Classical derived functors.

7◇1. Compute all $Tor_\nu(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ in the category of finitely generated $\mathbb{Z}-$*modules* for the following choices of $m, n \in \mathbb{Z}$:

a) m, n are coprime

b) $m = p^r$, $n = p^s$, where $p \geqslant 2$ is prime

c) $m = m' \cdot \prod_{\nu=1}^{k} p_\nu^r$, $n = n' \cdot \prod_{\nu=1}^{k} p_\nu^r$, where m', n' are coprime and all $p_\nu \geqslant 2$ are distinct, prime, and coprime with m', n'

7◇2. Let V be a vector space over a field k of zero characteristic, $S = \oplus S^n V$ be its symmetric exterior algebra, F be a free S-module of rank 1. Compute

a) $Ext^\nu(F, F)$ **b)** $Ext^\nu(F, k)$ **c)** $Ext^\nu(k, F)$ **d)** $Ext^\nu(k, k)$

e) $Tor_\nu(F, F)$ **f)** $Tor_\nu(F, k)$ **g)** $Tor_\nu(k, k)$

in the category of S-modules (where k is considered as S-module on which all $v \in V$ act by the zero operators).

7◇3. Let V be a vector space over a field k of zero characteristic, $\Lambda = \oplus \Lambda^n V$ be its exterior algebra, F be a free left Λ-module of rank 1. Compute

a) $Ext^\nu(F, F)$ **b)** $Ext^\nu(F, k)$ **c)** $Ext^\nu(k, F)$ **d)** $Ext^\nu(k, k)$

e) $Tor_\nu(F, F)$ **f)** $Tor_\nu(F, k)$ **g)** $Tor_\nu(k, k)$

in the category of left Λ-modules (where k is considered as Λ-module on which all $v \in V$ act by the zero operators).

7◇4 (bar-construction). Let A be an algebra with unity (possibly, non commutative) over a field k. Write $\overline{A}$ for a vector space A/k, where k is embedded in A as $k \cdot 1_A$, and write $\overline{a}$ for a class of $a \in A$ in $\overline{A}$. For each $k \geqslant 0$ consider a vector space $B_m \overset{\text{def}}{=} A \underset{k}{\otimes} \overbrace{\overline{A} \underset{k}{\otimes} \cdots \underset{k}{\otimes} \overline{A}}^{m} \underset{k}{\otimes} A$ (the tensor product of $m+2$ vector spaces over k) and define a differential $B_m \xrightarrow{d} B_{m-1}$ by prescription

$$\begin{aligned} d(a_0 \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_m \otimes a_{m+1}) = a_0 a_1 \otimes \overline{a}_2 \otimes \cdots \otimes \overline{a}_m \otimes a_{m+1} + \\ + \sum_{\nu=1}^{m-1} (-1)^\nu a_0 \otimes \cdots \otimes \overline{a_\nu a_{\nu+1}} \otimes \cdots \otimes a_{m+1} + \\ + (-1)^m a \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_{m-1} \otimes a_m a_{m+1} \end{aligned}$$

a) Show that d is well defined and satisfies $d^2 = 0$.

b) Compute the homologies of this complex.

7◇5. Construct a free resolution of a homogeneous ideal $I \subset k[\, a_0, a_1, \ldots, a_n \,]$ spanned by quadratic equations[1] coming from the matrix relation

$$\operatorname{rk} \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \\ \vdots & \vdots \\ a_{n-1} & a_n \end{pmatrix} = 1$$

at least for $n = 3, 4$.

[1]these are equations for the Veronese curve $\mathbb{P}_1 \hookrightarrow \mathbb{P}_n$