§5 Grassmannian varieties in more details

5.1 The Plücker quadric and grassmannian Gr(2,4). Let us fix a vector space V of dimension 4. The grassmannian $\mathrm{Gr}(2,V) = \mathrm{Gr}(2,4)$ parameterizes the vector subspaces $U \subset V$ of dimension 2, or equivalently, the lines $\ell \subset \mathbb{P}_3 = \mathbb{P}(V)$. The *Plücker embedding*

$$\mathfrak{u} : \mathrm{Gr}(2,4) \hookrightarrow \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V), \quad U \mapsto \Lambda^2 U, \quad (ab) \mapsto a \wedge b \tag{5-1}$$

sends every 2-dimensional subspace $U \subset V$ to the 1-dimensional subspace $\Lambda^2 U \subset \Lambda^2 V$, or equivalently, every line $(ab) \subset \mathbb{P}(V)$ to the point $a \wedge b \in \mathbb{P}(\Lambda^2 V)$. It assigns the bijection between the grassmannian $\mathrm{Gr}(2,4)$ and the *Plücker quadric*[1]

$$P \stackrel{\text{def}}{=} \{\omega \in \Lambda^2 V \mid \omega \wedge \omega = 0\}$$

which consists of all decomposable grassmannian quadratic forms $\omega = a \wedge b$, $a, b \in V$, see Example 4.9 on p. 58.

Let us fix a basis e_0, e_1, e_2, e_3 in V, the monomial basis $e_{ij} \stackrel{\text{def}}{=} e_i \wedge e_j$ in $\Lambda^2 V$, and write x_{ij} for the homogeneous coordinates in $\mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$ with respect to the latter basis. The computation

$$\Big(\sum_{i<j} x_{ij} \cdot e_i \wedge e_j\Big) \wedge \Big(\sum_{i<j} x_{ij} \cdot e_i \wedge e_j\Big) = 2\left(x_{01}x_{23} - x_{02}x_{13} + x_{03}x_{12}\right) \cdot e_0 \wedge e_1 \wedge e_2 \wedge e_3$$

shows that P is described by the non-degenerated quadratic equation $x_{02}x_{13} = x_{01}x_{23} + x_{03}x_{12}$.

Exercise 5.1. Check that the Plücker embedding (5-1) takes the subspace spanned by vectors $a = \sum \alpha_i e_i$, $b = \sum \beta_j e_j$ to the point with coordinates $x_{ij} = \alpha_i\beta_j - \alpha_j\beta_i$, that is, sends a matrix $\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{pmatrix}$ to the collection of its six 2×2-minors $x_{ij} = \det\begin{pmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{pmatrix}$.

In coordinate-free terms, the Plücker quadric is described as follows. There exists a unique up to proportionality bilinear form $\widetilde{q}$ on $\Lambda^2 V$ defined by prescription

$$\forall\, \omega_1, \omega_2 \in \Lambda^2 V \quad \omega_1 \wedge \omega_2 = \widetilde{q}(\omega_1, \omega_2) \cdot \delta\,, \tag{5-2}$$

where $\delta \in \Lambda^4 V \simeq \Bbbk$ is an arbitrary non zero vector[1]. This form is symmetric, because $\omega_1 \wedge \omega_2 = \omega_2 \wedge \omega_1$ for even grassmannian polynomials. Obviously, $P = V(q)$ for the quadratic form $q(\omega) = \widetilde{q}(\omega, \omega)$ corresponding to $\widetilde{q}$.

Lemma 5.1
Two lines $\ell_1, \ell_2 \subset \mathbb{P}_3$ are intersecting if and only if $\widetilde{q}(\mathfrak{u}(\ell_1), \mathfrak{u}(\ell_2)) = 0$ in $\mathbb{P}_5$.

Proof. Let $\ell_1 = \mathbb{P}(U_1)$, $\ell_2 = \mathbb{P}(U_2)$. If $U_1 \cap U_2 = 0$, then $V = U_1 \oplus U_2$ and we can choose a basis $e_0, e_1, e_2, e_3 \in V$ such that $\ell_1 = (e_0 e_1)$, $\ell_2 = (e_2 e_3)$. Then $\mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell_2) = e_0 \wedge e_1 \wedge e_2 \wedge e_3 \neq 0$. If $\ell_1 = (ab)$, $\ell_2 = (ac)$ are intersecting in a, then $\mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell_2) = a \wedge b \wedge a \wedge c = 0$. □

Remark 5.1. The injectivity of (5-1) becomes obvious[2] after Lemma 5.1. Indeed, for any two lines $\ell_1 \neq \ell_2$ on $\mathbb{P}_3$ there exists a third line ℓ which intersects ℓ_1 and does not intersect ℓ_2. Then $\mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell) = 0$ and $\mathfrak{u}(\ell_2) \wedge \mathfrak{u}(\ell) \neq 0$. This forces $\mathfrak{u}(\ell_1) \neq \mathfrak{u}(\ell_2)$.

[1]See formula (4-47) on p. 58.
[1]Since $\dim \Lambda^4 V = 1$, such a vector is unique up to proportionality.
[2]Compare with Exercise 4.27 on p. 58.

Corollary 5.1
For every point $p = \mathfrak{u}(\ell) \in P$, the intersection $P \cap T_pP$ in $\mathbb{P}_5$ consists of all points $\mathfrak{u}(\ell')$ such that $\ell \cap \ell' \neq \varnothing$ in $\mathbb{P}_3$.

Proof. This follows from Lemma 5.1 and Proposition 2.2 on p. 18. □

5.1.1 Nets and pencils of lines in $\mathbb{P}_3$. A family of lines on $\mathbb{P}_3$ is called a *net* if the Plücker embedding sends it to a plane $\pi \subset P \subset \mathbb{P}_5$. Every plane $\pi \subset P$ is spanned by a triple of non collinear points $p_i = \mathfrak{u}(\ell_i)$, $i = 1, 2, 3$, and lies in the intersection of tangent spaces to P at these points: $\pi \subset P \cap T_{p_1}P \cap T_{p_2}P \cap T_{p_3}P$. It follows from the Lemma 5.1 and Corollary 5.1 that the corresponding net of lines in $\mathbb{P}_3$ consists of all lines intersecting three given pairwise intersecting lines ℓ_1, ℓ_2, ℓ_3. Since three mutually intersecting lines have to be either concurrent or coplanar, there are exactly two different types of line nets in $\mathbb{P}_3$:

α-net consists of lines passing through a given point $a \in \mathbb{P}_3$ and corresponds to *α-plane* $\pi_\alpha(a) \subset P$ spanned by Plücker's images of three non-coplanar lines passing through a

β-net consists of lines laying in a given plane $\Pi \in \mathbb{P}_3$ and corresponds to *β-plane* $\pi_\beta(\Pi) \subset P$ spanned by Plücker's images of three non-concurrent lines laying in Π.

Any two planes of the same type have exactly one intersection point:

$$\pi_\beta(\Pi_1) \cap \pi_\beta(\Pi_2) = \mathfrak{u}(\Pi_1 \cap \Pi_2)\,, \quad \pi_\alpha(a_1) \cap \pi_\alpha(a_2) = \mathfrak{u}((a_1 O_2))\,.$$

Two planes of different types $\pi_\beta(\Pi)$, $\pi_\alpha(a)$ are either not intersecting (if $a \notin \Pi$) or intersecting along a line (if $a \in \Pi$). In the latter case the intersection line depicts the pencil of lines in $\mathbb{P}_3$ passing through a and laying in Π.

Exercise 5.2. Show that there are no other pencils of lines in $\mathbb{P}_3$, i.e., every line laying on $P \subset \mathbb{P}_5$ has the form $\pi_\beta(\Pi) \cap \pi_\alpha(a)$ for some $a \in \Pi \subset \mathbb{P}_3$.

Exercise 5.3. Convince yourself that the assignment $U \mapsto \operatorname{Ann} U$ establishes the bijection $\operatorname{Gr}(2, V) \xrightarrow{\sim} \operatorname{Gr}(2, V^*)$ sending α-planes to β-planes and vice versa.

5.1.2 Cell decomposition of P. Let us fix a point $p \in P$ and a hyperplane $H \simeq \mathbb{P}_3$ laying inside $T_pP \simeq \mathbb{P}_4$ and complementary to p within this $\mathbb{P}_4$. The intersection $C = P \cap T_pP$ is the simple cone with vertex p over a smooth quadric $G = H \cap P$, which can be thought of as the Segre quadric in $\mathbb{P}_3 = H$. Fix a point $p' \in G$ and write π_α, π_β for the planes spanned by p and two lines laying on G and passing through p'. Associated with these data is the following stratification of the Plücker quadric P by closed subvarieties shown on fig. 5◇1 on p. 62:

$$\begin{tikzcd} & & \pi_\alpha \arrow[dr, hook] & & \\ p \arrow[r, hook] & \pi_\alpha \cap \pi_\beta \arrow[ur, hook] \arrow[dr, hook] & & C \arrow[r, hook] & P \\ & & \pi_\beta \arrow[ur, hook] & & \end{tikzcd} \tag{5-3}$$

For every stratum σ of this stratification, the complement to the union of all strata contained in σ is naturally identified with an affine space. This leads to the following decomposition of $\mathrm{Gr}(2,4)$ in disjoint union of affine spaces:

$$\mathrm{Gr}(2,4) \;=\; \mathbb{A}^0 \sqcup \mathbb{A}^1 \sqcup \begin{pmatrix} \mathbb{A}^2 \\ \sqcup \\ \mathbb{A}^2 \end{pmatrix} \sqcup \mathbb{A}^3 \sqcup \mathbb{A}^4 \,.$$

The leftmost $\mathbb{A}^0$ is the point p. Then goes $\mathbb{A}^1$, which is the complement to p within the projective line $(pp') = \pi_\alpha \cap \pi_\beta$. Then go two affine planes $\mathbb{A}^2$, the complements to (pp') within the projective planes π_α and π_β respectively. Then goes $\mathbb{A}^3$, which is the complement to $\pi_\alpha \cup \pi_\beta$ within the cone $C = P \cap T_p P$, which is the linear join of G and p. This complement is isomorphic to the direct product of $\mathbb{A}^1$, which is the cone generator punctured at the vertex of cone, and $\mathbb{A}^2 = G \smallsetminus T_{p'} G$. The rightmost piece $\mathbb{A}^4 = P \smallsetminus C$. The identifications $G \smallsetminus T_{p'} G = \mathbb{A}^2$ and $P \smallsetminus T_{p'} P = \mathbb{A}^4$ made on the last two steps are based on the Lemma 5.2 following below.

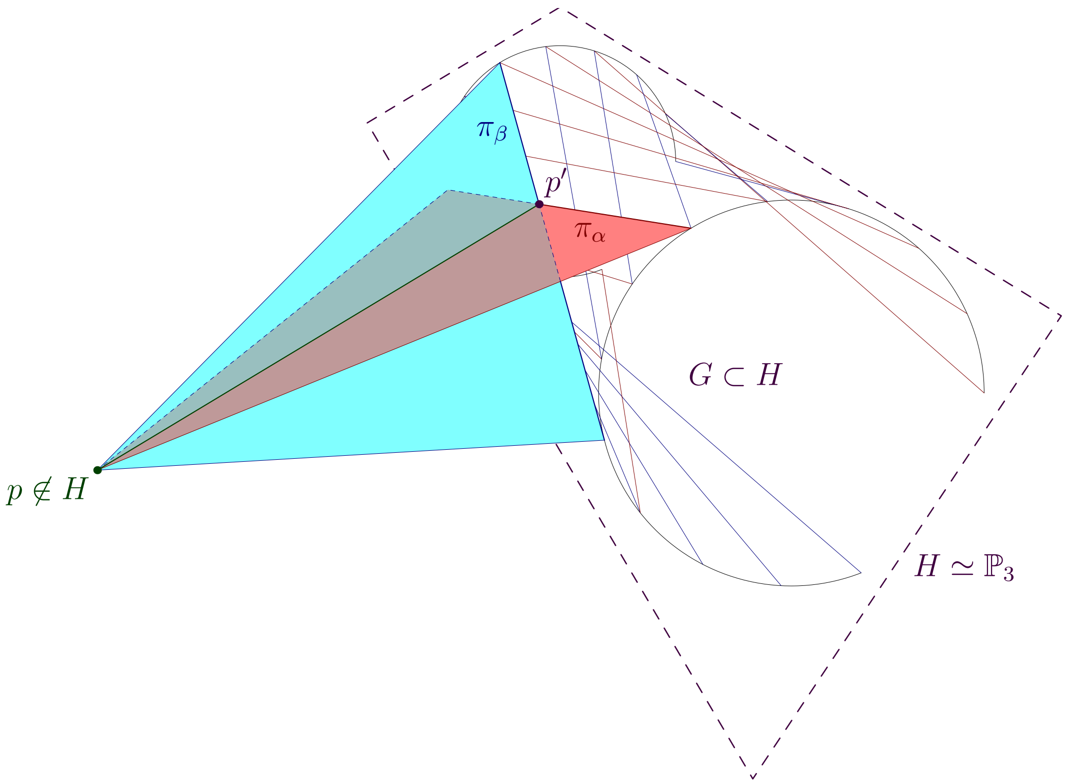


Fig. 5⋄1. The cone $C = P \cap T_p P$ viewed within $\mathbb{P}_4 = T_p P$.

Lemma 5.2
For every smooth quadric $Q \subset \mathbb{P}_n$, point $p \in Q$, and hyperplane $\Pi \not\ni p$, the projection $p : Q \to \Pi$ from p to Π establishes a bijection between $Q \smallsetminus T_p Q$ and $\mathbb{A}^{n-1} = \Pi \smallsetminus T_p Q$.

Proof. Every non-tangent line passing through p intersects Q in exactly one point other than p. All these lines stay in bijection with the points of $\Pi \smallsetminus T_p Q \simeq \mathbb{A}^{n-1}$. □

EXERCISE 5.4. If you have some experience in CW-topology, show that the integer homology groups of complex grassmannian Gr(2, 4) are

$$H_m\left(\mathrm{Gr}(2,\mathbb{C}^4),\,\mathbb{Z}\right)=\begin{cases}0 & \text{for odd } m\leqslant 7 \text{ and all } m>8\\ \mathbb{Z} & \text{for } m=0,\ 2,\ 6,\ 8\\ \mathbb{Z}\oplus\mathbb{Z} & \text{for } m=4\end{cases}$$

Try to compute the integer homologies $H_m\left(\mathrm{Gr}(2,\mathbb{R}^4),\,\mathbb{Z}\right)$ of the real grassmannian Gr(2, 4).

5.1.3 Lagrangian grassmannian LGr(2,4) and lines on a smooth quadric in $\mathbb{P}_4$. Let a vector space V of dimension 4 be equipped with a non-degenerated alternating bilinear form Ω. A line $\ell = (ab) \subset \mathbb{P}(V)$ is called *lagrangian* if $\Omega(u, w) = 0$ for all $u, w \in \ell$, or equivalently, if $\Omega(a, b) = 0$. The set of all lagrangian lines is called the *lagrangian grassmannian* and denoted by $\mathrm{LGr}(2, 4) \subset \mathrm{Gr}(2, 4)$. Let us show that the Plücker embedding sends LGr(2, 4) to a smooth hyperplane section of the Plücker quadric, that is, to a smooth quadric in $\mathbb{P}_4$.

Associated with Ω is the linear form $\Omega' : \Lambda^2 V \to \mathbb{k}$, $a\wedge b \mapsto \Omega(a, b)$. Let us fix a non-zero vector $\delta \in \Lambda^4 V$. Since the bilinear form $\widetilde{q}$ on $\Lambda^2 V$ defined in formula (5-2) on p. 60 is non-degenerate, its correlation map $\widehat{q} : \Lambda^2 V \xrightarrow{\sim} \left(\Lambda^2 V\right)^*$ is an isomorphism. Hence, there exists a unique grassmannian quadratic form $\omega = \widehat{q}^{-1}(\Omega') \in \Lambda^2 V$ such that

$$\forall\, a, b \in V \quad \omega \wedge a \wedge b = \Omega(a, b) \cdot \delta\,. \tag{5-4}$$

Write $W = \mathrm{Ann}\,\Omega' \subset \Lambda^2 V$ for the orthogonal complement to ω with respect to the Plücker quadratic form q. The projectivization $Z = \mathbb{P}(W) \simeq \mathbb{P}_4 \subset \mathbb{P}_5$ is the polar hyperplane of ω with respect to the Plücker quadric $P \subset \mathbb{P}(\Lambda^2 V)$.

EXERCISE 5.5. Verify that $\omega \notin P$.

Hence, the intersection $R = Z \cap P$ is a smooth quadric within $\mathbb{P}_4 = Z$. The points of this quadric stay in bijection with the lagrangian lines in $\mathbb{P}(V)$, because the formulas (5-4), (5-2) say together that a line $(ab) \subset \mathbb{P}_3$ is lagrangian if and only if $\widetilde{q}(\omega,\, a \wedge b) = 0$. Thus, $\mathrm{LGr}(2, 4) = R$ is a smooth quadric in $\mathbb{P}_4 = Z$.

It follows from the general theory developed in n° 2.6 on p. 24 that R does not contain planes but every point $r \in R$ is the vertex of cone $R \cap T_r R$, the linear join of r with a smooth conic in a plane complementary to p within $T_p R \simeq \mathbb{P}_3$.

DEFINITION 5.1 (THE FANO VARIETY OF A PROJECTIVE VARIETY)
The set of lines laying on a projective algebraic variety X is called the *Fano variety* of X and denoted by $F(X)$.

PROPOSITION 5.1
For every point $p \in \mathbb{P}(V)$, the lagrangian lines $\ell \subset \mathbb{P}(V)$ passing through p form a pencil. Sending p to this pencil assigns the bijection $\mathbb{P}(V) \xrightarrow{\sim} F\big(\mathrm{LGr}(2, V)\big)$.

PROOF. Every pencil of lines in $\mathbb{P}_3 = \mathbb{P}(V)$ is mapped by the Plücker embedding to a line $L \subset P$, which has the form[1] $L = \pi_p \cap \pi(\Pi)$ for some point p and plane Π in $\mathbb{P}_3$ such that $p \in \Pi$. In other words, L consists of all lines passing through p and laying in Π. For $L \subset R = P \cap Z$ all these lines

[1] See n° 5.1.1, especially Exercise 5.2 on p. 61.

are lagrangian. On the other hand, a line $(px) \subset \mathbb{P}(V)$ is lagrangian if and only if $\Omega(p,x) = 0$. Hence, every lagrangian line passing through p lies in the orthogonal plane to p with respect to the form Ω and therefore, belongs to the pencil L. This proves the first statement. The second is obvious from the discussion preceding the proposition. $\square$

5.2 The homogeneous, Plücker's, and affine coordinates on Gr(k,m). The general grassmannian $\mathrm{Gr}(k,m)$, which parameterizes the vector subspaces of dimension k in $V = \Bbbk^m$, is a straightforward generalization of the projective space $\mathbb{P}_{m-1} = \mathrm{Gr}(1,m)$ attached to V. If a basis $e_1, e_2, \dots, e_d$ in V is fixed, then a vector subspace $U \subset V$ with a basis $u = u_1, u_2, \dots, u_m$ can be described by the $k \times m$ matrix A_u formed by the coordinate rows of vectors u_i in the chosen basis of V. Every other basis $w_1, w_2, \dots, w_m$ in U has the form $(w_1, w_2, \dots, w_m) = (u_1, u_2, \dots, u_m) \cdot C_{uw}$, where $C_{wu} \in \mathrm{GL}_k(\Bbbk)$, and leads to the matrix $A_w = C_{uw}^t A_u$.

EXERCISE 5.6. Check this.

Thus, two $k \times m$ matrices A_u, A_w of rank k correspond to the same subspace $U \subset V$ if and only if $A_w = GA_u$ for some $k \times k$ matrix $G \in \mathrm{GL}_k(\Bbbk)$. For $k = 1$, this agrees with the description of $\mathbb{P}_{m-1} = \mathrm{Gr}(1,m)$ as the set of nonzero rows $(x_1, x_2, \dots, x_m) \in \Bbbk^m = \mathrm{Mat}_{1\times m}$ considered up to multiplication by nonzero constants $\lambda \in \Bbbk^* = \mathrm{GL}_1(\Bbbk)$. Thus, the matrix $A_u \in \mathrm{Mat}_{k\times m}$, formed by coordinate rows of some basis vectors $u_1, u_2, \dots, u_k \in U$ and considered up to the left multiplication by matrices $G \in \mathrm{GL}_k$, is the direct analog of homogeneous coordinates on the projective space.

The Plücker embedding $\mathfrak{u} : \mathrm{Gr}(k,V) \hookrightarrow \mathbb{P}(\Lambda^k V)$ takes a subspace $U \subset V$ of dimension k to the subspace $\Lambda^m U \subset \Lambda^m V$ of dimension 1. For every basis $u_1, u_2, \dots, u_m$ in U, the grassmannian monomial $u_1 \wedge u_2 \wedge \dots \wedge u_m$ spans $\mathfrak{u}(U)$.

EXERCISE 5.7 (PLÜCKER COORDINATES). Verify that for every $I = (i_1, i_2, \dots, i_k)$, the coefficient α_I in the expansion $u_1 \wedge u_2 \wedge \dots \wedge u_m = \sum_I \alpha_I e_I$ equals the $k \times k$ minor situated in the columns $i_1, i_2, \dots, i_k$ of matrix A_u.

Thus, the $\binom{m}{k}$ homogeneous coordinates of the point $\mathfrak{u}(U) \in \mathbb{P}(\Lambda^k V)$ with respect to the basis formed by the grassmannian monomials e_I are the determinants $a_I = \det A_I$ of $k \times k$ submatrices $A_I \subset A_u$. They called the *Plücker coordinates* of the subspace $U \subset V$. Two subspaces $U, W \subset V$ of dimension k coincide if and only if their Plücker coordinates are proportional.

EXERCISE 5.8. Is there a rational 2×4 matrix with minors A) 2, 3, 4, 5, 6, 7 B) 3, 4, 5, 6, 7, 8? If such matrices exist, write some of them explicitly. If not, explain why.

5.2.1 Affine charts. For every subspace $T \subset V$ of codimension k, the set

$$\mathcal{U}_T \overset{\text{def}}{=} \{W \subset V \mid \dim W = k\,,\ W \cap T = 0\}$$

is called the *affine chart* provided by T on the grassmannian $\mathrm{Gr}(k,V)$. For every $U \in \mathcal{U}_T$, the set $\mathcal{U}_T$ is naturally identified with the affinization $\mathbb{A}(\mathrm{Hom}(U,T))$ of the vector space of linear maps $\tau : U \to T$ as follows. We have the direct sum decomposition $V = T \oplus U$ and $\mathcal{U}_T$ consists of all those subspaces $W \subset U$ isomorphically projected onto U along T. Thus, every $W \in \mathcal{U}_T$ is the graph of linear map $\tau_W : U \to T$ sending a vector $u \in U$ to the unique vector $t \in T$ such that $u + t \in W$, and vice versa, for every linear map $\tau : U \to T$, its graph $W_\tau = \{u + \tau(u) \mid u \in U\}$ is a linear subspace in V isomorphically projected onto U along T.

For every $U \in \mathcal{U}_T$, the projection $V \twoheadrightarrow T$ along U assigns the isomorphism $\pi_T : V/U \xrightarrow{\sim} T$. It provides us with the linear isomorphism $\alpha_T : \mathrm{Hom}(U, V/U) \xrightarrow{\sim} \mathrm{Hom}(U,T)$, $\tau \mapsto \pi_T \circ \tau$, which allows to consider all affine charts $\mathcal{U}_T$ containing a given point $U \in \mathrm{Gr}(k,V)$ as affine spaces over the same vector space $\mathrm{Hom}(U, V/U)$ independent on T. Thus, locally, in a neighborhood of every point U, the grassmannian $\mathrm{Gr}(k,V)$ looks as an affine space over the vector space $\mathrm{Hom}(U,V/U)$ of dimension $k\times(m-k)$. This vector space is called the *tangent space* to the grassmannian $\mathrm{Gr}(k,V)$ at the point U and is denoted by $\mathcal{T}_U\mathrm{Gr}(k,V)$.

EXAMPLE 5.1 (AFFINE CHARTS ON $\mathbb{P}_{m-1}=\mathrm{Gr}(1,m)$ REVISITED)
Every codimension 1 subspace $T\subset V$ has the form $T=\mathrm{Ann}\,\xi$ for a non-zero covector $\xi\in V^*$ uniquely up to proportionality determined by T. Defined in n° 1.2 on p. 5 were affine charts U_ξ on $\mathbb{P}_{m-1}=\mathbb{P}(V)$. For all ξ such that $\mathrm{Ann}\,\xi=T$, the charts U_ξ consist of the same points, the dimension 1 subspaces $\Bbbk\cdot u\subset V$ such that $u\notin T$. Exactly the same subspaces form the chart $\mathcal{U}_T$ on $\mathrm{Gr}(1,V)$. This chart is an affine space associated with the vector space $\mathrm{Hom}(\Bbbk,T)\simeq T$. A particular choice of dimension 1 subspace $\Bbbk\cdot u\in\mathcal{U}_T$ fixes the origin in this affine space. Under this choice, every dimension 1 subspace $\Bbbk\cdot w$ laying in $\mathcal{U}_T$, i.e., such that $\xi(w)\neq 0$, can be identified with the linear map $\tau_w:\ \Bbbk\cdot u\to\mathrm{Ann}\,\xi=T,\ u\mapsto w\cdot\xi(u)/\xi(w)-u$. Note that this map depends only on the subspaces $\Bbbk\cdot u$, $\Bbbk\cdot w$, and T in V but not on the choice of $u\in\Bbbk\cdot u$, $w\in\Bbbk\cdot w$, and $\xi\in\mathrm{Ann}\,T$.

5.2.2 The standard affine charts on Gr(k,m). For every collection I of increasing indexes $1\leqslant i_1<i_2<\cdots<i_k\leqslant m$, write $E_I,E_{\hat I}\subset\Bbbk^m$ for the complementary subspaces spanned by the basis vectors e_i, $i\in I$, and e_j, $j\notin I$, respectively. The affine chart $\mathcal{U}_{E_{\hat I}}$, which consists of all dimension k subspaces $U\subset\Bbbk^m$ isomorphically projected onto E_I along $E_{\hat I}$, is called the *standard I-chart* on grassmannian $\mathrm{Gr}(k,m)$ and denoted by $\mathcal{U}_I$.

For every subspace $U\subset V$ laying in the chart $\mathcal{U}_I$, write $u^{(I)}=u_1^{(I)},u_2^{(I)},\dots,u_k^{(I)}$ for the basis of U projected along $E_{\hat I}$ to the basis $e_{i_1},e_{i_2},\dots,e_{i_k}$ of E_I. The matrix $A^{(I)}\overset{\text{def}}{=}A_{u^{(I)}}$, formed by the coordinate rows of these vectors, has the identity $k\times k$ submatrix in the columns $i_1,i_2,\dots,i_k$. We conclude that among the matrices A_u representing a subspace $U\in\mathcal{U}_I$, there exists the unique matrix having the identity submatrix in I-columns. We write $A^{(I)}(U)$ for this matrix and use the $k(m-k)$ elements staying outside the I-columns of $A^{(I)}(U)$ as the *standard affine coordinates* of U in the chart $\mathcal{U}_I$.

Clearly, a point $U\in\mathrm{Gr}(k,m)$ represented by a matrix $A=A_u\in\mathrm{Mat}_{k\times m}(\Bbbk)$ lies in $\mathcal{U}_I$ if and only if the $k\times k$ submatrix $A_I\subset A$ situated in I-columns of A is invertible. In this case, $A^{(I)}(U)=A_I^{-1}A$. Thus, the standard chart $\mathcal{U}_I$ consists of those U whose Ith Plücker coordinate is not zero. The matrices $A^{(I)}=A^{(I)}(U)$ and $A^{(J)}=A^{(J)}(U)$ producing the local affine coordinates of a point $U\in\mathcal{U}_I\cap\mathcal{U}_J$ in the standard charts $\mathcal{U}_I$, $\mathcal{U}_J$ are related as $A^{(I)}=(A_I^{(J)})^{-1}A^{(J)}$. Hence, the standard affine coordinates of the same subspace $U\subset V$ in different charts are rational functions of each other.

EXERCISE 5.9. Make it sure that the standard affine charts and local affine coordinates on $\mathrm{Gr}(1,m)=\mathbb{P}_{m-1}$ are exactly those introduced in Example 1.2 on p. 8.

EXERCISE 5.10. If you had deal with differential (respectively, analytic[2]) geometry, check that real (respectively complex) grassmannians are smooth (respectively holomorphic) manifolds.

5.3 The cell decomposition for Gr(k,m). The Gaussian elimination method shows that every subspace $U\subset V$ admits a unique basis $u=u_1,u_2,\dots,u_m$ with the *reduced echelon* matrix A_u, i.e., the leftmost nonzero element in every row of A_u stays strictly to the right of such element in the

[2]Also known as holomorphic.

previous row, equals 1, and is the only nonzero element of its column.

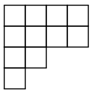
EXERCISE 5.11. Convince yourself that the rows of different reduced echelon $k \times m$ matrices span different subspaces in $\Bbbk^m$.

Thus, there exist a bijection between $\mathrm{Gr}(k, m)$ and the set of reduced echelon $k \times m$ matrices of rank m. The latter splits in disjoint union of affine spaces as follows. Write $J = j_1, j_2, \dots, j_k$ for successive numbers of those columns containing the starting units of rows in a reduced echelon matrix A, and call this increasing sequence of integers the *shape* of A. Every reduced echelon $k \times m$ matrix A of shape I contains the identity submatrix in the J-columns, and has exactly

$$k(m-k) - (j_1 - 1) - (j_2 - 2) - \cdots - (j_m - m) = \dim \mathrm{Gr}(k, m) - \sum_{\nu=1}^{m} (j_\nu - \nu)$$

free cells which may contain arbitrary elements of $\Bbbk$. Thus, these matrices form an affine space of codimension $\sum_{\nu=1}^{m} (j_\nu - \nu)$ in $\mathrm{Gr}(k, m)$. It is denoted by α_J and called an *affine Schubert cell*. The whole grassmannian splits in disjoint union of $\binom{m}{k}$ such cells: $\mathrm{Gr}(k, m) = \bigsqcup_J \alpha_J$.

5.3.1 Young diagram notations. Besides the strictly increasing sequences of integers, the *partitions* are also commonly used for indexing the Schubert cells. A *partition* λ is a non-increasing sequence of non-negative integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 0$ visualized as the *Young diagram*, the pile of horizontal cellular strips of lengths $\lambda_1, \lambda_2, \dots, \lambda_m$ aligned to the left in the non-increasing top-down order. For example, the partition $(4, 4, 2, 1)$ has the Young diagram



(5-5)

The total number of cells in a diagram λ is denoted by $|\lambda| \overset{\text{def}}{=} \sum \lambda_i$ and called the *weight* of λ. Thus, the partitions of weight n enumerate the ways to break a set of n mutually elements in a union of disjoint subsets. The total number of non-empty parts is called the *height* of partition and denoted by $h(\lambda) = \max(k \mid \lambda_k > 0)$. The cardinality λ_1 of biggest part is called the *width* of the partition. For example, the diagram (5-5) has weight 11, height 4, and width 4.

We say that a reduced echelon matrix A has the shape λ for some partition $\lambda = \lambda_1, \lambda_2, \dots, \lambda_k$ if for every $i = 1, 2, \dots, k$, the starting unit in the ith from the bottom row of A stays in the λ_ith cell to the right of the leftmost possible position. This means that $\lambda_{k+1-\nu} = j_\nu - \nu$ for every $\nu = k + 1 - i = 1, 2, \dots, k$. Note that the codimension of the affine Schubert cell α_λ equals the weight $|\lambda|$ of Young diagram λ.

EXERCISE 5.12. Convince yourself that the prescription $j_1, j_2, \dots, j_k \mapsto \lambda_1, \lambda_2, \dots, \lambda_k$ such that $\lambda_{k+1-\nu} = j_\nu - \nu$ for all $1 \leqslant \nu \leqslant k$ establishes a bijection between the sequences of k strictly increasing integers in range $[0, m]$ and the Young diagrams ot height $\leqslant k$ and width $\leqslant m - k$.

For example, the affine Schubert cell $\alpha_{4421} \subset \mathrm{Gr}(4, 10)$ corresponding to the diagram (5-5) consists of subspaces $U \subset \Bbbk^{10}$ represented by reduced echelon matrices of the shape

$$\begin{pmatrix} 0 & 1 & * & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{pmatrix}.$$

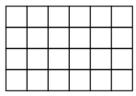
Colored in red are the leftmost possible positions for the starting units of reduced echelon 4×10 matrix. Colored in blue are the actual starting units. Being read bottom-up, they are sifted by 4, 4, 2, and 1 cell to the right of red cell. The grassmannian $\mathrm{Gr}(4,10)$ has dimension 24, the codimension of $\alpha_{4421} \simeq \mathbb{A}^{13}$ equals $11 = 4+4+2+1$.

The zero partition $(0,0,0,0)$ has empty Young diagram meaning that the starting units stay in the leftmost possible positions. It describes the largest Schubert cell α_0 of dimension 24 which consists of subspaces $U \subset \Bbbk^{10}$ represented by matrices of the shape

$$\begin{pmatrix} 1 & 0 & 0 & 0 & * & * & * & * & * & * \\ 0 & 1 & 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 1 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \end{pmatrix}$$

Thus, the cell α_0 coincides with the standard affine chart $\mathcal{U}_{1234} \subset \mathrm{Gr}(4,10)$.

The maximal possible for $\mathrm{Gr}(4,10)$ Young diagram $(6,6,6,6)$ exhausts the whole rectangle



and describes one point cell, the coordinate subspace $E_{7,8,9,10} \subset \Bbbk^{10}$ spanned by the rows of matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

5.3.2 The closed Schubert cycles. We write $\lambda \subseteq \mu$ if the diagram λ is contained in the diagram μ sharing the same upper left corner. Consider a pair of such diagrams and a subspace $W \subset \Bbbk^m$ such that $W \in \alpha_\mu$ in $\mathrm{Gr}(k,m)$. Let A be the reduced echelon matrix of W, B the reduced echelon matrix of shape λ corresponding to the origin of affine cell α_λ, i.e., all element of B but the starting units of rows equal zero. For every $t = (t_0 : t_1) \in \mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$ except for $t = (1:0)$, the reduced echelon form of matrix $C_t = t_0 A + t_1 B$ has the shape λ but $C_\infty = A$ is of shape μ. The subspace $U_t \subset \Bbbk^m$ spanned by the rows of matrix C_t draws a rationally parameterized curve in $\mathrm{Gr}(k,m) \subset \mathbb{P}(\Lambda^k \Bbbk^m)$ as t runs through $\mathbb{P}_1$. All points of this curve but $U_\infty = W \in \alpha_\mu$ belong to the affine Schubert cell α_λ. We conclude that the affine cell α_μ lies in the closure of α_λ for all $\mu \supseteq \lambda$. For every Young diagram λ contained in the rectangle $k \times (m-k)$, the union $\sigma_\lambda = \bigsqcup_{\mu \supseteq \lambda} \alpha_\mu$ is called the (*closed*) *Schubert cycle* of grassmannian $\mathrm{Gr}(k,m)$.

Write $E_{\geqslant n} \subset \Bbbk^m$ for the coordinate subspace spanned by $e_n, e_{n+1}, \dots, e_m$, and $E_{<n}$ for the complementary coordinate subspace. Then, in J-notations, σ_J consists of those subspaces $U \subset \Bbbk^m$ mapped by the projection $\pi_\nu : \Bbbk^m \twoheadrightarrow E_{<j_\nu}$ along $E_{\geqslant j_\nu}$ to a subspace of dimension $\leqslant \nu - 1$ for every $1 \leqslant \nu \leqslant k$, or equivalently, of those U intersecting $\ker \pi_\nu = E_{\geqslant j_\nu}$ in a subspace of dimension at least $k+1-\nu$. Thus, $\sigma_J = \{U \subset \Bbbk^m \mid \dim(U \cap E_{\geqslant j_\nu}) \geqslant k+1-\nu \text{ for } \nu = 1,\dots,k\}$. This is translated in λ-notations as $\sigma_\lambda = \{U \subset \Bbbk^m \mid \dim(U \cap E_{\geqslant k+1-i+\lambda_i}) \geqslant i \text{ for } i = 1,\dots,k\}$.

EXERCISE 5.13. Convince yourself that for $\Bbbk = \mathbb{R}, \mathbb{C}$, the Schubert cycles are closed submanifolds of the grassmannian $\mathrm{Gr}(k,m)$.

EXAMPLE 5.2 (THE SCHUBERT CELLS ON $\mathrm{Gr}(2,4)$)
In $\mathbb{P}_3 = \mathbb{P}(\Bbbk^4)$, consider the point $a = (0:0:0:1)$ and plane $\Pi = V(x_0)$. Then the strata of stratification from formula (5-3) on p. 61 are the Plücker images of Schubert cycles on $\mathrm{Gr}(2,4)$.

Namely, in the notations of n° 5.1.2, the α-plane $\pi_\alpha(a)$ on the Plücker quadric $P \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2\Bbbk^4)$ is the Plücker image of Schubert cycle σ_{20}, i.e., the closure of affine cell α_{11} formed by reduced echelon matrices $\left(\begin{smallmatrix}1 & * & * & 0\\ 0 & 0 & 0 & 1\end{smallmatrix}\right)$. The β-plane $\pi_\beta(\Pi)$ is the cycle σ_{11}, the closure of affine cell α_{20} formed by matrices $\left(\begin{smallmatrix}0 & 1 & 0 & *\\ 0 & 0 & 1 & *\end{smallmatrix}\right)$. Their intersection $\pi_\alpha(a) \cap \pi_\beta(\Pi) = (pp')$ equals σ_{21}, the closure $\alpha_{21} \sqcup \alpha_{22}$ of the cell α_{21} formed by matrices of shape $\left(\begin{smallmatrix}0 & 1 & * & 0\\ 0 & 0 & 0 & 1\end{smallmatrix}\right)$. The dimension zero cycle $\sigma_{22} = \alpha_{22}$ is the point $p = (0 : 0 : 0 : 0 : 0 : 1) \in \mathbb{P}_5 = \mathbb{P}(\Lambda^2\Bbbk^4)$, the Plücker image of matrix $\left(\begin{smallmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{smallmatrix}\right)$. The cone $T_p \cap P = \sigma_{10}$ is the closure of α_{10}, the affine cell formed by matrices $\left(\begin{smallmatrix}1 & * & 0 & *\\ 0 & 0 & 1 & *\end{smallmatrix}\right)$. The biggest cycle σ_{00} is the whole Plücker quadric P.

Exercise 5.14. Check all these statements carefully.

5.3.3 The homology of complex grassmannians and Schubert calculus. Write

$$\Lambda(k,m) \stackrel{\text{def}}{=} \bigoplus_i H_i\big(\mathrm{Gr}(k,\mathbb{C}^m),\mathbb{Z}\big)$$

for the total integer homology group of the complex grassmannian considered as a (real) topological manifold. The (open) affine Schubert cells α_λ provide $\mathrm{Gr}(k,m)$ with the cell decomposition which consists of even dimensional cells only. Hence, all boundary maps in the chain complex constructed by means of this chain decomposition vanish. Therefore, the closed Schubert cycles $\sigma_\lambda = \overline{\alpha}_\lambda$ form a basis of $\Lambda(k,m) = \bigoplus_i H_i$ over $\mathbb{Z}$. E.g., for the Plc̈ker quadric $P = \mathrm{Gr}(2,\mathbb{C}^4) \subset \mathbb{P}(\mathbb{C}^5)$ of real dimension 8, we have $H_0 = H_2 = H_6 = H_8 = \mathbb{Z}$, $H_4 = \mathbb{Z} \oplus \mathbb{Z}$, and all the homology of odd dimension vanishes. This agrees with Exercise 5.4 on p. 63.

Topological intersection of cycles provides $\Lambda(k,m)$ with the structure of commutative ring closely connected with the ring Λ_m of symmetric polynomials in m variables, which is the polynomial ring $\Lambda_m = \mathbb{Z}[\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m] \subset \mathbb{Z}[x_1,x_2,\dots,x_m]$ generated by the *elementary* symmetric polynomials[1] $\varepsilon_k(x_1,x_2,\dots,x_m)$. Namely, there is the surjective homomorphism of commutative rings $\Lambda_m \twoheadrightarrow \Lambda(k,m)$ sending the *Schur polynomial*[2] $s_\lambda(x_1,x_2,\dots,x_m)$ to the Schubert cycle σ_λ. The kernel ideal of this homomorphism is spanned by *complete* symmetric polynomials[0] $\eta_{m-k+1},\dots,\eta_m$ of degrees in range $[m-k+1,m]$. All known[0] proofs of these statements are indirect and besides the

[1]Recall that ε_n is sum of all *multilinear* monomials of total degree n in $x_1,x_2,\dots,x_m$.

[2]The Schur polynomial $s_\lambda \in \mathbb{Z}[x_1,x_2,\dots,x_m]$ is defined either as the quotient of determinant

$$\Delta_\lambda = \det(x_j^{\lambda_i+m-i}) = \det\begin{pmatrix} x_1^{\lambda_1+m-1} & x_2^{\lambda_1+m-1} & \cdots & x_m^{\lambda_1+m-1}\\ x_1^{\lambda_2+m-2} & x_2^{\lambda_2+m-2} & \cdots & x_m^{\lambda_2+m-2}\\ \vdots & \vdots & \cdots & \vdots\\ x_1^{\lambda_{m-1}+1} & x_2^{\lambda_{m-1}+1} & \cdots & x_n^{\lambda_{m-1}+1}\\ x_1^{\lambda_m} & x_2^{\lambda_m} & \cdots & x_n^{\lambda_m}\end{pmatrix}$$

by the Vandermonde determinant $\Delta_{0,\dots,0}$ or as the sum of all monomials in $x_1,x_2,\dots,x_m$ obtained as follows: fill the cells of diagram λ by (possibly repeated) variables $x_1,x_2,\dots,x_m$ in such a way that indexes strictly increase top-down in columns and non-strictly increase from left to right in rows, then multiply them altogether to one monomial of total degree $|\lambda|$. E.g, for the one-column diagram of height h, we get $s_{1,1,\dots,1} = \varepsilon_h$. The coincidence of two descriptions is non-trivial and known as the *Jacobi–Trudi identity*. For details, see W. Fulton, *Young Tableaux with Applications to Representation Theory and Geometry*, CUP, 1997.

[0]Recall that the complete symmetric polynomial η_n equals the sum of all degree n monomials in $x_1,x_2,\dots,x_m$ at all.

[0]At least, to me.

geometry of grassmannians, use sophisticated combinatorics of symmetric functions. The geometric part of the proof establishes two basic intersection rules:

1) The intersection of cycles σ_λ, σ_μ of complementary codimensions $|\lambda|+|\mu| = k(m-k)$ is not zero if and only if the diagrams λ, μ are complementary[0], and in this case, the intersection consists of one point, that is, equals $\sigma_{k,\dots,k}$.

2) The *Pieri rules*: for any integer n and diagram λ, $\sigma_\lambda\sigma_{(n,0,\dots,0)} = \sum \sigma_\mu$ and $\sigma_\lambda\sigma_{(\underbrace{1,\dots,1}_{n})} = \sum \sigma_\nu$, where μ, ν run through the Young diagrams obtained by adding n cells to λ in such a way that all added cells appear in different rows of μ and in different columns of ν. If there are no such diagrams, the intersection is zero.

The proofs can be found, e.g., in: P. Griffits, J. Harris, *Principles of Algebraic Geometry, I*. It follows from the determinantal definition of Schubert polynomials that they form a basis over $\mathbb{Z}$ in the additive group of symmetric polynomials, because the alternating sums

$$\Delta_\lambda = \det(x_j^{\lambda_i+m-i}) = \sum_{g\in S_m} \mathrm{sgn}(g) x_{g(1)}^{\lambda_1+m-1} x_{g(2)}^{\lambda_2+m-2} \dots x_{g(m)}^{\lambda_m}$$

obviously form a basis in the additive group of alternating polynomials in $x_1, x_2, \dots, x_m$, and dividing by the Vandermonde determinant maps this group isomorphically to the additive group of symmetric polynomials.

EXERCISE 5.15. Show that every alternating polynomial in $x_1, x_2, \dots, x_m$ is divisible by the Vandermonde determinant in the polynomial ring $\mathbb{Z}[x_1, x_2, \dots, x_m]$.

The combinatorial part of the proof verifies that the multiplication of Schur polynomials also satisfies the Pieri rules, which are particular cases of the Littlewood–Richardson rule for multiplying arbitrary Schur polynomials[0]. It is easy to see that the Pieri rules completely determine the multiplicative structure in the both rings Λ_m, $\Lambda(k, m)$. This proves that the map $\Lambda_m \to \Lambda(k, m)$, $s_\lambda \mapsto \sigma_\lambda$, is a well defined surjective homomorphism of rings. The description of its kernel comes from the intersection rule (1) for the Schubert cycles of complementary dimensions. The details of this story, known as the *Schubert calculus*, can be found in the cited book of P. Griffits and J. Harris and in the *Intersection Theory* book of W. Fulton.

EXAMPLE 5.3 (THE INTERSECTION THEORY ON $\mathrm{Gr}(2, 4)$)
As we have seen in Example 5.2, the Schubert cycles on $\mathrm{Gr}(2, 4)$ can be realized as

$\sigma_{10}(\ell) = \{\ell' \subset \mathbb{P}_3 \mid \ell \cap \ell' \neq \varnothing\}$ for a line $\ell \subset \mathbb{P}_3$,
$\sigma_{20}(a) = \{\ell' \subset \mathbb{P}_3 \mid \ell' \ni a\}$ for a point $a \in \mathbb{P}_3$,
$\sigma_{11}(\Pi) = \{\ell' \subset \mathbb{P}_3 \mid \ell' \subset \Pi\}$ for a plane $\Pi \subset \mathbb{P}_3$,
$\sigma_{21}(a, \Pi) = \sigma_{20}(a) \cap \sigma_{11}(\Pi)$ for $a \in \Pi \subset \mathbb{P}_3$,
$\sigma_{22}(\ell) = \{\ell\}$, a line $\ell \subset \mathbb{P}_3$ considered as a point of $\mathrm{Gr}(2, 4)$.

Certainly, $\sigma_{ij}\sigma_{k\ell} = 0$ for $i+j+k+\ell = \operatorname{codim}\sigma_{ij} + \operatorname{codim}\sigma_{k\ell} > 4$. We have seen in Example 5.2 that $\sigma_{20}(a_1) \cap \sigma_{20}(a_2) = \sigma_{22}((a_1a_2))$, $\sigma_{11}(\Pi_1) \cap \sigma_{11}(\Pi_2) = \sigma_{22}(\Pi_1 \cap \Pi_2)$, whereas for $a \notin \Pi$,

[0] That is, can be fitted together without holes and overlaps to assemble $k \times (m-k)$ rectangle.

[0] See already cited W. Fulton's book on Young diagrams, or Sec. 4.5 in: A.L.Gorodentsev, *Algebra II. Textbook for Students of Mathematics*, Springer, 2017. The Pieri rules can be proven independently on the Littlewood–Richardson rule by formal algebraic manipulations with determinants, see, e.g., Section 3.6 of *loc. cit.*

$\sigma_{20}(a) \cap \sigma_{11}(\Pi) = \varnothing$. By the same geometric reasons, for a line ℓ and a plane Π intersecting at a point b, we have $\sigma_{10}(\ell) \cap \sigma_{11}(\Pi) = \sigma_{21}(b, \Pi)$. Dually, for a line ℓ and a point $a \notin \ell$, we have $\sigma_{10}(\ell) \cap \sigma_{20}(a) = \sigma_{21}(a, \Pi)$, where Π is the plane passing trough a and ℓ. Similarly, for a point a in a plane Π, and a line ℓ intersecting Π in a point $b \neq a$, we get $\sigma_{10}(\ell) \cap \sigma_{21}(a, \Pi) = \sigma_{22}((a, b))$. For a generic choice of lines $\ell_1, \ell_2 \subset \mathbb{P}_3$ the intersection $\sigma_{10}(\ell_1) \cap \sigma_{10}(\ell_2)$, which consists of all lines intersecting both ℓ_1, ℓ_2, is the Segre quadric laying in $\mathbb{P}_3 = T_{\mathrm{u}(\ell_1)}P \cap T_{\mathrm{u}(\ell_1)}P$ as it was shown in fig. 5◇1 on p. 62. However, when the lines ℓ_1, ℓ_2 are intersecting but still different, the intersection $\sigma_{10}(\ell_1) \cap \sigma_{10}(\ell_2)$ splits in the union of the α-net $\sigma_{20}(a)$ centered at the intersection point $a = \ell_1 \cap \ell_2$ and the β-net $\sigma_{11}(\Pi)$, where Π is the plane containing ℓ_1, ℓ_2. Since the integer homology classes of all cycles just mentioned are not changed under continuous moving of the points, lines, and planes in $\mathbb{P}_3$ used to construct the realizations of these cycles within $\mathrm{Gr}(2, 4)$, we conclude that nonzero products of the Schubert cycles in $\mathrm{Gr}(2, 4)$ are exhausted by

$$\sigma_{10}^2 = \sigma_{20} + \sigma_{11}\,, \quad \sigma_{10}\sigma_{20} = \sigma_{10}\sigma_{11} = \sigma_{21}\,, \quad \sigma_{10}\sigma_{21} = \sigma_{20}^2 = \sigma_{11}^2 = \sigma_{22}\,,$$

and $\sigma_{00}\sigma_{ij} = \sigma_{ij}$ for all Young diagrams (ij) went in the square 2×2. As a byproduct, we get a «topological» solution of Exercise 2.14 on p. 24: for a generic choice of 4 mutually non-intersecting lines in $\mathbb{P}_3$, the set of lines intersecting them all represents the homology class of topological fourfold self-intersection $\sigma_{10}^4 = (\sigma_{20} + \sigma_{11})^2 = \sigma_{20}^2 + \sigma_{11}^2 = 2\sigma_{22}$, that is, consists of two lines.

Comments to some exercises

EXRC. 5.2. (Comp. with general theory from n° 2.6 on p. 24.) The cone $C = P \cap T_p P$ consist of all lines passing through p and laying on P. On the other hand, it consists of all lines joining its vertex p with a smooth quadric $G = C \cap H$ cut out of C by any 3-dimensional hyperplane $H \subset T_p P$ complementary to p inside $T_p P \simeq \mathbb{P}_4$. Thus, any line on P passing through p has a form $(pp') = \pi_\alpha \cap \pi_\beta$, where $p' \in G$ and π_α, π_β are two planes spanned by p and two lines laying on the Segre quadric G and passing through p' (see fig. 5⋄1 on p. 62).

EXRC. 5.4. See n° 5.3.3 on p. 68.

EXRC. 5.5. If $\omega \in P$, then $Z = T_\omega P$ and $\omega = \mathfrak{u}(\ell)$ for some lagrangian line $\ell \subset \mathbb{P}(V)$. Then all lines in $\mathbb{P}_3$ intersecting ℓ have to be lagrangian as well. This forces Ω to be degenerated.

EXRC. 5.6. The relations $w = e \cdot A_w^t$, $u = e \cdot A_u^t$, $w = u \cdot C_{uw}$, where e, u, w are the row matrices whose elements are the corresponding basis vectors, force $A_w^t = A_u^t C_{uw}$.

EXRC. 5.7. See Example 4.3 on p. 47.

EXRC. 5.8. Use the Plücker relation (4-47) on 58 and appropriate congruence reasons avoiding the complete enumeration of 720 matchings between a_{ij} and the given 6 numbers.

EXRC. 5.15. Since an alternating polynomial, considered as a polynomial in x_j with coefficients in the polynomial ring on the remaining variables, has the root $x_j = x_i$, it is divisible by $(x_i - x_j)$ for all $i \neq j$.