Spectral sequences

6◇1 (coupled pairs). An exact triangle of module homomorphisms
$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$
is called *a coupled pair* and denoted by (D,E,i,j,k). Show that $d = jk : E \to E$ satisfies $d^2 = 0$ and a triangle
$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$
with $D' = \operatorname{im} i \subset D$, $E' = \ker d / \operatorname{im} d$, $i' = i|_{\operatorname{im} i}$, $j' : i(x) \longmapsto j(x)$, $k' : x \pmod{\operatorname{im} d} \longmapsto k(x)$ is well defined and exact, i. e. (D',E',i',j',k') is a coupled pair as well (it is called *the derived pair* of (D,E,i,j,k)).

6◇2. Let (D_r,E_r,i_r,j_r,k_r) be $(r-1)$-th successive derivative of a coupled pair (D_1,E_1,i_1,j_1,k_1). Show that $D_r = \operatorname{im} i_1^{r-1}$, $E_r = k_1^{-1}(\operatorname{im} i_1^{r-1})/j_1(\ker i_1^r)$ and there is an exact sequence
$$0 \to \operatorname{im} i_1^{r-1}/\operatorname{im} i_1^r \to E_r \to \ker i_1^r / \ker i_1^{r-1} \to 0$$

Bigraded coupled pairs & spectral sequences. A coupled pair (D,E,i,j,k) is called *upper-style bigraded*, if $D = \bigoplus\limits_{p,q\in\mathbb{Z}} D^{p,q}$, $E = \bigoplus\limits_{p,q\in\mathbb{Z}} E^{p,q}$ and $\deg i = (-1,1)$, $\deg j = (0,0)$, $\deg k = (1,0)$, that is
$$\begin{aligned} i &= \oplus i^{(p,q)}\,, \text{ where } i^{(p,q)} : D^{p,q} \to D^{p-1,q+1} \\ j &= \oplus j^{(p,q)}\,, \text{ where } j^{(p,q)} : D^{p,q} \to E^{p,q} \\ k &= \oplus k^{(p,q)}\,, \text{ where } k^{(p,q)} : E^{p,q} \to D^{p+1,q} \end{aligned}$$

Similarly, a *bottom-style bigraded* coupled pair (D,E,i,j,k) has $D = \bigoplus\limits_{p,q\in\mathbb{Z}} D_{p,q}$, $E = \bigoplus\limits_{p,q\in\mathbb{Z}} E_{p,q}$ and homogeneous morphisms of degrees $\deg i = (1,-1)$, $\deg j = (0,0)$, $\deg k = (-1,0)$. These two grading styles go to each other by changing signs of the both indices. We will use upper-style notations in what follows. Let (D_r,E_r,i_r,j_r,k_r) with $r = 1, 2, \dots$ be a sequence of successive derivatives for an initial bigraded coupled pair $(D,E,i,j,k) = (D_1,E_1,i_1,j_1,k_1)$. For each r, we write direct summands $E_r^{p,q}$, of E_r, into the cells of an infinite rectangular table and number these cells by the pairs $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ (taking p as the horizontal coordinate). The sequence of these tables E_1, E_2, E_3, $\dots$ is called *a spectral sequence* of bigraded coupled pair (D,E,i,j,k).

6◇3. Show that r-th table E_r carries graded differential $d_r^{(p,q)} = (j_r \circ k_r)^{(p,q)} : E_r^{p,q} \to E_r^{p+r,q+1-r}$ whose cohomologies $E_{r+1}^{p,q} = \ker d_r^{(p,q)} / \operatorname{im} d_r^{p-r,q-1+r}$ coincide with the entries of $(r+1)$-th table (see the diagram (3) below).

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r, "k"'] & D^{p-1,q+2} \arrow[r, "j"] & E^{p-1,q+2} \arrow[r, "k"'] & D^{p,q+2} \arrow[r, "j"] & E^{p,q+2} \arrow[r, "k"'] & D^{p+1,q+2} \arrow[r, "j"] & E^{p+1,q+2} \arrow[r, "k"'] & D^{p+2,q+2} \arrow[r, "j"] & \cdots \\
\cdots \arrow[r, "k"'] & D^{p-1,q+1} \arrow[r, "j"] & E^{p-1,q+1} \arrow[r, "k"'] & D^{p,q+1} \arrow[r, "j"] \arrow[ull, "i"] & E^{p,q+1} \arrow[r, "k"'] & D^{p+1,q+1} \arrow[r, "j"] \arrow[ull, "i"] & E^{p+1,q+1} \arrow[r, "k"'] & D^{p+2,q+1} \arrow[r, "j"] \arrow[ull, "i"] & \cdots \\
\cdots \arrow[r, "k"'] & D^{p-1,q} \arrow[r, "j"] & E^{p-1,q} \arrow[r, "k"'] & D^{p,q} \arrow[r, "j"] \arrow[ull, "i"] & E^{p,q} \arrow[r, "k"'] & D^{p+1,q} \arrow[r, "j"] \arrow[ull, "i"] & E^{p+1,q} \arrow[r, "k"'] & D^{p+2,q} \arrow[r, "j"] \arrow[ull, "i"] & \cdots \\
\cdots \arrow[r, "k"'] & D^{p-1,q-1} \arrow[r, "j"] & E^{p-1,q-1} \arrow[r, "k"'] & D^{p,q-1} \arrow[r, "j"] \arrow[ull, "i"] & E^{p,q-1} \arrow[r, "k"'] & D^{p+1,q-1} \arrow[r, "j"] \arrow[ull, "i"] & E^{p+1,q-1} \arrow[r, "k"'] & D^{p+2,q-1} \arrow[r, "j"] \arrow[ull, "i"] & \cdots
\end{tikzcd} \tag{3}$$

6◇4. Let $E_1^{p,q}$ vanish for all $q \ll 0$ (uniformly in p) and for all $p \ll 0$ (uniformly in q).

a) What does this mean in terms of $D^{p,q}$ with fixed $(p+q)=n$ and $p \ll 0$ and $p \gg 0$?

b) Show that $\forall\, n\ \exists\, N$ such that $E_r^{p,q} = E_{r+1}^{p,q}$ for all $r > N$ and all p, q with $p+q=n$ (these stabilized terms are called *limit terms* and denoted by $E_\infty^{p,q}$).

c) Describe $E_\infty^{p,q}$ and $\bigoplus\limits_{p+q=n} E_\infty^{p,q}$ in terms of the map $D \xrightarrow{i} D$.

Limits. We say that a spectral sequence $E_r^{p,q}$ converges to E_∞^n and write $E_r^{p,q} \Rightarrow E_\infty^n$, if two conditions hold:

1. $E_r^{p,q}$ becomes stabilize, i.e. $\forall\, p, q\ \exists\, N = N(p,q)$ such that both outgoing and incoming differentials at (p,q)-cell vanish in all terms E_r with $r > N$; then the stable entry $E_{N+1}^{p,q} = E_{N+2}^{p,q} = \dots$ is denoted by $E_\infty^{p,q}$
2. E_∞^n has a decreasing filtration $F^p E_\infty^n$ whose factors coincide with stable $E_\infty^{p,n-p}$, i.e. $E_\infty^n = \bigcup\limits_p F^p E_\infty^n$, where $\cdots \supset F^{p-1}E_\infty^n \supset F^p E_\infty^n \supset F^{p+1}E_\infty^n \supset \cdots$ and $F^p E_\infty^n / F^{p+1} E_\infty^n = E_\infty^{p,n-p}$

6◇5 (filtered complexes). Let each term K^m, of a complex $\cdots \xrightarrow{d} K^m \xrightarrow{d} K^{m+1} \xrightarrow{d} \cdots$, have a decreasing filtration $K^m = F^0K^m \supset F^1K^m \supset F^2K^m \supset \cdots$ such that $d(F^pK^m) \subset F^pK^{m+1}$ for all p, m, that is, for each p there is a complex $F^pK^\bullet$ with the same differential d.

a) Show that for each p there is a factor complex $G^pK^\bullet$ whose m'th term equals $F^pK^m/F^{p+1}K^m$ and the differential is induced by d.

b) For each p, construct a long exact sequence of cohomologies
$$\cdots\ H^i(F^{p+1}K^\bullet) \to H^i(F^pK^\bullet) \to H^i(G^pK^\bullet) \to H^{i+1}(F^{p+1}K^\bullet) \to \ \cdots$$

c) Construct a bigraded coupled pair with $D_1^{p,q} = H^{p+q}(F^pK^\bullet)$, $E_1^{p,q} = H^{p+q}(G^pK^\bullet)$ and describe $E_r^{p,q}$-terms of all its successive derivatives.

d) Show that if the complex is bounded and all the filtrations are uniformly bounded, then there exists a spectral sequence $E_r^{p,q} \Rightarrow H^n(K^\bullet)$ with $E_1^{p,q} = H^{p+q}(G^pK^\bullet)$.

6◇6 (double complexes). A *a double complex* is a bigraded object $K^{\bullet\bullet} = \oplus K^{p,q}$ equipped with two differentials $K^{p,q} \xrightarrow{d_1} K^{p+1,q}$, $K^{p,q} \xrightarrow{d_2} K^{p,q+1}$ that satisfy $d_1^2 = d_2^2 = 0$ and supercommute, i.e. $d_1d_2 + d_2d_1 = 0$. Show that: **a)** objects $Tot^n(K) \overset{\text{def}}{=} \bigoplus\limits_{p+q=n} K^{p,q}$ equipped with the maps $Tot^n(K) \xrightarrow{d_1+d_2} Tot^{n+1}(K)$ form a complex (it is called *a total complex* or *a contraction* of the double complex $K^{\bullet\bullet}$)

b) the complex $Tot^\bullet(K)$ has two decreasing filtrations:
$$^{I}F^p\,Tot^m(K) = \bigoplus_{\substack{\nu \geqslant p\\ p+q=n}} K^{p,q} \qquad \text{and} \qquad {}^{II}F^q\,Tot^m(K) = \bigoplus_{\substack{\nu \geqslant q\\ p+q=n}} K^{p,q}$$

c) p-th factor complex $^{I}G^p\,Tot^\bullet(K) = \bigoplus\limits_{\substack{\nu \geqslant p\\ p+q=n}} K^{p,q}$, of the first filtration, coincides with the p-th column of the bicomplex $K^{\bullet\bullet}$ considered as a complex w.r.t. the differential d_2 (we'll write $H_{d_2}^q(K^{p,\bullet})$ for its q-th cohomology)

d) the differentials d_1 induce well defined morphisms $H_{d_2}^q(K^{p,\bullet}) \xrightarrow{d_1} H_{d_2}^q(K^{p+1,\bullet})$ between these cohomologies, that is, produce a complex of them (we'll write $H_{d_1}^p(H_{d_2}^q(K))$ for its p-th cohomology)

e) prb. 6◇5 leads to a spectral sequence $^{I}E_r^{p,q} \Rightarrow H^n(Tot^\bullet(K))$ with $^{I}E_2^{p,q} = H_{d_1}^p(H_{d_2}^q(K))$

f) there is the similar spectral sequence $^{II}E_r^{p,q} \Rightarrow H^n(Tot^\bullet(K))$ with $^{II}E_2^{p,q} = H_{d_2}^q(H_{d_1}^p(K))$ coming from the second filtration on $Tot^\bullet(K)$

6◇7. Produce the both spectral sequences of a double complex from appropriate coupled pairs without any usage of prb. 6◇5

6◇8. Let a complex $K_\bullet$ have only one non zero homology $H_0(K_\bullet) = M$ and let a module N be such that $Ext^\nu(K_i, N) = 0$ for all i and all $\nu \geqslant 1$. Using a Cartan - Eilenberg resolution for $K_\bullet$, show that $Ext^m(M,N)$ coincides with m-th cohomology group of the complex $\mathrm{Hom}(K_\bullet, N)$.