Some canonical constructions.

5◇1. Let $\mathcal{N}$ be an arbitrary partially ordered set considered as a category[1]. Show that any family of $\mathcal{N}$-indexed sets $\mathcal{N} \xrightarrow{X} \mathcal{S}et$ has $\varinjlim X$.

HINT. A right queue of X is a sequence of elements $x_\nu \in X_\nu$ indexed by some $S \subset \operatorname{Ob}\mathcal{N}$ such that all $\mu > \nu$ belong to S as soon as $\nu \in S$ and $\varphi_{\mu\nu}(x_\mu) = x_\nu \ \forall \mu, \nu \in S$. Two right queues $\{x_\alpha\}$, $\{y_\beta\}$ are called equivalent, if $\forall\, x_\alpha, y_\beta \ \exists\, \gamma > \alpha, \beta : \varphi_{\alpha\gamma}(x_\alpha) = \varphi_{\beta\gamma}(y_\beta)$. Check that a set of all equivalence classes of right queues satisfies the universal properties defining $\varinjlim X$.

5◇2. Consider the poset $\mathcal{N} = \mathbb{N}$ with its standard order and the system of Abelian groups $A_n = \mathbb{Z}/p^n\mathbb{Z}$. Describe

a) $\varprojlim A_n$ w. r. t. inverse system of factorizations $\psi_{nm} : \mathbb{Z}/p^n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z}$ (defined $\forall\, m < n$).

b) $\varinjlim A_n$ w. r. t. direct system of inclusions $\varphi_{mn} : \mathbb{Z}/p^m\mathbb{Z} \xhookrightarrow{[1]\mapsto[p^{n-m}]} \mathbb{Z}/p^n\mathbb{Z}$ (again $\forall\, m < n$).

HINT. $\varprojlim A_n = \mathbb{Z}_p$ is the set of all p-adic integers and $\varinjlim A_n \subset \mathbb{Q}/\mathbb{Z}$ consists of (mod $\mathbb{Z}$)-classes of fractions z/p^ℓ whose denominator is a power of p (so called p-rational numbers).

5◇3. Consider the poset $\mathcal{N} = \mathbb{N}$ with the partial ordering prescribed by the divisibility and Abelian groups $A_n = \mathbb{Z}/n\mathbb{Z}$. Describe:

a) $\varprojlim A_n$ w. r. t. inverse system of factorizations $\psi_{nm} : \mathbb{Z}/n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}$ (defined $\forall\, m|n$).

b) $\varinjlim A_n$ w. r. t. direct system of inclusions $\varphi_{mn} : \mathbb{Z}/m\mathbb{Z} \xhookrightarrow{[1]\mapsto[n/m]} \mathbb{Z}/n\mathbb{Z}$ (again $\forall\, m|n$).

HINT. $\varinjlim A_n = \mathbb{Q}/\mathbb{Z}$ and $\varprojlim A_n = \prod\limits_p \mathbb{Z}_p$ is the product of all rings of p-adic integer numbers (so called *completion* of $\mathbb{Z}$).

5◇4 (projective and injective objects). An object P of an Abelian category $\mathcal{A}$ is called *projective*, if any arrow $P \xrightarrow{\psi} C$ (started from P), can be lifted along any *epi*morphism $B \overset{\beta}{\twoheadrightarrow} C$ (i. e. $\exists$ $P \xrightarrow{\varphi} B$ such that $\beta\circ\varphi = \psi$). Dually, an object I of $\mathcal{A}$ is called *injective*, if any arrow $M \xrightarrow{\varphi} I$ (ended at I) can be extended along any *mono*morphism $M \xhookrightarrow{\alpha} N$ (i. e. $\exists$ $N \xrightarrow{\psi} I$ such that M coincides with $\psi\circ\alpha = \varphi$). Show that:

a) P is projective iff the functor $\operatorname{Hom}(P\,,\,*)$ preserves exactness of any exact sequence of morphisms;

b) I is injective iff the functor $\operatorname{Hom}(*\,,\,I)$ preserves exactness of any exact sequence of morphisms.

5◇5 (projective modules). Show that the following conditions on a module P over a commutative ring K with unity are pairwise equivalent:

a) P is projective; **b)** there exists a module Q such that $P \oplus Q$ is free;

c) any epimorphism $F \overset{\pi}{\twoheadrightarrow} P$, from a free module F onto P, admits a section, i. e. a homomorphism $P \xrightarrow{\varphi} F$ such that $\pi\circ\varphi = \mathrm{Id}_P$.

5◇6. Show that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules.

HINT. Use transfinite induction (or the Zorn lemma): take maximal w. r. t. inclusions submodule $N \subset M' \subset M$ such that $N \xrightarrow{\varphi} \mathbb{Q}$ can be extended onto M'. If there is $m \notin M'$, then a lager submodule $M'' \supset M'$, which is spanned by m and M', can have only one new relation besides ones coming from M' and this relation looks like $zm = m'$ for some $z \in \mathbb{Z}, m' \in M'$; use this to extend φ onto M''.

5◇7. For any $\mathbb{Z}$-module M and any non zero element $m \in M$ construct some $\mathbb{Z}$-module homomorphism $M \xrightarrow{\varphi} \mathbb{Q}/\mathbb{Z}$ such that $\varphi(m) \neq 0$.

5◇8. For any commutative ring K with unity, let $E_K = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ be Abelian group of all $\mathbb{Z}$-linear[2] homomorphisms considered as K-module w. r. t. the action $\varphi \overset{x}{\longmapsto} x\varphi$, of $x \in K$, defined by prescription $x\varphi(y) \overset{\text{def}}{=} \varphi(xy)$. Show that:

[1] a partially ordered set considered as a category is usually abbreviated as *poset*

[2] i. e. preserving only the Abelian group structure

a) E_K is injective K-module;

b) for any non zero element $m \in M$ of any K-module M there is K-module homomorphism $M \xrightarrow{\varphi} E_K$ such that $\varphi(m) \neq 0$;

c) a direct product $M_I \stackrel{\text{def}}{=} \prod_{\varphi} [E_K]_\varphi$ (of distinct copies of E_K labelled by all non zero K-module homomorphisms $M \xrightarrow{\varphi} E_K$) is injective for any K-module M and contains M as a submodule.

5◇9 (projective resolutions). A complex of projective objects

$$\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \longrightarrow 0$$

(maybe unbounded from the left side) is called *a projective resolution* of an object M, if $H_0(P_\bullet) = M$ and all other homologies vanish[3]. Let $P_\bullet$, $Q_\bullet$ be some projective resolutions for objects M, N respectively. Show that:

a) for any morphism $M \xrightarrow{\overline{\varphi}} N$ there exists a morphism of complexes $P_\bullet \xrightarrow{f} Q_\bullet$ that induces $\overline{\varphi}$ on the homology;

b) this morphism is unique up to a homotopy;

c) if $M = N$, then $P_\bullet$ and $Q_\bullet$ are *homotopy equivalent* (i. e. there are two morphisms of complexes $P_\bullet \xrightarrow{\alpha} Q_\bullet$, $Q_\bullet \xrightarrow{\beta} P_\bullet$ such that both $\alpha\beta - \mathrm{Id}_{Q_\cdot}$ and $\beta\alpha - \mathrm{Id}_{P_\cdot}$ are zero homotopic).

5◇10 (injective resolutions). A complex of injective objects

$$0 \longrightarrow M \xrightarrow{d} I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \longrightarrow \cdots$$

(maybe unbounded from the right side) is called *an injective resolution* of an object M, if $H^0(I^\bullet) = M$ and all other cohomologies vanish[4]. Let $I^\bullet$, $J^\bullet$ be some injective resolutions for objects M, N respectively. Show that:

a) for any morphism $M \xrightarrow{\overline{\varphi}} N$ there exists a morphism of complexes $I^\bullet \xrightarrow{f} J^\bullet$ that induces $\overline{\varphi}$ on the homology;

b) this morphism is unique up to a homotopy;

c) if $M = N$, then $I^\bullet$, $J^\bullet$ are *homotopy equivalent* (i. e. there are two morphisms of complexes $I^\bullet \xrightarrow{\alpha} J^\bullet$, $J^\bullet \xrightarrow{\beta} I^\bullet$ such that both $\alpha\beta - \mathrm{Id}_{J^\cdot}$ and $\beta\alpha - \mathrm{Id}_{I^\cdot}$ are zero homotopic).

5◇11 (resolutions of modules). Show that any module M over an arbitrary commutative ring with unity has both projective and injective resolutions.

5◇12 (Cartan – Eilenberg resolutions). For any complex of modules $M^\bullet$ construct two double complexes: **a)** $I^{\bullet,\bullet}$, of injective modules, and **b)** $P^\bullet_\bullet$, of projective modules, (with bi-grading situated in upper half plane of $\mathbb{Z}^2$) such that their rows are exact and their columns give injective (resp. projective) resolutions for the modules $M^\bullet$. (The total complexes associated with these double complexes are called injective (resp. projective) *Cartan – Eilenberg resolutions* for the complex $M^\bullet$.)

HINT. Start with the case where $M^\bullet$ is an *exact triple* $0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M_2 \longrightarrow 0$ and then reduce generic case to this by cutting a generic (non exact) complex into a series of concatenatable exact triples.

5◇13*. Show that:

a) each morphism of complexes can be extended to a morphism of their Cartan - Eilenberg resolutions;

b) this extension is unique up to a homotopy;

c) any two Cartan – Eilenberg resolutions of a given complex are homotopy equivalent.

5◇14*. Let φ, ψ be two morphisms of complexes of **a)** injective **b)** projective objects. Is it true that their induced maps $\overline{\varphi}$, $\overline{\psi}$ on homologies coincide iff φ and ψ are homotopic?

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[3]this means that we have an *exact* sequence $\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{d} M \longrightarrow 0$

[4]this means that we have an *exact* sequence $0 \longrightarrow M \xrightarrow{d} I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \longrightarrow \cdots$