Complexes and their (co)homologies

(all the problems below are about modules over a commutative ring k)

Notations. A sequence of module homomorphisms $\cdots \xrightarrow{d^{i-1}} V^i \xrightarrow{d^i} V^{i+1} \xrightarrow{d^{i+1}} \cdots$ (shortly denoted as $(V^\bullet, d^\bullet)$) is called *a complex*, if $\ker(d^i) \supset \mathrm{im}\,(d^{i-1})$ for all i. So, d satisfies $d^2 = 0$; it is called *a differencial*. The factors $H^i(V^\bullet) \stackrel{\text{def}}{=} \ker(d^i)/\mathrm{im}\,(d^{i-1})$ are called *cohomologies* of the complex $V^\bullet$.

This was the *co*homological indexation style. The homological indexation style puts the indexes at the bottom: $(V_\bullet, \partial_\bullet)$, inverts arrows: $V_i \xrightarrow{\partial_i} V_{i-1}$, and calls $H_i(V_\bullet) = \ker(\partial_i)/\mathrm{im}\,(\partial_{i-1})$ the *homologies*. (The both styles are automatically translated to each other by changing signs of indexes and putting them from the top to the bottom).

Bigraded collection of modules $B^{p,q}$ equipped with two collections of maps

$$\begin{tikzcd} B^{p+1,q} & \\ B^{p,q} \arrow[u, "d_I^{p,q}"] \arrow[r, "d_{II}^{p,q}"] & B^{p,q+1} \end{tikzcd}$$

is called a *bicomplex*, if $(d_I + d_{II})^2 = 0$, that is $d_I^2 = d_{II}^2 = d_I d_{II} + d_{II} d_I = 0$ (in particular, $(B^{\bullet,q},\, d_I^{\bullet,q})$ is a complex for any fixed q and $(B^{p,\bullet},\, d_{II}^{p,\bullet})$ is a complex for any fixed p). Each bicomplex produces an ordinary complex called *total* (or *diagonal*) complex for $B^{\bullet,\bullet}$. Its entries are $Tot^n\,(B^{\bullet,\bullet}) \stackrel{\text{def}}{=} \bigoplus\limits_{p+q=n} B^{p,q}$ and the differential $d_{\mathrm{Tot}} \stackrel{\text{def}}{=} d_I + d_{II}$.

4◇1 (morphisms of complexes). Given two complexes $(V^\bullet,\, d_V^\bullet)$, $(W^\bullet,\, d_W^\bullet)$, a *morphism of complexes* $V^\bullet \xrightarrow{\varphi^\bullet} W^\bullet$ is a sequence of module homomorphisms $V^i \xrightarrow{\varphi^i} W^i$ satisfying $d_W^i \circ \varphi^{i-1} = \varphi^i \circ d_W^{i-1}$ for all i. Show that for each morphism of complexes $V^\bullet \xrightarrow{\varphi^\bullet} W^\bullet$

a) $(\ker(\varphi^i),\, d_V)$ and $(\mathrm{coker}\,(\varphi^i),\, d_W(\mathrm{mod}\ \mathrm{im}\,(\varphi^i))$ are well defined complexes as well;

b) there are well defined morphisms $H^i(V^\bullet) \xrightarrow{\widehat{\varphi}^i} H^i(W^\bullet)$ induced by $\varphi^\bullet$.

4◇2 (long (co)homology sequence). Show that each short exact sequence of complexes

$$0 \longrightarrow K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} M^\bullet \longrightarrow 0$$

induces a long exact sequence of their cohomologies

$$\cdots \longrightarrow H^i(K^\bullet) \xrightarrow{\widehat{\varphi}} H^i(L^\bullet) \xrightarrow{\widehat{\psi}} H^i(M^\bullet) \xrightarrow{\delta} H^{i+1}(K^\bullet) \xrightarrow{\widehat{\varphi}} H^{i+1}(L^\bullet) \longrightarrow \cdots$$

4◇3 (homotopies). A morphism of complexes $V^\bullet \xrightarrow{\varphi^\bullet} W^\bullet$ is called *contractible* (or *zero homotopic*), if there exists a sequence of degree -1 maps: $V^i \xrightarrow{\gamma^i} W^{i-1}$:

$$\begin{tikzcd} \cdots \arrow[r, "d_V^{i-2}"] & V^{i-1} \arrow[r, "d_V^{i-1}"] \arrow[d, "\varphi^{i-1}"] & V^i \arrow[r, "d_V^i"] \arrow[d, "\varphi^i"] \arrow[dl, "\gamma^i"'] & V^{i+1} \arrow[r, "d_V^{i+1}"] \arrow[d, "\varphi^{i+1}"] \arrow[dl, "\gamma^{i+1}"'] & V^{i+2} \arrow[r, "d_V^{i+2}"] \arrow[d, "\varphi^{i+2}"] \arrow[dl, "\gamma^{i+2}"'] & \cdots \\ \cdots \arrow[r, "d_W^{i-2}"] & W^{i-1} \arrow[r, "d_W^{i-1}"] & W^i \arrow[r, "d_W^i"] & W^{i+1} \arrow[r, "d_W^{i+1}"] & W^{i+2} \arrow[r, "d_W^{i+2}"] & \cdots \end{tikzcd}$$

(called a *contracting homotopy*) such that $\varphi = d_W \circ \gamma + \gamma \circ d_V$. Show that:

a) each contractible map of complexes induces (via prb. 4◇1) the zero map on cohomologies;

b) if two morphisms of complexes $\varphi, \psi : V^\bullet \longrightarrow W^\bullet$ are *homotopy equivalent* (i. e. $\varphi - \psi$ is zero homotopic) then their induced morphisms of cohomologies coincide: $\widehat{\varphi} = \widehat{\psi}$.

4◇4 (tensoring). Given two complexes $(V^\bullet, d_V)$, $(W^\bullet, d_W)$, show that modules $T^{p,q} = V^p \underset{k}{\otimes} W^q$ form a bicomplex w. r. t. differentials $d_I = d_V \otimes 1$ and $d_{II} = 1 \otimes d_W$, which send send $v \otimes w \in T^{p,q}$ to $d_I(v \otimes w) = (d_V v) \otimes w \in T^{p+1,q}$ and $d_{II}(v \otimes w) = (-1)^p v \otimes (d_W w) \in T^{p,q+1}$ (the corresponding total complex is called *a tensor product* of the initial complexes and is denoted by $V^\bullet \underset{k}{\otimes} W^\bullet$).

4◇5. Given $f \in k$, write $L^\bullet_f$ for a two-terms complex $k \xrightarrow{x \mapsto xf} k$ situated at degrees -1, 0 (so, its non zero cohomology is only $H^0(L^\bullet_f) = k/(f)$). Given a complex of k-modules $(V^\bullet, d_V)$, construct a long exact sequence of cohomologies

$$\cdots\ H^i(V^\bullet) \xrightarrow{x \mapsto xf} H^i(V^\bullet) \longrightarrow H^i(V^\bullet \underset{k}{\otimes} L^\bullet) \xrightarrow{\delta} H^{i+1}(V^\bullet) \xrightarrow{x \mapsto xf} H^{i+1}(V^\bullet)\ \cdots$$

HINT. construct an exact triple of complexes $0 \longrightarrow V[-1]^\bullet \longrightarrow V^\bullet \underset{k}{\otimes} L^\bullet \longrightarrow V^\bullet \longrightarrow 0$, where $V[-1]^\bullet$ is a (left) *degree shift* of $V^\bullet$: its i-th entry $V[-1]^i \overset{\text{def}}{=} V^{i-1}$ and $d^i_{V[-1]} = (-1)^i d^{i-1}_V$.

4◇6 (Koszul complex). Consider rank r free module $k^{\oplus r}$ with basis $\{e_1, e_2, \dots, e_r\}$ and write $\Lambda^m\left(k^{\oplus r}\right)$ for its m-th exterior power (this is a free module of rank $\binom{r}{m}$ spanned by $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$). Given a sequence of elements $(f_1, f_2, \dots, f_r)$, of k, show that:

a) the tensor product $\overset{r}{\underset{\nu=1}{\otimes}} L^\bullet_{f_\nu}$, of two-terms complexes defined in prb. 4◇5, is a complex

$$0 \longrightarrow \Lambda^r\left(k^{\oplus r}\right) \xrightarrow{\partial_r} \cdots \xrightarrow{\partial_3} \Lambda^2\left(k^{\oplus r}\right) \xrightarrow{\partial_2} \Lambda^1\left(k^{\oplus r}\right) \xrightarrow{\partial_1} k \longrightarrow 0$$

(written in bottom-style notations) whose differential $\partial = \sum_{\alpha=1}^{r} f_\alpha \dfrac{\partial}{\partial e_\alpha}$ acts on the basic vectors as $\partial_\nu(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\nu}) = \sum_{m=1}^{\nu} (-1)^{m-1} f_{i_m} \cdot e_{i_1} \wedge \cdots \wedge e_{i_{m-1}} \wedge e_{i_{m+1}} \wedge \cdots \wedge e_{i_\nu}$ (this complex is called *Koszul complex* associated with (f_ν)).

b) If the sequence $(f_1, f_2, \dots, f_r)$ is *regular* (i. e. for each i the class $f_i \pmod{(f_1, f_2, \dots, f_{i-1})}$ is not a zero divisor in the factor ring $k/(f_1, f_2, \dots, f_{i-1})$), then the non zero homologies of the Koszul complex are exhausted by $H^0 = k/(f_1, f_2, \dots, f_r)$, that is it provides a free resolution for the factor module $k/(f_1, f_2, \dots, f_r)$.

4◇7 (Hom-complexes). Given two complexes $(V^\bullet, d_V)$, $(W^\bullet, d_W)$, show that:

a) Modules $M^q_p = \mathrm{Hom}_k(V^p, W^q)$, of all k-linear maps, form a bicomplex (written in mixed indexation style) w. r. t. differentials sending $V^p \xrightarrow{\varphi} W^q \in M^q_p$ to

$$\partial(\varphi) \overset{\text{def}}{=} (-1)^{q-p} \varphi \circ d_V \in M^q_{p-1} \qquad \text{and} \qquad d(\varphi) \overset{\text{def}}{=} d_W \circ \varphi \in M^{q+1}_p$$

(the corresponding total complex $\mathrm{GrHom}^n_k(V^\bullet, W^\bullet) \overset{\text{def}}{=} \underset{q-p=n}{\oplus} M^q_p$ is called a complex *of graded homomorphisms* from $V^\bullet$ to $W^\bullet$).

b) The zero cohomology $H^0\left(\mathrm{GrHom}^\bullet(V, W)\right)$ is formed by homotopy equivalence classes of morphisms of complexes[1] $V^\bullet \xrightarrow{\varphi} W^\bullet$.

[1] similarly, the elements of $H^d\left(\mathrm{GrHom}^\bullet(V, W)\right)$ are called *homotopy classes of degree d morphisms of complexes*, that is homotopy classes of morphisms of complexes $V^\bullet \xrightarrow{\varphi} W[d]^\bullet$