $\mathbb{P}_3 = \mathbb{P}(V)$ form the projective space $\mathbb{P}_N = \mathbb{P}(S^d V^*)$ of dimension $N = \frac{1}{6}(d+1)(d+2)(d+3) - 1$. The lines in $\mathbb{P}_3$ form the Grassmannian $\mathrm{Gr}(2,4) = \mathrm{Gr}(2,V)$, which is isomorphic to the smooth 4-dimensional projective Plücker quadric[1]

$$P = \{\omega \in \Lambda^2 V \mid \omega \wedge \omega = 0\}$$

in $\mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$ by means of the Plücker embedding, which maps a line $(a,b) \subset \mathbb{P}_3$ to the decomposable Grassmannian quadratic form $a \wedge b \in \mathbb{P}_5$. Consider the incidence variety

$$\Gamma \overset{\text{def}}{=} \{\, (S, \ell) \in \mathbb{P}_N \times \mathrm{Gr}(2,4) \mid \ \ell \subset S \,\}\ .$$

EXERCISE 9.8. Convince yourself that $\Gamma \subset \mathbb{P}_N \times \mathrm{Gr}(2,4)$ is a projective algebraic variety.

The projection $\pi_2 : \Gamma \twoheadrightarrow Q_P$ is surjective and all its fibers are projective spaces of the same constant dimension. Indeed, the line ℓ given by the equations $x_0 = x_1 = 0$ lies on a surface $V(f)$ if and only if $f = x_2 \cdot g + x_3 \cdot h$ belongs to the image of the $\Bbbk$-linear map

$$\psi : S^{d-1}V^* \oplus S^{d-1}V^* \to S^d V^*\,,\ (g,h) \mapsto x_2 g + x_3 h\,.$$

This image is isomorphic to the quotient of the space $S^{d-1}V^* \oplus S^{d-1}V^*$ by the subspace

$$\ker \psi = \{(g,h) = (x_3 q, -x_2 q) \mid q \in S^{d-2}V^*\}\,.$$

Since $\dim S^{d-1}V^* = \frac{1}{6}d(d+1)(d+2)$ and $\dim \ker \psi = \frac{1}{6}(d-1)d(d+1)$, the degree-$d$ surfaces containing ℓ form a projective space of dimension

$$\frac{1}{6}\Big(\, 2\,d(d+1)(d+2) - (d-1)d(d+1)\,\Big) - 1 = \frac{1}{6}\,d(d+1)(d+5) - 1\,.$$

[1] Compare with Problem 17.20 of Algebra I

We conclude that Γ is an irreducible projective variety of dimension

$$\dim \Gamma = \frac{1}{6} d(d+1)(d+5) + 3\,.$$

The image of projection $\pi_1 : \Gamma \to \mathbb{P}_N$ consists of all surfaces containing at least one line. It follows from the above analysis that $\pi_1(\Gamma)$ is an irreducible closed submanifold of $\mathbb{P}_N$.

EXERCISE 9.9. For every integer $d \geqslant 3$ find a degree-d surface $S \subset \mathbb{P}_3$ containing just a finite number of lines.

The exercise shows that for $d \geqslant 3$, the projection π_1 has a nonempty fiber of dimension zero. Therefore, a generic nonempty fiber of π_1 is finite, and $\dim \pi_1(\Gamma) = \dim \Gamma$ for $d \geqslant 3$. Since the difference $N - \dim \Gamma = \frac{1}{6}\big((d+1)(d+2)(d+3) - d(d+1)(d+5)\big) - 4 = d - 3$, every cubic surface in $\mathbb{P}_3$ contains a line, and the set of cubic surfaces with a finite number of lines lying on them contains a dense Zariski open subset of $\mathbb{P}_N$. At the same time, there are no lines on a generic surface of degree $d \geqslant 4$.

9.4 Application: 27 lines on a smooth cubic surface. Let $S \subset \mathbb{P}_3$ be a smooth cubic surface provided by equation $F(x) = 0$. We are going to show that there are exactly 27 lines laying on S and the configuration of these lines does not depend on S up to permutations of the lines.

9.4.1 The 10 lines associated with a given line. To construct the lines laying on S, we consider one such a line $\ell \subset S$, which exists by the previous Example 9.2, and intersect S with the planes passing through ℓ.

LEMMA 9.2
A reducible plane section of S splits into a union of either a line and a smooth conic or a triple of distinct lines. In other words, it does not contain a double line component.

PROOF. Let a plane section $\pi \cap S$ contain a double line ℓ. In coordinates where π has the equation $x_2 = 0$ and ℓ is given by $x_2 = x_3 = 0$, the equation of S acquires the form

$$F(x) = x_2 Q(x) + x_3^2 L(x) = 0$$

for some linear L and quadratic Q. Let a be an intersection point of ℓ with the quadric $Q(x) = 0$. The relations $x_2(a) = x_3(a) = Q(a) = 0$ force all partial derivatives $\partial F/\partial x_i$ vanish at a. Thus, S is singular at a. □

COROLLARY 9.8
For a point $p \in S$, there may be at most three lines lying on S and passing through p, and all such lines must be coplanar.

PROOF. All lines passing through $p \in S$ and lying on S lie inside $S \cap T_p S$, which is a plane cubic that may split into a union of at most three lines. □

LEMMA 9.3
For every line $\ell \subset S$, there are exactly five distinct planes $\pi_1, \pi_2, \dots, \pi_5$ containing ℓ and intersecting S in a triple of lines. Let $\pi_i \cap S = \ell \cup \ell_i \cup \ell_i'$, then $\ell_i \cap \ell_j = \ell_i \cap \ell_j' = \ell_i' \cap \ell_j' = \varnothing$ for all $i \neq j$, and every line on S that does not intersect ℓ must intersect exactly one of the lines ℓ_i, ℓ_i' for every $i = 1, \dots, 5$.

PROOF. Fix a basis $\{e_0, e_1, e_2, e_3\}$ in V such that $\ell = (e_0 e_1)$ is given by equations $x_2 = x_3 = 0$. Then the equation of S acquires the form

$$L_{00}(x_2, x_3) \cdot x_0^2 + 2\,L_{01}(x_2, x_3) \cdot x_0 x_1 + L_{11}(x_2, x_3) \cdot x_1^2 + \\ + 2\,Q_0(x_2, x_3) \cdot x_0 + 2\,Q_1(x_2, x_3) \cdot x_1 + R(x_2, x_3) = 0\,, \quad \text{(9-6)}$$

where $L_{ij}, Q_\nu, R \in k[x_2, x_3]$ are homogeneous of degrees $1, 2, 3$ respectively. Let us parameterize the pencil of plains π_ϑ passing through ℓ by the points

$$e_\vartheta \overset{\text{def}}{=} \pi_\vartheta \cap (e_2, e_3) = \vartheta_2 e_2 + \vartheta_3 e_3 \in (e_2 e_3)$$

and write $(t_0 : t_1 : t_2)$ for the homogeneous coordinates in the plane $\pi_\vartheta = (e_0 e_1 e_\vartheta)$ with respect to the basis e_0, e_1, e_ϑ. The equation for the plane conic $\left(\pi_\vartheta \cap S\right) \setminus \ell$ is obtained by the substitution $x = (t_0 : t_1 : \vartheta_2 t_3 : \vartheta_3 t_3)$ in the equation (9-6) and canceling the common factor t_3. The resulting conic has the Gram matrix

$$G = \begin{pmatrix} L_{00}(\vartheta) & L_{01}(\vartheta) & Q_0(\vartheta) \\ L_{01}(\vartheta) & L_{11}(\vartheta) & Q_1(\vartheta) \\ Q_0(\vartheta) & Q_1(\vartheta) & R(\vartheta) \end{pmatrix}$$

whose determinant $D(\vartheta)$ is the following homogeneous degree-5 polynomial in $\vartheta = (\vartheta_2 : \vartheta_3)$

$$L_{00}(\vartheta)L_{11}(\vartheta)R(\vartheta) + 2\,L_{01}(\vartheta)Q_0(\vartheta)Q_1(\vartheta) - L_{11}(\vartheta)Q_0^2(\vartheta) - L_{00}(\vartheta)Q_1^2(\vartheta) - L_{01}(\vartheta)^2 R(\vartheta)\,.$$

It has five roots, and we have to show that all these roots are simple. Every root corresponds to a splitting of the conic into a pair of lines ℓ', ℓ''. There are two possibilities: either the intersection point $\ell' \cap \ell''$ lies on ℓ or it lies outside ℓ.

In the first case, we can fix a basis in order to have $\ell' = (e_0 e_2)$ and $\ell'' = \left(e_0\,(e_1 + e_2)\right)$. These lines are given by the equations $x_3 = x_1 = 0$ and $x_3 = (x_1 - x_2) = 0$, and the splitting appears for $\vartheta = (1 : 0)$. The multiplicity of this root equals the highest power of ϑ_3 dividing $D(\vartheta_2, \vartheta_3)$. Since $\ell, \ell', \ell'' \subset S$, the equation (9-6) has the form $x_1 x_2 (x_1 - x_2) + x_3 \cdot q(x)$ for some quadratic $q(x)$. Thus, elements of G that may be not divisible by ϑ_3 are exhausted by $L_{11} \equiv x_2 \pmod{\vartheta_3}$ and $Q_1 \equiv -x_2^2\,/\,2 \pmod{\vartheta_3}$. So, $D(\vartheta_2, \vartheta_3) \equiv -L_{00} Q_1^2 \pmod{\vartheta_3^2}$. This term is of order one in t_3 if the monomials $x_1 x_2^2$ and $x_0^2 x_2$ appear in (9-6) with non zero coefficients. The first of these two monomials is the only monomial that gives a nonzero contribution in $\partial F / \partial x_1$ computed at $e_2 \in S$ and the second in $\partial F / \partial x_2$ at $e_0 \in S$. Hence, they have to appear in F.

In the second case, we fix a basis in order to have $\ell' = (e_0 e_2)$, $\ell'' = (e_1 e_2)$, the lines given by the equations $x_3 = x_1 = 0$ and $x_3 = x_0 = 0$. The splitting happens again for $\vartheta = (1 : 0)$. The equation (9-6) turns to $x_0 x_1 x_2 + x_3 \cdot q(x)$. A nonzero modulo ϑ_3 contribution may come only from $L_{01} \equiv x_2\,/\,2 \pmod{\vartheta_3}$. Thus, $D(\vartheta_2, \vartheta_3) \equiv -L_{01}^2 R \pmod{\vartheta_3^2}$ is of the first order in t_3 if $x_2^2 x_3$ and $x_0 x_1 x_2$ appear in (9-6). The first is the only monomial giving a non zero contribution to $\partial F / \partial x_3$ computed at $e_2 \in S$. Thus, it does appear. The second does too, because otherwise F would be divisible by x_3.

All the remaining statements of the lemma follow immediately from Corollary 9.8, Lemma 9.2 and the fact that every line in $\mathbb{P}_3$ intersects every plane. □

LEMMA 9.4

Any four mutually nonintersecting lines on S do not lie simultaneously on a quadric, and there exist either one or two (but no more!) lines on S intersecting each of the four lines.

PROOF. If the four given lines lie on some quadric Q, then Q is smooth and the lines belong to the same ruling family[1]. Every line from the second ruling family lies on S, because a line passing through four distinct points of S must lie on S. Hence, $Q \subset S$ and therefore, S is reducible. It remains to apply Exercise 2.14. □

9.4.2 The configuration of all 27 lines. Fix two nonintersecting lines a, $b \subset S$ and consider the five pairs of lines ℓ_i, ℓ_i' provided by Lemma 9.3 applied to the line $\ell = a$. Write ℓ_i for the lines that do meet b, and ℓ_i' for the remaining lines, which do not. There are five more lines ℓ_i'' coupled with ℓ_i by the Lemma 9.3 applied to the line $\ell = b$. Every line ℓ_i'' intersects b but neither a nor ℓ_j for $j \neq i$. Thus, ℓ_i'' intersects all ℓ_j' with $j \neq i$. Every line $c \subset S$, different from the 17 lines just constructed, intersects neither a nor b. At the same time, for each i, it must intersect either ℓ_i or ℓ_i'. By Lemma 9.4, the lines intersecting $\geqslant 4$ of the ℓ_i's are exhausted by a and b. Let c intersect $\leqslant 2$ of the ℓ_i's, say ℓ_1', ℓ_2', ℓ_3' and either ℓ_4' or ℓ_5. In both cases, we already have two distinct lines a, ℓ_5'' other than c intersecting all the four lines. This contradicts to Lemma 9.4. We conclude that c intersects exactly three of the five lines ℓ_i.

LEMMA 9.5
The remaining lines $c \subset S$ stay in bijection with 15 triples $\{i, j, k\} \subset \{1, 2, 3, 4, 5\}$.

PROOF. For every triple of lines ℓ_i, there is at most one line c other than a intersecting the three given lines and the remaining two lines ℓ_j', because these five lines are mutually nonintersecting. On the other hand, it follows from Lemma 9.3 that for every i, there are exactly 10 lines on S intersecting the line ℓ_i. Four of them are a, b, ℓ_i', ℓ_i''. Each of the other six lines must intersect exactly two of the remaining four ℓ_j's. So, we have a bijection between these six lines and the $6 = \binom{4}{2}$ pairs of ℓ_j's. □

COROLLARY 9.9
Every smooth cubic surface $S \subset \mathbb{P}_3$ contains exactly 27 lines and their incidence matrix[2] is the same for all S up to reordering the lines. □

EXERCISE 9.10*. Write $G \subset S_{27}$ for the group of all permutations of the 27 lines that preserve all pairwise incidences between them. Consider the field of 4 elements $\mathbb{F}_4 \stackrel{\text{def}}{=} \mathbb{F}_2[\omega]/(\omega^2+\omega+1)$, where $\mathbb{F}_2 = \mathbb{Z}/(2)$. The extension $\mathbb{F}_2 \subset \mathbb{F}_4$ is equipped with the conjugation automorphism[3] $z \longmapsto \overline{z} \stackrel{\text{def}}{=} z^2$, which lives $\mathbb{F}_2$ fixed and permutes two roots of the polynomial $\omega^2 + \omega + 1$. Show that the *unitary*[4] 4×4 matrices with elements in $\mathbb{F}_4$, considered up to proportionality, form a (normal) subgroup of index 2 in G, and find the order of G.

[1]See n° 2.5.1 on p. 23.
[2]That is, the matrix of size 27×27 whose rows and columns stay in bijection with the lines, and the element in a position (i, j) equals 1 if $\ell_i \cap \ell_j \neq \varnothing$ and 0 otherwise.
[3]It is quite similar to the complex conjugation in the extension $\mathbb{R} \subset \mathbb{C}$.
[4]That is, satisfying $\overline{M} \cdot M^t = E$.

Comments to some exercises

EXRC. 9.1. Let $X_1, X_2 \subset X$ be two closed irreducible subsets, and $U \subset X$ an open set such that both intersections $X_1 \cap U$, $X_2 \cap U$ are nonempty. Then $X_1 = X_2 \iff X_1 \cap U = X_2 \cap U$, because $X_i = \overline{X_i \cap U}$.

EXRC. 9.3. Chose some basis in H and write the coordinates of the basis vectors together with the coordinates of a variable point $p \in \mathbb{P}_n$ as the rows of $(n-d+1) \times (n+1)$-matrix. Then the condition $p \in H$ is equivalent to vanishing of all the minors of maximal degree $n-d+1$ in these matrix. The latter are quadratic bilinear polynomials in the homogeneous coordinates of p and the Plücker coordinates[1].

EXRC. 9.5. The set $\Gamma \subset \mathbb{P}_{N_0} \times \cdots \times \mathbb{P}_{N_n} \times \mathbb{P}_n$ is given by the equations

$$f_0(p) = f_1(p) = \cdots = f_n(p) = 0$$

on $f_i \in \mathbb{P}_{N_i}$ and $p \in \mathbb{P}_n$, linear homogeneous in each f_i and homogeneous of degrees d_i in p.

EXRC. 9.6. Take $n+1$ hyperplanes intersecting at one point and exponentiate their linear equations in the prescribed degrees.

EXRC. 9.7. Consider the product $\mathbb{P}_{n_1} \times \mathbb{P}_{n_2} \times \cdots \times \mathbb{P}_{n_m}$ and write $x^{(i)} = (x_0^{(i)} : x_1^{(i)} : \ldots : x_{n_i}^{(i)})$ for the set of homogeneous coordinates on the i
divs th factor $\mathbb{P}_{n_i}$. Modify the proof of Lemma 8.1 on p. 103 to show that any closed submanifold $Z \subset \mathbb{P}_1 \times \mathbb{P}_2 \times \cdots \times \mathbb{P}_m$ can be described by appropriate system of global polynomial equations $f_\nu\left(x^{(1)}, x^{(2)}, \ldots, x^{(n)}\right) = 0$, homogeneous in every group of variables $x^{(i)}$. Then assume that Z is irreducible of codimension 1, show that there exists an irreducible polynomial $q\left(x^{(1)}, x^{(2)}, \ldots, x^{(n)}\right)$ vanishing on Z, and use the dimensional argument to check that $Z = V(q)$ is the zero set of q. Finally, use the strong Nullstellensatz to show that for irreducible polynomials q_1, q_2, the equality $V(q_1) = V(q_2)$ forces q_1, q_2 to be proportional.

EXRC. 9.8. Identify $\mathrm{Gr}(2,4)$ with the Plücker quadric $P \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$ by sending a line $(a,b) \subset \mathbb{P}_3$ to the point $a \wedge b \in \mathbb{P}_5$. The line (a,b) lies on the surface $V(f) \subset \mathbb{P}_3$ if and only if the polynomial f vanishes identically on the linear span of vectors a, b, which is the linear support of the Grassmannian polynomial $a \wedge b$ and coincides with the image of the map $V^* \to V$, $\xi \mapsto \xi \llcorner (a \wedge b)$, contracting a covector $\xi \in V^*$ with the first tensor factor of $(a \otimes b - b \otimes a)/2 \in \mathrm{Skew}^2 V$. Verify that the identical vanishing of the function $\xi \mapsto f(\xi \llcorner (a \wedge b))$ can be expressed by a system of bihomogeneous equations on the coefficients of f and the Plücker coordinates x_{ij} of the bivector $a \wedge b = \sum_{0 \leqslant i < j \leqslant 3} x_{ij} e_i \wedge e_j$.

EXRC. 9.9. Show that the affine surface $x_1 x_2 \ldots x_n = 1$ contains no affine lines and its projective closure intersects the hyperplane of infinity in n lines $x_i = 0$.

EXRC. 9.10. Hint: use the fact that over $\mathbb{F}_4$, the Fermat cubic form $\sum x_i^3$, whose zero set is a smooth cubic surface, coincides with the standard Hermitian inner product $\sum x_i \overline{x}_i$. The final answer is $|G| = 51\,840 = 2^7 \cdot 3^4 \cdot 5$.

[1] Recall that they equal the top degree minors of the transition matrix from some basis in H to the the standard basis in V, see Example 8.4 on p. 101.