§8 Algebraic manifolds

Everywhere in §8 we assume on default that the ground field $\Bbbk$ is algebraically closed.

8.1 Definitions and examples. The definition of an algebraic manifold follows the same template as the definitions of manifold in topology and differential geometry. It can be outlined as follows. A *manifold* is a topological space X such that every point $x \in X$ possesses an open neighborhood $U \ni x$, called a *local chart*, which is equipped with the homeomorphism $\varphi_U : X_U \xrightarrow{\sim} U$ identifying some standard local model X_U with U, and any two local charts $\varphi_U : X_U \xrightarrow{\sim} U$, $\varphi_W : X_W \xrightarrow{\sim} W$ are *compatible*, meaning that the homeomorphism between open subsets $\varphi_U^{-1}(U \cap W) \subset X_U$ and $\varphi_W^{-1}(U \cap W) \subset X_W$ provided by the composition $\varphi_W^{-1} \circ \varphi_U$ is a regular isomorphism. In topology and differential geometry, the local model $X_U = \mathbb{R}^n$ does not depend on U, and the regularity of the *transition homeomorphism*

$$\varphi_{WU} \stackrel{\text{def}}{=} \varphi_W^{-1} \circ \varphi_U\big|_{\varphi_U^{-1}(U\cap W)} : \varphi_U^{-1}(U \cap W) \xrightarrow{\sim} \varphi_W^{-1}(U \cap W), \tag{8-1}$$

means that it will be a diffeomorphism of open subsets in $\mathbb{R}^n$ in the differential geometry, and means nothing besides to be a homeomorphism in the topology. In algebraic geometry, the local model X_U is an arbitrary algebraic variety that may depend on $U \subset X$ and an a affine algebraic variety. Thus, an algebraic manifold may look locally, say, as a union of a line and a plane in $\mathbb{A}^3$, crossing or parallel, and this picture may vary from chart to chart. The regularity of homeomorphism (8-1), in algebraic geometry, means that the maps φ_{WU}, $\varphi_{UW} = \varphi_{WU}^{-1}$ are described in affine coordinates by some rational functions, which are regular within both open sets $f_U^{-1}(U \cap W)$, $\varphi_W^{-1}(U \cap W)$. This provides every algebraic manifold X with a well defined sheaf $\mathcal{O}_X$ of *regular rational functions* with values in the ground field $\Bbbk$, in the same manner as the *smooth* functions on a manifold are introduced in differential geometry.

Let us now give precise definitions. Given a topological space X, an *affine chart* on X is a homeomorphism $\varphi_U : X_U \xrightarrow{\sim} U$ between an affine algebraic variety X_U over $\Bbbk$, considered with the Zariski topology, and an open subset $U \subset X$, considered with the topology induced from X. Two affine charts $\varphi_U : X_U \xrightarrow{\sim} U$, $\varphi_W : X_W \xrightarrow{\sim} W$ on X are called *compatible* if the pullback map $\varphi_{WU}^* : f \mapsto f \circ \varphi_{WU}$, provided by the transition homeomorphism (8-1), establishes a well defined isomorphism of $\Bbbk$-algebras[1]

$$\varphi_{WU}^* : \mathcal{O}_{X_W}\left(\varphi_W^{-1}(U \cap W)\right) \xrightarrow{\sim} \mathcal{O}_{X_U}\left(\varphi_U^{-1}(U \cap W)\right) .$$

An open covering $X = \bigcup U_\nu$ by mutually compatible affine charts $U_\nu \subset X$ is called an *algebraic atlas* on X. Two algebraic atlases are declared to be *equivalent* if their union is an algebraic atlas as well. A topological space X equipped with an equivalence class of algebraic atlases is called *an algebraic manifold* or *algebraic variety*[2]. An algebraic manifold is said to be of *finite type* if it allows a finite algebraic atlas.

EXERCISE 8.1. Verify that any algebraic manifold of finite type is a Noetherian topological space in the sense of Remark 7.1. on p. 90 and therefore admits a unique decomposition into a finite union of the irreducible components.

[1]Recall that for an open set W in an affine algebraic variety Z, we write $\mathcal{O}_Z(W) = \{f \in \Bbbk(Z) \mid W \subset \operatorname{Dom}(f)\}$ for the $\Bbbk$-algebra of rational functions on Z regular everywhere in W, see n° 7.3.1 on p. 91 for details.

[2]without the epithet «affine»

Example 8.1 (projective spaces)
The projective space $\mathbb{P}_n = \mathbb{P}\left(\Bbbk^{n+1}\right)$ with homogeneous coordinates $x = (x_0 : x_1 : \dots : x_n)$ is covered by the $(n+1)$ standard affine charts $U_i = \{(x_0 : x_1 : \dots : x_n) \mid x_i \neq 0\}$, $0 \leqslant i \leqslant n$. Write $X_i = \mathbb{A}(\Bbbk^n)$ for the affine space with coordinates[1] $t_i = \left(t_{i,0}, \dots, t_{i,i-1}, t_{i,i+1}, \dots, t_{i,n}\right)$. For each i, there exists a bijection

$$\varphi_i : X_i \xrightarrow{\sim} U_i, \quad t_i \mapsto \left(t_{i,0} : \dots : t_{i,i-1} : 1 : t_{i,i+1} : \dots : t_{i,n}\right). \tag{8-2}$$

Preimage of the intersection $U_i \cap U_j$ under this bijection is the principal open set $\mathcal{D}\left(t_{i,j}\right) \subset X_i$.

Exercise 8.2. Verify that the transition map $\varphi_{ji} = \varphi_j^{-1}\varphi_i : \mathcal{D}\left(t_{i,j}\right) \xrightarrow{\sim} \mathcal{D}\left(t_{j,i}\right)$, $t_i \mapsto t_{i,j}^{-1} \cdot t_j$, establishes the regular isomorphism between affine algebraic varieties

$$\mathcal{D}\left(t_{i,j}\right) = \operatorname{Spec}_{\mathrm{m}} \Bbbk\left[t_{i,j}^{-1}, t_{i,0}, \dots, t_{i,i-1}, t_{i,i+1}, \dots, t_{i,n}\right], \tag{8-3}$$

$$\mathcal{D}\left(t_{j,i}\right) = \operatorname{Spec}_{\mathrm{m}} \Bbbk\left[t_{j,i}^{-1}, t_{j,0}, \dots, t_{j,j-1}, t_{j,j+1}, \dots, t_{j,n}\right]. \tag{8-4}$$

Therefore, transferring the Zariski topology from $X_i \simeq \mathbb{A}^n$ to U_i by means of the bijection (8-2) provides $\mathbb{P}_n$ with a well defined topology whose restriction on $U_i \cap U_j$ does not depend on what source, X_i or X_j, it comes from. In this topology, all bijections (8-2) certainly are homeomorphisms. Thus, $\mathbb{P}_n$ is an algebraic manifold of finite type locally isomorphic to the affine space $\mathbb{A}^n$.

Example 8.2 (Grassmannians)
Recall[2] that the set of all k-dimensional vector subspaces in a given vector space V over $\Bbbk$ is called the *Grassmannian* $\operatorname{Gr}(k, V)$, and for the coordinate space $V = \Bbbk^m$ we write $\operatorname{Gr}(k, m)$ instead of $\operatorname{Gr}(k, \Bbbk^m)$. We have seen in n° 5.2 on p. 64 that the points of $\operatorname{Gr}(k, m)$ can be viewed as the orbits of $k \times m$ matrices of rank k under the natural action of $\mathrm{GL}_k(\Bbbk)$ by left multiplication. The orbit of the matrix x corresponds to the subspace $U_x \subset \Bbbk^m$ spanned by the rows of x, and x is recovered from U_x up to the action $\mathrm{GL}_k(\Bbbk)$ as the matrix whose rows are the coordinates of some linearly independent vectors $u_1, u_2, \dots, u_k \in U_x$ in the standard basis of $\Bbbk^m$. This leads to the following covering of $\operatorname{Gr}(k, m)$ by $\binom{m}{k}$ affine charts $U_I \simeq \mathbb{A}^{k(m-k)}$, called *standard* and numbered by increasing collections of indexes $I = (i_1, i_2, \dots, i_k)$, $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m$. Write $s_I(x)$ for the $k \times k$ submatrix of $k \times m$ matrix x formed by the columns with numbers $i_1, i_2, \dots, i_k$, and U_I for the set of $\mathrm{GL}_k(\Bbbk)$-orbits of all matrices x with $\det s_I(x) \neq 0$. Every such an orbit contains a unique matrix z with $s_I(z) = E$, namely, $z = s_I(x)^{-1} \cdot x$.

Exercise 8.3. Convince yourself that U_I consists of those k-dimensional subspaces $W \subset \Bbbk^m$ which are isomorphically projected onto the coordinate k-plane spanned by the standard basis vectors $e_{i_1}, e_{i_2}, \dots, e_{i_k}$ along the transversal coordinate $(m-k)$-plane spanned by the remaining standard basis vectors.

Write $X_I = \mathrm{Mat}_{k\times(m-k)}(\Bbbk) \simeq \mathbb{A}^{k(m-k)}$ for the affine space of $k\times(m-k)$ matrices whose columns are numbered in order by the collection of indexes $\overline{I} = \{1, 2, \dots, m\} \smallsetminus I$, complementary to I. There is a bijection $\varphi_I : X_I \xrightarrow{\sim} U_I$, $t \mapsto \mathrm{GL}_k(\Bbbk) \cdot \varphi_I(t)$, where the $k \times m$ matrix $\varphi_I(t)$ has $s_I\left(\varphi_I(t)\right) = E$, and $s_{\overline{I}}(\varphi_I(t)) = t$, i.e., it is obtained from t by the order-preserving insertion of the columns

[1] The first index i is the order number of the chat, the second index numbers the coordinates within the ith chart and takes n values $0 \leqslant \nu \leqslant n$, $\nu \neq i$.

[2] See n° 4.6.4 on p. 58.

of E between the columns of t in such the way that the columns of E are assigned the numbers $i_1, i_2, \dots, i_k$ in the resulting $k \times m$ matrix.

EXERCISE 8.4. Verify that the inverse bijection maps $x \mapsto s_{\overline{I}}\left(s_I(x)^{-1} \cdot x\right)$, and the result does not depend on the choice of x in the orbit $\mathrm{GL}_k(\Bbbk) \cdot x$.

Therefore, $\varphi_I^{-1}\left(U_I \cap U_J\right) = \mathcal{D}\Big(\det s_J\big(\varphi_I(t)\big)\Big)$ is the principal open set in X_I. The transition map $\varphi_{JI} = \varphi_J^{-1}\varphi_I$ sends $\mathcal{D}\Big(\det s_J\big(\varphi_I(t)\big)\Big) \subset X_I$ to $\mathcal{D}\Big(\det s_I\big(\varphi_J(t)\big)\Big) \subset X_J$ by the rule $t \mapsto s_{\overline{J}}\Big(s_J^{-1}\big(\varphi_I(t)\big) \cdot \varphi_I(t)\Big)$ and gives a regular isomorphism of affine algebraic varieties. The inverse isomorphism takes $t \mapsto s_{\overline{I}}\left(s_I^{-1}\left(\varphi_J(t)\right) \cdot \varphi_J(t)\right)$.

EXERCISE 8.5. Check this.

The same arguments as in the previous example show that $\mathrm{Gr}(k, n)$ is an algebraic variety of finite type locally isomorphic to the affine space $\mathbb{A}^{k(m-k)} = \mathbb{A}\left(\mathrm{Mat}_{k\times(m-k)}(\Bbbk)\right)$. Note that for $k = 1$, $m = n + 1$, the standard algebraic atlas $\{U_I\}$ on $\mathrm{Gr}(k, m)$ is precisely the standard atlas $\{U_i\}$ on $\mathbb{P}_n$ described in Example 8.1.

EXAMPLE 8.3 (DIRECT PRODUCT OF ALGEBRAIC MANIFOLDS)
The set-theoretical direct product of algebraic manifolds X, Y is canonically equipped with the algebraic atlas formed by the mutual direct products $U \times W$ of affine charts $U \subset X$, $W \subset X$. Thus, $X \times Y$ is an algebraic manifold.

8.2 Regular and rational maps. Given an algebraic manifold X, a function $f : X \to \Bbbk$ is called *regular* at a point $x \in X$ if there exist an affine chart $\varphi_W : X_W \xrightarrow{\sim} W$ with $x \in W$ and a rational function $\widetilde{f} \in \Bbbk(X_W)$ such that $\varphi_W^{-1}(x) \in \mathrm{Dom}(\widetilde{f})$ and $\varphi_W^* f(z) = \widetilde{f}(z)$ for all $z \in \mathrm{Dom}\,\widetilde{f}$. For an open subset $U \subset X$, the regular everywhere in U functions $U \to \Bbbk$ form a $\Bbbk$-algebra denoted by $\mathcal{O}_X(U)$ and called the *algebra of regular functions* on U. The assignment $U \mapsto \mathcal{O}_X(U)$ provides the topological space X with the sheaf of $\Bbbk$-algebras, called the *structure sheaf*[1] or the *sheaf of regular functions* on X.

EXERCISE 8.6. For any affine chart $\varphi_U : X_U \xrightarrow{\sim} U$ on X, verify that the pullback of the regular functions along φ_U assigns the isomorphism $\varphi_U^* : \mathcal{O}_X(U) \xrightarrow{\sim} \Bbbk[X_U]$.

A map of algebraic manifolds $f : X \to Y$ is called a *regular morphism* if f is continuous and for any open $U \subset Y$, the pullback of regular functions along $f|_{f^{-1}(U)}$ gives a well defined homomorphism of $\Bbbk$-algebras $f|_U^* : \mathcal{O}_Y(U) \to \mathcal{O}_X\left(\varphi^{-1}(U)\right)$, $h \mapsto h \circ f$.

EXERCISE 8.7. Identify $\mathcal{O}_X(X)$ with the set of regular morphisms $X \to \mathbb{A}^1$.

8.2.1 Closed submanifolds. Let X be an algebraic manifold. Any closed subset $Z \subset X$ possesses the natural structure of algebraic manifold. Namely, for any affine chart $\varphi_U : X_U \xrightarrow{\sim} U$, the set $\varphi_U^{-1}(Z \cap U)$ is closed in the affine algebraic variety X_U and therefore, has the natural structure of affine algebraic variety with the coordinate algebra

$$\Bbbk[X_U] / \varphi_U^* I(Z \cap U) \simeq \mathcal{O}_X(U) / I(Z \cap U),$$

where $I(Z \cap U) = \{f \in \mathcal{O}_X(U) \mid f(z) = 0 \text{ for all } z \in Z \cap U\}$. The affine charts

$$\varphi_U^{-1}(Z \cap U) \xrightarrow{\sim} Z \cap U \subset Z$$

[1] See n° 7.3.1 on p. 91.

certainly form an algebraic atlas on Z. The assignment $U \mapsto I(Z \cap U)$ defines a sheaf of ideals on X denoted by $\mathcal{I}_Z \subset \mathcal{O}_X$ and called the *ideal sheaf* of the *closed submanifold* $Z \subset X$.

Every sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ determines a closed submanifold $V(\mathcal{J}) \subset X$ whose intersection with any affine chart $U \subset X$ is the zero set of the ideal $\mathcal{J}(U) \subset \mathcal{O}_X(U) \simeq \Bbbk[X_U]$ in the affine algebraic variety X_U. Note that the ideal sheaf $\mathcal{I}(V(\mathcal{J})) = \sqrt{\mathcal{J}}$ has not to coincide with the sheaf $\mathcal{J}$ of equations describing the submanifold $V(\mathcal{J})$.

A regular morphism $f : X \to Y$ is called a *closed immersion* if $f(X) \subset Y$ is a closed submanifold of Y and f establishes an isomorphism between X and $f(X)$.

EXERCISE 8.8. Convince yourself that an algebraic manifold X admits a closed immersion in affine space if and only if X is an affine algebraic variety in the sense of n° 6.7 on p. 77.

8.2.2 Families of manifolds. Any regular morphism $\pi : X \to Y$ can be viewed as a family of closed submanifolds $X_y = \pi^{-1}(y) \subset X$ parametrized by the points $y \in Y$. In this case Y is referred to as the *base* of family π. Given two families $\pi : X \to Y$, $\pi' : X' \to Y$ over the same base Y, a regular morphism $\varphi : X \to X'$ is called a *morphism of families* or *morphism over* Y if $\pi = \pi' \circ \varphi$, i.e., if φ maps X_y to X'_y for all $y \in Y$. A family $\pi : X \to Y$ is called *constant* or *trivial* if it is isomorphic over Y to the canonical projection $\pi_Y : X_0 \times Y \to Y$ from the direct product of the base and some fixed manifold X_0.

8.2.3 Rational maps. Let X be an algebraic manifold and $U \subset X$ an open dense subset. A regular morphism $\varphi : U \to Y$ is called a *rational map* from X to Y. Given such a map, we write $\varphi : X \dashrightarrow Y$ although this discards the information about U. A regular morphism $\psi : W \to Y$ is called an *extension* of φ if $W \supset U$ and $\psi|_U = \varphi$. The union of all open sets $W \supset U$ on which φ can be extended, is called the *domain* of rational map $\varphi : X \dashrightarrow Y$ and denoted $\mathrm{Dom}(\varphi)$.

EXERCISE 8.9 (CREMONA'S QUADRATIC INVOLUTION). Verify that the prescription

$$(x_0 : x_1 : x_2) \mapsto (x_0^{-1} : x_1^{-1} : x_2^{-1})$$

determines a rational map $\varkappa : \mathbb{P}_2 \dashrightarrow \mathbb{P}_2$ whose domain is the whole of $\mathbb{P}_2$ except three points. Find these points and describe the image of $\varkappa$.

Despite its name, a rational map $\varphi : X \dashrightarrow Y$ is not a map «from X» in the set-theoretical sense, because φ may be undefined at some points. In particular, the composition of rational maps may be undefined, e.g., if the image of the first map falls outside the domain of the second. However, the rational maps often appear in various applications and play an important role within the algebraic geometry itself. For example, the tautological projection $\mathbb{A}(V) \dashrightarrow \mathbb{P}(V)$, which sends a point of $\mathbb{A}(V)$ provided by a vector $v \in V$ to the point of $\mathbb{P}(V)$ provided by the same vector, is a surjective rational map regular everywhere outside the origin.

8.3 Separated manifolds. The standard atlas on $\mathbb{P}_1$ consists of two charts

$$\varphi_i : \mathbb{A}^1 \xrightarrow{\sim} U_i \subset \mathbb{P}_1 , \quad i = 0, 1 .$$

Their intersection is visible within each chart as the complement to origin

$$\varphi_0^{-1}(U_0 \cap U_1) = \varphi_1^{-1}(U_0 \cap U_1) = \mathbb{A}^1 \smallsetminus \{O\} = \{t \in \mathbb{A}^1 \mid t \neq 0\} .$$

The charts are glued together along this intersection by means of the transition map

$$\varphi_{01} : t \mapsto 1/t . \tag{8-5}$$

If, instead of rational map (8-5), we use much simpler gluing rule

$$\tilde{\varphi}_{01} : t \mapsto t , \tag{8-6}$$

we get another manifold looking as an affine line with the double origin: ———:——— . Such kind of pathology is called *non-separateness*. It has appeared because the gluing rule (8-6) considered as the binary relation on $\mathbb{A}^1$, i.e., as the subset of $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, is not closed. Namely, it is provided by the line $x = y$ without the point $x = y = 0$. This gluing rule can be completed by continuity up to the whole line $x = y$, whereupon the double point disappears.

In general situation, the separateness phenomenon is formalized as follows. By the universal property of the direct product, for any two affine charts U_0, U_1 on an algebraic manifold X, the inclusions $U_0 \hookleftarrow U_0 \cap U_1 \hookrightarrow U_1$ produce the inclusion $U_0 \cap U_1 \hookrightarrow U_0 \times U_1$ whose image is the intersection of the affine chart $U_0 \times U_1$ on $X \times X$ with the diagonal $\Delta_X = \{(x, x) \in X \times X \mid x \in X\}$. In other words, the gluing rule for charts U_0, U_1, considered as a subset of $U_0 \times U_1$, is $\Delta \cap U_0 \times U_1$. For example, the gluing rule (8-5) corresponds to the immersion $(\mathbb{A}^1 \smallsetminus O) \hookrightarrow \mathbb{A}^2$, $t \mapsto (t\,,\, t^{-1})$, whose image $\Delta_{\mathbb{P}_1} \cap U_0 \times U_1$ is a closed subset of $U_0 \times U_1 \simeq \mathbb{A}^2$, namely, the hyperbola $xy = 1$. In contrast, the trivial transition map (8-6) produces the immersion $(\mathbb{A}^1 \smallsetminus O) \hookrightarrow \mathbb{A}^2$, $t \mapsto (t\,,\, t)$, whose image is not closed in $\mathbb{A}^2$. An algebraic manifold X is called *separated* if the diagonal $\Delta_X \subset X \times X$ is closed in $X \times X$. In more expanded form, this means that for every pair of affine charts $U, W \subset X$, the canonical map $U \cap W \hookrightarrow U \times W$ is a closed immersion.

For example, both $\mathbb{A}^n$ and $\mathbb{P}_n$ are separated, because the diagonals in $\mathbb{A}^n \times \mathbb{A}^n$ and $\mathbb{P}_n \times \mathbb{P}_n$ are described by the polynomial equations $x_i = y_i$ and $x_i y_j = x_j y_i$ respectively[1]. Every closed submanifold $X \subset Y$ in a separated manifold Y is separated as well, because the diagonal of $X \times X$ is the preimage of the diagonal $\Delta_Y \subset Y \times Y$ under the regular map $X \times X \hookrightarrow Y \times Y$ provided by the inclusion $X \hookrightarrow Y$. In particular, all affine and projective varieties are separated and have finite type.

8.3.1 Closeness of the graph of a regular map Let $\varphi : X \to Y$ be a regular morphism of algebraic manifolds. The preimage of the diagonal $\Delta_Y \subset Y \times Y$ under the map $\varphi \times \mathrm{Id}_Y : X \times Y \to Y \times Y$ is called the *graph* of φ and denoted Γ_φ. As a set, $\Gamma_\varphi = \{(x, f(x)) \in X \times Y \mid x \in X\}$. If Y is separated, the graph of any regular morphism $\varphi : X \to Y$ is closed. For example, the graph of a regular morphism of affine algebraic varieties $\varphi : \operatorname{Spec}_{\mathrm{m}}(A) \to \operatorname{Spec}_{\mathrm{m}}(B)$ is described by a system of equations $1 \otimes f = \varphi^*(f) \otimes 1$ in $A \otimes B$, where f runs through B.

8.4 Projective varieties. An algebraic manifold X is called *projective* if it admits a closed immersion into projective space, i.e., is isomorphic to a closed submanifold of $\mathbb{P}_n$ for some $n \in \mathbb{N}$.

EXERCISE 8.10. Verify that the solution set of every system of homogeneous polynomial equations in the homogeneous coordinates in $\mathbb{P}_n$ is a closed submanifold of $\mathbb{P}_n$.

EXAMPLE 8.4 (PLÜCKER COORDINATES)
The *Plücker embedding* from n° 4.6.4 on p. 58

$$p_{k,V} : \mathrm{Gr}(k, V) \hookrightarrow \mathbb{P}(\Lambda^k V), \quad U \mapsto \Lambda^k U , \tag{8-7}$$

[1]The first formula relates $2n$ affine coordinates $(x_1, \dots, , x_n, y_1, \dots , y_n)$ in $\mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}$, whereas the second deals with two collections of homogeneous coordinates $(x_0 : x_1 : \dots : x_n)$, $(y_0 : y_1 : \dots : y_n)$ on $\mathbb{P}_n \times \mathbb{P}_n$ (note that they cannot be combined together in one collection). We will see in Exercise 8.12 on p. 102 that the latter equations actually determine a closed submanifold of $\mathbb{P}_n \times \mathbb{P}_n$ in the sense of n° 8.2.1.

maps the Grassmannian $\mathrm{Gr}(k,V)$ isomorphically onto projective algebraic variety determined in $\mathbb{P}(\Lambda^k V)$ by the quadratic Plücker's relations from formula (4-44) on p. 57. In the matrix notations from Example 8.2 on p. 98, the Plücker embedding maps $k\times m$ matrix x_U, formed by the coordinate rows of some basis vectors in $U \subset \Bbbk^n$ expanded through the standard basis vectors $e_i \in \Bbbk^n$, to the point of $\mathbb{P}(\Lambda^k \Bbbk^m)$ whose Ith homogeneous coordinate in the basis formed by the monomials

$$e_I = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}$$

equals $\det s_I(x_U)$, the degree-k minor of x_U situated in the columns with numbers from I.

EXERCISE 8.11. Check this and convince yourself that the Plücker embedding is regular.

The collection of $\binom{k}{n}$ minors $\det s_I(x_U)$ is called the *Plücker coordinates* of the subspace $U \subset \Bbbk^n$. Since the pullbacks of the coordinate functions on $\mathbb{P}(\Lambda^k \Bbbk^n)$ are the polynomials in the affine coordinates on the Grassmannian, the map (8-7) is a regular closed immersion of the Grassmannian into projective space. Therefore, the Grassmannians, as well as all their closed submanifolds, are projective algebraic varieties.

EXERCISE 8.12. Show that the direct product of projective manifolds is projective, and use this to prove that every subset in $\mathbb{P}_{n_1} \times \mathbb{P}_{n_2} \times \cdots \times \mathbb{P}_{n_m}$ defined by a system of polynomial equations in homogeneous coordinates such that every equation is homogeneous in every set of coordinates is a projective algebraic variety.

EXAMPLE 8.5 (BLOWUP OF POINT ON $\mathbb{P}_n$)

Write $E \simeq \mathbb{P}_{n-1}$ for the projective space formed by all lines in $\mathbb{P}_n$ passing through a given point $p \in \mathbb{P}_n$. The incidence graph $\mathcal{B}_p = \{(q,\ell) \in \mathbb{P}_n \times E \mid q \in \ell\}$ is called the *blowup* of the point $p \in \mathbb{P}_n$. The projection $\sigma_p : \mathcal{B}_p \twoheadrightarrow \mathbb{P}_n$ is one-to-one over $\mathbb{P}_n \smallsetminus p$, whereas the preimage of p

$$\sigma_p^{-1}(p) = \{p\} \times E \subset \mathbb{P}_n \times E$$

coincides with the whole space E. This fiber is called the *exceptional divisor*[1] of the blowup. The second projection $\varrho_E : \mathcal{B}_p \twoheadrightarrow E$ represents $\mathcal{B}_p$ as a *line bundle* over E, i.e., the family of projective lines $(pq) \subset \mathbb{P}_n$ parametrized by the points $q \in E$. This line bundle is called the *tautological line bundle* over the projective space E. It follows from Exercise 8.12 that $\mathcal{B}_p$ is a projective algebraic manifold. Indeed, choose some homogeneous coordinates in $\mathbb{P}_n$ such that $p = (1 : 0 : \ldots : 0)$, and identify E with the projective hyperplane $V\left(x_0\right) = \{(0 : t_1 : \ldots : t_n)\} \subset \mathbb{P}_n$ by mapping a line $\ell \ni p$ to the point $t = \ell \cap V(x_0)$. Then the collinearity of points p, q, t is equivalent to the following system of homogeneous quadratic equations on the pair $(q,\lambda) \in \mathbb{P}_n \times E$:

$$\mathrm{rk}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ q_0 & q_1 & \cdots & q_n \\ 0 & t_1 & \cdots & t_n \end{pmatrix} = 2 \quad \text{or} \quad q_i t_j = q_j t_i\,,\ 1 \leqslant i < j \leqslant n\,.$$

Geometrically, the blowup of $p \in \mathbb{P}_n$ can be imagined as the replacement of the point p by the projective space E glued to the space $\mathbb{P}_n$, punctured at p, in such a way that every line $\ell \subset \mathbb{P}_n$ approaching p passes through the point $\ell \in E$.

[1] Given an irreducible algebraic manifold X, a (*Weil*) *divisor* on X is an element of the free abelian group generated by all closed irreducible submanifolds of codimension 1 in X (the dimensions of algebraic varieties will be discussed in §9)

LEMMA 8.1
Every closed submanifold $X \subset \mathbb{P}_n$ can be described as a set of solutions to some system of homogeneous polynomial equations in homogeneous coordinates in $\mathbb{P}_n$.

PROOF. We write $(x_0 : x_1 : \dots : x_n)$ for the homogeneous coordinates in $\mathbb{P}_n$ and use the notations from Example 8.1 on p. 98 for the standard affine charts $U_i \subset \mathbb{P}_n$ and the standard affine coordinates $t_{i,j}$ therein. For each i, the intersection $X \cap U_i$ is the zero set $V(I_i)$ of some ideal I_i in the polynomial ring in n variables $t_{i,\nu} = x_\nu / x_i$, $0 \leqslant \nu \leqslant n$, $\nu \neq i$. Every polynomial f in this ring can be rewritten as $\overline{f}(x_0, x_1, \dots, x_n) / x_i^d$, where $d = \deg f$ and $\overline{f} \in \Bbbk[x_0, x_1, \dots, x_n]$ is homogeneous of degree d and turns to f for $x_i = 1$, $x_j = t_{i,j}$, $j \neq i$:

$$\overline{f}\left(t_{i,0}, \dots, t_{i,i-1}, 1, t_{i,i+1}, \dots, t_{i,n}\right) = f\left(t_{i,0}, \dots, t_{i,i-1}, t_{i,i+1}, \dots, t_{i,n}\right) .$$

Let us fix generators $f_{i,\alpha}$ of the ideal I_i and write $\overline{f}_{i,\alpha} \in \Bbbk[x_0, x_1, \dots, x_n]$ for their homogenizations just described. Then X coincides with the solution set Z of the system of polynomial equations $x_i \cdot \overline{f}_{i,\alpha}(x_0, x_1, \dots, x_n) = 0$, where $0 \leqslant i \leqslant n$ and for each i, the index α numbers the chosen generators $f_{i,\alpha}$ of the ideal I_i. To check this, it is enough to establish the coincidence $Z \cap U_i = X \cap U_i$ for every i. In terms of the affine coordinates $t_{i,j}$ on U_i, the intersection $U_i \cap V\left(x_i \cdot \overline{f}\right)$ is described by the equation

$$\overline{f}\left(t_{i,0}, \dots, t_{i,i-1}, 1, t_{i,i+1}, \dots, t_{i,n}\right) = f\left(t_{i,0}, \dots, t_{i,i-1}, t_{i,i+1}, \dots, t_{i,n}\right) = 0 .$$

Hence, U_i intersects the set of common zeros of the polynomials $x_i \cdot \overline{f}_{i,\alpha}$, whose i coincides with i of the chart, exactly along the set $X \cap U_i$. Therefore, $Z \cap U_i \subset X \cap U_i$. It remains to check that every homogeneous polynomial $x_j \cdot \overline{f}_{j,\beta}$ with $j \neq i$ vanishes on $X \cap U_i$ as well. The first factor x_j vanishes along the hyperplane $V(t_{i,j}) \subset U_i$. The principal open set in $X \cap U_i$ complementary to this hyperplane lies within $X \cap U_i \cap U_j \subset X \cap U_j$. As we have already seen, the second factor $\overline{f}_{j,\beta}$ vanishes on $X \cap U_j$. $\square$

EXAMPLE 8.6 (AN ILLUSTRATION TO THE PROOF OF LEMMA 8.1)
The zero set of the homogeneous polynomial $x_0 x_1 x_2$ on $\mathbb{P}_2$ is the union of three lines complementary to the standard affine charts. The affine equations of this set in the charts U_0, U_1, U_2 are, respectively, $t_{0,1} t_{0,2} = 0$, $t_{1,0} t_{1,2} = 0$, $t_{2,0} t_{2,1} = 0$. Let $X \subset \mathbb{P}_2$ be the closed submanifold locally described by these equations. Being applied to this X, the previous proof transforms the left hand sides of the local affine equations to the homogeneous polynomials $\overline{f}_{0,1} = x_1 x_2$, $\overline{f}_{1,1} = x_0 x_2$, $\overline{f}_{2,1} = x_0 x_1$, and then serves $x_0 \cdot \overline{f}_{0,1} = 0$, $x_1 \cdot \overline{f}_{1,1} = 0$, $x_2 \cdot \overline{f}_{2,1} = 0$ as the global homogeneous equations for X. They all coincide with the initial equation $x_0 x_1 x_2 = 0$ in our case.

8.5 Closeness of projective morphisms. Projective varieties behave similarly to the compact manifolds in the differential geometry in the sense that every regular map from a projective manifold X to an arbitrary separated algebraic manifold Y is *closed* meaning that the image of every closed subset $Z \subset X$ is closed in Y. The proof is based on the following lemma.

LEMMA 8.2
The projection $\pi : \mathbb{P}_m \times \mathbb{A}^n \twoheadrightarrow \mathbb{A}^n$ is closed, i.e., $\pi(Z) \subset \mathbb{A}^n$ is closed for every closed $Z \subset \mathbb{P}_m \times \mathbb{A}^n$.

PROOF. Write $x = (x_0 : x_1 : \dots : x_m)$ and $t = (t_1, t_2, \dots, t_n)$ for the homogeneous and affine coordinates on $\mathbb{P}_m$ and $\mathbb{A}^n$ respectively. Let a closed subset $Z \subset \mathbb{P}_m \times \mathbb{A}^n$ be described by a system of polynomial equations $f_\nu(x,t) = 0$, homogeneous in x. Then $\pi(Z) \subset \mathbb{A}^n$ consists of all $p \in \mathbb{A}^n$ such that the system of homogeneous equations $f_\nu(x,p) = 0$ in x has a non zero solution. The latter holds if and only if the coefficients of the homogeneous forms $f_\nu(x,p)$ satisfy the system of resultant polynomial equations defined in n° 6.8 on p. 79. Since the coefficients of the forms $f_\nu(x,p)$ are polynomials in p, we conclude that $\pi(Z)$ is described by polynomial equations. □

COROLLARY 8.1
Let X be a projective algebraic variety. Then for all algebraic manifolds Y, the projection $X \times Y \twoheadrightarrow Y$ is closed.

PROOF. It is enough to prove this statement separately for every affine chart of Y instead of the whole Y. Thus, we may assume that Y is affine. In this case, $X \times Y$ is the closed subset in $\mathbb{P}_m \times \mathbb{A}^n$, and the projection in question is the restriction of the projection $\mathbb{P}_m \times \mathbb{A}^n \twoheadrightarrow \mathbb{A}^n$, which is closed, on this closed subset. Therefore, it closed as well. □

THEOREM 8.1
Every regular morphism $\varphi : X \to Y$ from a projective variety X to a separated manifold Y is closed.

PROOF. Write $\Gamma_\varphi \subset X \times Y$ for the graph[1] of the regular map $\varphi : X \to Y$. It is closed, because Y is separated[2]. For every $Z \subset X$, the image $\varphi(Z) \subset Y$ coincides with the image of the intersection $\Gamma_\varphi \cap (Z \times Y) \subset X \times Y$ under the projection $X \times Y \twoheadrightarrow Y$. If Z is closed in X, the product $Z \times Y$ is closed in $X \times Y$. Since X is projective, the projection $X \times Y \twoheadrightarrow Y$ maps the closed set $\Gamma_\varphi \cap (Z \times Y) \subset X \times Y$ to the closed set $\varphi(Z) \subset Y$. □

COROLLARY 8.2
Every regular map from a connected[3] projective variety X to an affine algebraic variety Y contracts X to one point of Y. In particular, $\mathcal{O}_X(X) = \Bbbk$ is exhausted by constants.

PROOF. Let $Y \subset \mathbb{A}^n$ and $\varphi : X \to Y$ be such a regular map. Composing it with the projections of Y to the n coordinate axes of $\mathbb{A}^n$ reduces the statement to the case $Y = \mathbb{A}^1$. Composing a regular map $X \to \mathbb{A}^1$ with the inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{P}_1$ as the standard affine chart U_0 gives a nonsurjective regular map $X \to \mathbb{P}_1$, whose image must be a proper connected Zariski closed subset, that is, one point. □

8.6 Finite projections. A regular morphism of algebraic manifolds $\varphi : X \to Y$ is called *finite* if for every affine chart $U \subset Y$, the preimage $W = \varphi^{-1}(U)$ is an affine chart on X, and the restricted map $\varphi_W : W \to U$ is a finite morphism of affine algebraic varieties in the sense of n° 7.4.3 on p. 94. It follows from Proposition 7.12 on p. 94 that every finite morphism $\varphi : X \to Y$ is closed, and the restriction of φ to a closed submanifold $Z \subset X$ remains a finite morphism. Moreover, if X is irreducible and $Z \subsetneq X$ is a proper closed subset, then $\varphi(Z) \subsetneq Y$ is a *proper* closed subset of Y as well.

EXERCISE 8.13. Prove that the composition of finite morphisms is finite.

[1] See n° 8.3.1 on p. 101.
[2] See the same n° 8.3.1 on p. 101.
[3] That is, indecomposable into disjoint union of two nonempty closed subsets.

PROPOSITION 8.1
For a proper closed subset $X \subsetneq \mathbb{P}_n$, a point $p \notin X$, and a hyperplane $H \not\ni p$, a finite regular morphism $\pi_p : X \to H$ is provided by the projection from p to H.

PROOF. Let $U \subset H$ be an affine chart. Fix some homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$ on $\mathbb{P}_n$ such that the hyperplane $H = V(x_0)$ is formed by the points $q = (0 : q_1 : \ldots : q_n) \in \mathbb{P}_n$, the chart $U \subset H$ is formed by the points $u = (0 : u_1 : \ldots : u_{n-1} : 1) \in \mathbb{P}_n$, and the point $p = (1 : 0 : \ldots : 0)$. Let X be described by homogeneous equations $f_\nu(x) = 0$ in these coordinates. Since $p \notin X$, the preimage $\pi_p^{-1}(U)$ is cut out of X by the punctured cone C ruled by the projective lines (pu), $u \in U$, with the punctured point p. Every such line is described by the parametric equation $u + pt$, $t \in \Bbbk$, and the cone C is an affine algebraic variety isomorphic to $\mathbb{A}^n = U \times \mathbb{A}^1$. The isomorphism maps a point $(u,t) \in U \times \mathbb{A}^1$ to the point $x = u + tp \in \mathbb{P}_n$ laying on the cone C. The intersection $C \cap X = \pi_p^{-1}(U)$ is described in the coordinates (u,t) on C by the equations

$$f_\nu(tp+u) = \alpha_0^{(\nu)}(u)\,t^m + \alpha_1^{(\nu)}(u)\,t^{m-1} + \cdots + \alpha_m^{(\nu)}(u) = 0 \tag{8-8}$$

and therefore, it is an affine algebraic variety, i.e., an affine chart on X. It remains to show that the coordinate algebra $\Bbbk[C \cap X]$ is integral over $\Bbbk[U] = \Bbbk[u_1, u_2, \ldots, u_{n-1}]$. By the construction, $\Bbbk[C \cap X] = \Bbbk[t,\, u_1, u_2, \ldots, u_{n-1}]/I$, where I is generated by the polynomials (8-8). This algebra is generated over $\Bbbk[U]$ by one element t. It is enough to check that t is integral over $\Bbbk[U]$, i.e., that the ideal I contains a monic polynomial in t. Such a polynomial exists if and only if the leading coefficients $\alpha_0^{(\nu)}(u)$ of the polynomials (8-8) generate the nonproper ideal in $\Bbbk[U]$. By the weak Nullstellensatz, the latter means that the coefficients $\alpha_0^{(\nu)}(u)$ have no common zeros in U. But this is guaranteed by the condition $p \notin X$. Indeed, if all the coefficients $\alpha_0^{(\nu)}(u)$ simultaneously vanish at some point u_0, then the homogenizations of equations (8-8)

$$f_\nu(\vartheta_0 p + \vartheta_1 u_0) = \alpha_0^{(\nu)}(u_0)\vartheta_0^m + \alpha_1^{(\nu)}(u_0)\vartheta_0^{m-1}\vartheta_1 + \cdots + \alpha_m^{(\nu)}(u_0)\vartheta_1^m = 0\,,$$

which describe the intersection of X with the whole unpunctured projective line (p, u_0), have the common root $(\vartheta_0 : \vartheta_1) = (1 : 0)$ on this line. This means that $p \in X$ despite the assumption made in the Proposition. □

COROLLARY 8.3
Every projective variety admits a regular finite surjection onto projective space.

PROOF. Let $X \subset \mathbb{P}_n$ be a projective variety. Make a finite projection $\pi_1 : X \to H_1$ from some point $p_1 \in \mathbb{P}_n \smallsetminus X$ to some hyperplane $H_1 \subset \mathbb{P}_n$. If $\pi_1(X) \neq H_1$, make the second finite projection $\pi_2 : \pi_1(X) \to H_2$ from some point $p_2 \in H_1 \smallsetminus \pi_1(X)$ to some hyperplane $H_2 \subset H_1$, etc. □

COROLLARY 8.4
Every affine algebraic variety admits a regular finite surjection onto affine space.

PROOF. Consider an affine variety $X \subsetneq \mathbb{A}^n$ and embed $\mathbb{A}^n$ into $\mathbb{P}_n$ as the standard affine chart U_0. Write $H_\infty = \mathbb{P}_n \smallsetminus U_0$ for the hyperplane at infinity and $\overline{X} \subset \mathbb{P}_n$ for the projective closure of X. Pick a point $p \in H_\infty \smallsetminus \overline{X}$ and a hyperplane $L \not\ni p$. The projection $\pi_p : \overline{X} \to L$ from p to L looks within the chart U_0 as the parallel projection of $X = \overline{X} \smallsetminus H_\infty$ to the affine hyperplane $U_0 \cap L = L \smallsetminus H_\infty$ in the direction of the vector p. By the Proposition 8.1, this parallel projection is a finite morphism of affine algebraic varieties. If it is not surjective, we repeat the procedure within the target hyperplane, as in the proof of Corollary 8.3. □

EXERCISE 8.14. Check that $\overline{X} \cap H_\infty \neq H_\infty$ for $X \neq \mathbb{A}^n$.

EXAMPLE 8.7 (NOETHER'S NORMALIZATION)
Consider a polynomial $f \in \Bbbk[x_1, x_2, \dots, x_n]$ of positive degree and write it as

$$f = f_0 + f_1 + \cdots + f_d\,,$$

where every f_k is homogeneous of degree k. Let $\overline{X} = V(\overline{f}) \subset \mathbb{P}_n$ be the projective of affine hypersurface $X = V(f) \subset \mathbb{A}^n$, where $\mathbb{A}^n$ is identified with the standard affine chart $x_0 = 1$ in $\mathbb{P}_n$, $(x_0 : x_1 : \ \dots \ : x_n)$ are homogeneous coordinates on $\mathbb{P}_n$, $(x_1, x_2, \dots, x_n)$ are affine coordinates on $\mathbb{A}^n$ and $\overline{f} = f_0 x_0^d + f_1 x_0^{d-1} + \cdots + f_{d-1} x_0 + f_d$. An infinitely far point $p = (0 : p_1 : p_2 : \dots : p_n)$ does not lie on $\overline{X}$ if and only if $f_d(p_1, p_2, \dots, p_n) \neq 0$. Over an infinite field $\Bbbk$, such a point p can be always chosen. After renumbering the coordinates and rescaling p, we can assume that

$$p = (0 : p_1 : \dots : p_{n-1} : 1)\,.$$

Within the affine chart $\mathbb{A}^n$, the projection from p to the affine hyperplane $x_n = 0$ is looking as the parallel projection $\pi_p : X \to \mathbb{A}^{n-1}$ along the vector $p = (p_1, \dots, p_{n-1}, -1)$. It takes

$$(x_1, x_2, \dots, x_n) \mapsto \left(x_1 - p_1 x_n,\, x_2 - p_2 x_n,\, \dots\,,\, x_{n-1} - p_{n-1} x_n,\, 0\right)\,.$$

The pullback homomorphism $\pi_p^* : \Bbbk[x_1, x_2, \dots, x_{n-1}] \to \Bbbk[X] = \Bbbk[x_1, x_2, \dots, x_n]/(f)$ takes

$$x_i \mapsto t_i \overset{\text{def}}{=} x_i - p_i x_n \in \Bbbk[X]\,, \quad \text{for } 1 \leqslant i \leqslant n-1\,.$$

Since the class of x_n in $\Bbbk[X]$ is annihilated by the polynomial

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= f\left(t_1 + p_1 x_n,\, t_2 + p_2 x_n,\, \dots\,,\, t_{n-1} + p_{n-1} x_n, x_n\right) = \\ &= a_0 x_n^d + a_1 x_n^{d-1} + \cdots + a_{d-1} x_n + a_d \end{aligned}$$

whose coefficients $a_i \in \Bbbk[t_1, t_2, \dots, t_{n-1}]$ and the leading one $a_0 = f_d(p_1, \dots, p_{n-1}, 1) \in \Bbbk$ is invertible, the variable x_n and therefore the coordinate algebra $\Bbbk[X]$ is integral over $\pi_p^* \Bbbk[\mathbb{A}^n]$. Thus, the projection $\pi_p : X \to \mathbb{A}^{n-1}$ is finite, that agrees with Proposition 8.1. This claim is known as the *Noether*[1] *normalization lemma*. Over an algebraically closed field $\Bbbk$, the projection π_p is obviously surjective, because for a given point $q \in \mathbb{A}^{n-1}$, mapped to q by π_p is every point $\left(q_1 + \lambda p_1,\, q_2 + \lambda p_2,\, \dots\,,\, q_{n-1} + \lambda p_{n-1}, \lambda_n\right)$, where λ is a root of the degree-d polynomial

$$f\left(q_1 + p_1 t,\, q_2 + p_2 t,\, \dots\,,\, q_{n-1} + p_{n-1} t, t\right) \in \Bbbk[t]\,.$$

Thus, over an algebraically closed field, every affine algebraic hypersurface $V(f) \subset \mathbb{A}^n$ of positive degree admits a finite surjective parallel projection onto a hyperplane. Note that this forces

$$\operatorname{tr}\deg \Bbbk[X] = n - 1\,. \tag{8-9}$$

EXERCISE 8.15. Prove this by direct arguments not using Proposition 7.12.

[1] In honor of Emmy Noether, who proved it in 1926.

Exrc. 8.2. If $x_i x_j \neq 0$, then $t_{j,\nu} = x_\nu / x_j = \left(x_\nu : x_i\right) / \left(x_j : x_i\right) = t_{i,\nu} / t_{i,j}$ (for $\nu = i$ we put $t_{i,i} = 1$). Therefore, $\varphi_{ji}^* : t_{j,\nu} \mapsto t_{i,\nu} / t_{i,j}$. The inverse to φ_{ji}^* homomorphism $\Bbbk\left[\mathcal{D}\left(t_{i,j}\right)\right] \to \Bbbk\left[\mathcal{D}\left(t_{j,i}\right)\right]$ acts by the same rule $t_j^{(i)} \mapsto 1/t_i^{(j)}$, $t_{i,\nu} \mapsto t_{j,\nu}/t_{j,i}$.

Exrc. 8.3. Every such W has a unique basis $w_1, w_2, \dots, w_k$ projected to $e_{i_1}, e_{i_2}, \dots, e_{i_k}$. Write x_W for the matrix formed by the coordinates of vectors $w_1, w_2, \dots, w_k$ written in rows. Then $s_I\left(x_W\right) = E$.

Exrc. 8.5. Note that the elements of $k \times m$ matrix $s_J^{-1}\left(\varphi_I(t)\right) \cdot \varphi_I(t)$ are the rational functions of the elements of matrix t with the denominators equal to $\det s_J\left(\varphi_I(t)\right)$. In particular, they all are regular in $\mathcal{D}\left(\det s_J\left(\varphi_I(t)\right)\right)$.

Exrc. 8.6. This follows from the definition of regular function and Corollary 7.2 on p. 92.

Exrc. 8.9. The definition of $\varkappa$ can be rewrite as $(x_0 : x_1 : x_2) \mapsto (x_1 x_2 : x_0 x_2 : x_0 x_1)$. It makes clear that $\varkappa$ is undefined only at the points $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$ and takes all values except for these points.

Exrc. 8.10. Given a homogeneous polynomial $\overline{f}(x_0, x_1, \dots, x_n)$, write $V(f) \subset \mathbb{P}_n$ for the set of its zeros. In the notations of Example 8.1 on p. 98, the intersection $V(f) \cap U_i$ is described in terms of the affine coordinates t_i within th chart U_i by the polynomial equation

$$\overline{f}\left(t_{i,0}, \dots, t_{i,i-1}, 1, t_{i,i+1}, \dots, t_{i,n}\right) = 0\,.$$

Exrc. 8.12. Use the Segre embedding $\mathbb{P}_{n_1} \times \mathbb{P}_{n_2} \times \cdots \times \mathbb{P}_{n_m} \hookrightarrow \mathbb{P}_N$ described in n° 4.1.2 on p. 40 and analyzed in more details in Example 4.10 on p. 58.

Exrc. 8.13. If $A \subset B$ and $B \supset C$ are two integral extensions of commutative rings, then the extension $A \subset C$ is integral as well by Proposition 6.1 on p. 73.