PROGRAM OF THE EXAM

1. **Linear algebra:** ▷Generators, relations and free resolutions. ▷Theorem on compatible basises for finitely generated $\mathbb{Z}$-modules: for any submodule N of a finitely generated free $\mathbb{Z}$-module M there are $z_i \in \mathbb{N}$ and a basis $\{e_i\}$ for M such that $z_i | z_j$ for $i < j$ and $\{z_i \cdot e_i\}$ form a free basis for N. ▷Representation of module homomorphisms by matrices. ▷Image of multiplication by the determinant of a matrix is contained in the image of multiplication by this matrix.

2. **Tensor algebra:** ▷Tensor products of vector spaces, modules, and algebras. ▷Basis for the tensor product of vector spaces. ▷Tensor product of two modules presented by generators $\{e_i\}$ and $\{f_j\}$, which satisfy, respectively, the linear relations spanned by $\{\varrho_\mu(e)\}$ and $\{\sigma_\nu(f)\}$, is generated by $e_i \otimes f_j$, which satisfy the linear relations spanned by $\{e_i \otimes \sigma_\nu(f)\}$ and $\{\varrho_\mu(e) \otimes f_j\}$. ▷Tensor algebra of a vector space, contractions and duality. ▷Linear span of a tensor. ▷Symmetric and exterior algebra of a vector space, polarizations, contractions and partial derivatives of (skew) polynomials (in zero characteristics). ▷Linear span of a (skew) polynomial, Plücker relations for decomposable Grassmannian polynomials and quadratic relations for symmetric homogeneous polynomials that are pure powers of linear forms. ▷Laplace determinantal identities and Taylor expansion for $\det(\lambda A + \mu B)$

3. **Finitely generated algebras over a field:** ▷Integer elements and integer extensions of rings. ▷If $A \subset B$ is an integer extension and B is an integer domain, then A is a field $\iff$ B is a field. ▷Gauss – Kronecker – Dedekind lemma. ▷If a finitely generated algebra A over a field k is a field, then A is finite dimensional as a vector space over k and each $a \in A$ is algebraic over k.

4. **Polynomial ideals:** ▷Noetherian rings, Hilbert's base theorem. ▷Hilbert's Nullstellensatz. ▷Homogeneous ideals and the resultant equations for a system of homogeneous polynomials. ▷Minimal graded free resolution of a homogeneous ideal. ▷Hilbert theorem about the syzygies.

5. **Categories and functors:** ▷Functors and their natural transformations; category of functors. ▷(Co) representable functors and universal constructions. ▷Fibered products and coproducts. ▷Projective and inductive limits; inductive limits in the category of sets; localization $A[f^{-1}]$ as the inductive limit of morphisms of free rank 1 modules $A \cdot \frac{1}{f^n} \xhookrightarrow{x \mapsto x/f^{m-n}} A \cdot \frac{1}{f^m}$; p-adic integers $\mathbb{Z}_p$ as the projective limit of reductions $\mathbb{Z}/p^m\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$. ▷Additive and Abelian categories; direct sums; (co) kernels. ▷Commutative diagram searching technique; 5 homomorphisms lemma; snake lemma.

6. **Complexes and homologies:** ▷Koszul sign rules in graded tensor calculus. ▷Exact triple of complexes leads to a long exact sequence of their homologies. ▷Hom-complex of two complexes, homotopies and homotopy equivalence. ▷Tensor product of two complexes. ▷Bicomplexes and their total complexes. ▷Spectral sequence of a filtered complex. ▷Two spectral sequences associated with a bicomplex. ▷Spectral sequences via exact coupled pairs.

7. **Resolutions of modules and classical derived functors:** ▷Projective modules and projective resolutions. ▷Uniqueness of a projective resolution up to a homotopy. ▷Injective modules; injectivity of $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$; injective modules over an arbitrary ring A; I is injective iff for any ideal $J \subset A$ and any A-linear map $J \xrightarrow{\varphi} I$ there is $m \in I$ such that $\varphi(x) = xm\ fainxJ$. ▷Injective resolutions: existance and niqueness up to a homotopy. ▷Functors $\mathrm{Tor}_i(M, N)$ and $\mathrm{Ext}^i(M, N)$; equivalence of 3 definitions: via resolving M, via resolving N, and via resolving the both. ▷Cartan – Eilenberg resolutions for a complex. ▷How to compute $\mathrm{Tor}_i(M, N)$ and $\mathrm{Ext}^i(M, N)$ using non-canonical acyclic resolutions.

8. Koszyl complexes: ▷Koszul complexes of a vector space and Koszul complex associated with a sequence of ring elements. ▷Exactness of a Koszul complex associated with a regular sequence. ▷Čhech complex[1] of a principal open covering as an injective limit of Koszul complexes; Čhech cohomologies of invertible sheaves on $\mathbb{P}_n$.

9. Sheaves[2]: ▷Presheaves, stalks and sheaves. ▷Vector bundles and locally trivial $\mathscr{O}$-modules. ▷Line bundles and the Picard group; Picard groups of a factorial affine variety, of the projective spaces, and of the Grassmannians.

REAL EXAMPLES OF THE EXAM PROBLEMS (GIVEN IN THE PREVIOUS YEARS)

1. Show that the index of a subgroup spanned in $\mathbb{Z}^n$ by n linearly independent over $\mathbb{Q}$ vectors $\{e_i\}$ is equal to a volume of the parallelepiped spanned by e_i's.

2 (Sylvester's determinantal identity). Let A, B be two $n \times n$ - matrices. Fix some set K of k columns in A. Show that $\det A \cdot \det B = \sum \det A' \cdot \det B'$, where A', B' are obtained by order preserving interchanging K-columns in A with some set K' of k columns in B and the sum is running over all choices of this K'.

3. Show that any finite dimensional (as a vector space) algebra over an arbitrary field has only a finite set of prime ideals and all these ideals are maximal.

4 (local rings). Let a Noetherian ring A have a unique proper maximal ideal $\mathfrak{m} \subset A$. Show that:
- **a)** $\mathfrak{m}M \neq M$ for any non zero finitely generated A-module M;
- **b)** if (mod $\mathfrak{m}^2$)-classes of $e_1, e_2, \dots, e_n \in \mathfrak{m}$ span a vector space $\mathfrak{m}/\mathfrak{m}^2$ over $k = A/\mathfrak{m}$, then e_i's span $\mathfrak{m}$ over A;
- **c)** any finitely generated projective A-module is free.

5. Construct a minimal graded free resolution for the homogeneous ideal of Veronese's twisted cubic curve $C \subset \mathbb{P}_3$.

6. Let principal open sets $\mathscr{D}(f_\nu)$, $\nu = 1, \dots, n$, cover an affine (maybe reducible) algebraic variety $X = Spec_m A$. Show that the Koszul complex $\bigotimes_{\nu=1}^{n} \left(A \xrightarrow{x \mapsto f_\nu x} A\right)$ is exact.

7. On the set $\mathbb{N}$, of positive integers, consider a partial ordering given by the divisibility relation: $n \preceq m \overset{\text{def}}{\iff} n|m$. In the category of Abelian groups compute the injective limit $\varinjlim \mathbb{Z}/n\mathbb{Z}$ w.r.t. a direct system of arrows $\mathbb{Z}/n\mathbb{Z} \xhookrightarrow{[1] \mapsto [m/n]} \mathbb{Z}/m\mathbb{Z}$ given for any ordered pair $n \preceq m$.

8. Let $\mathfrak{m}, I \subset \mathbb{C}[x,y,z]$ be the ideals spanned by $(x, y.z)$ and x respectively. In the category of finitely generated $\mathbb{C}[x,y,z]$-modules compute all $\operatorname{Ext}^\nu(\mathfrak{m}, I)$ and all $\operatorname{Tor}_\nu(\mathfrak{m}, I)$.

9. For ordered $I = (i_0, i_1, \dots, i_k) \subset (0, 1, \dots, n)$ let $x_I = \prod\limits_{i \in I} x_i \in k[x_0, x_1, \dots, x_n]$. We write A_I for an algebra of fractions $k[x_0, x_1, \dots, x_n][x_I^{-1}]$ and consider a direct sum $C^k \overset{\text{def}}{=} \bigoplus\limits_{\sharp I = k} A_I$ (so, elements of C^k are the collections $f = \{f_{(i_0, i_1, \dots, i_k)}\}$ of $f_{(i_0, i_1, \dots, i_k)} \in A_{(i_0, i_1, \dots, i_k)}$). Define a map $C^k \xrightarrow{d} C^{k+1}$ by prescription: $(df)_J = \sum\limits_{\nu=0}^{k+1} (-1)^\nu f_{J \setminus \{j_\nu\}}$ (it is well defined, because all $A_{J \setminus \{j_\nu\}} \subset A_J$). Show that $0 \longrightarrow C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \cdots \xrightarrow{d} C^n \longrightarrow 0$ is a complex of (infinite dimensional) vector spaces over k with non zero cohomologies exhausted by $H^0 = k[x_0, x_1, \dots, x_n] \subset C^0$ and H^n, whose basis over k consists of (mod $\operatorname{im} d$)-classes of all monomials $x_0^{m_0} x_1^{m_1} \dots x_n^{m_n} \in A_{(0,1,\dots,n)}$ that have all $m_i < 0$.

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[1] all this topic is not necessary for passing the exam

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