Multilinear algebra.

All the problems below are about finite dimensional vector spaces over an arbitrary field k of zero characteristic

3◇1. Construct canonical isomorphism $U^* \otimes W \xrightarrow{\sim} \operatorname{Hom}(U, W)$. Given two operators $U \xrightarrow{\varphi} V$, $V \xrightarrow{\psi} W$ with tensor expansions $\varphi = \sum u_\nu^* \otimes v_\nu \in U^* \otimes V$, $\psi = \sum v_\mu^* \otimes w_\mu \in V^* \otimes W$, write some precise tensor expansion for their composition $\psi \circ \varphi \in U^* \otimes W$.

HINT. $\xi \otimes w$ is rank 1 operator taking $u \longmapsto \xi(u) \cdot w$

3◇2. Construct a canonical isomorphism $\operatorname{Hom}(U, \operatorname{Hom}(V, W)) \simeq \operatorname{Hom}(U \otimes V, W)$.

3◇3. Let x_ν, e_ν be dual bases of V^* and V. Does the *Kasimir tensor* $\sum x_\nu \otimes e_\nu \in V^* \otimes V$ depend on the choice of dual basises?

3◇4. There is a canonical correlation $\widehat{q}$ on the space $\operatorname{Hom}(V, V)$ given by the composition of canonical isomorphisms $\operatorname{Hom}(V, V) \simeq V^* \otimes V \xrightarrow{\tau} (V \otimes V^*)^* \simeq \operatorname{Hom}(V, V)^*$ where τ takes $\xi \otimes v$ to a linear form $v' \otimes \xi' \longmapsto \xi(v')\xi'(v)$. As usually, it corresponds to some bilinear form $q(A, B) \overset{\text{def}}{=} \widehat{q}A(B)$. Is this form symmetric? How to compute it in terms of matrices?

3◇5. Let $\dim V = 4$. Show that a skew quadratic form $\omega \in \Lambda^2 V$ is a product of two liner forms iff $\omega \wedge \omega = 0$.

HINT. Use the Grassmannian version of the Laplace theorem: in appropriate coordinates ω equals either $e_1 \wedge e_2$ or $e_1 \wedge e_2 + e_3 \wedge e_4$.

3◇6. Is there 2×4 - matrix with 2×2 - minors **a)** $\{2, 3, 4, 5, 6, 7\}$ **b)** $\{3, 4, 5, 6, 7, 8\}$?

3◇7. Show that the following conditions on $\omega \in \Lambda^d V$ are pairwise equivalent:

a) ω is decomposable[1] **b)** $\omega \wedge v = 0 \ \forall v \in \operatorname{span}(f)$;

c) $\left(\partial_{e_{i_1}} \partial_{e_{i_2}} \dots \partial_{e_{i_{d-1}}} \omega\right) \wedge \omega = 0 \quad \forall \{i_1, i_2, \dots, i_{d-1}\} \subset \{1,\ 2\ \dots\ ,\ n\}$ and some base $\{e_\nu\}$ of V.

3◇8* (skew exponent). For a *decomposible* skew polynomial $\omega \in \Lambda^{2m} V$ of *even* degree put $e^\omega \overset{\text{def}}{=} 1 + \omega$ and extend this definition onto the whole of $\Lambda^{2m} V$ by the rule $e^{\sum \omega_i} = \prod e^{\omega_i}$. Show that:

a) this definition of e^f does not depend on either a choice of the decomposition $f = \sum \omega_i$ or an ordering of factors in the right hand side.

b) the exponential map gives an inclusion $\Lambda^{even} V \hookrightarrow \Lambda^{even} V$ that takes sums to products.

c) Does it satisfy an expected property $\partial_v e^f = e^f \wedge \partial_v f$?

3◇9. Let $A = (a_{ij})$ be $n \times n$ - matrix whose entries are considered as independent variables. Fix a collection of m matrix elements $a_{i_\nu j_\nu}$, $1 \leqslant \nu \leqslant m$. In symmetric polynomial ring $k[a_{ij}]$ (in n^2 variables) compute the partial derivative $\frac{\partial^m \det A}{\partial a_{i_1 j_1} \partial a_{i_2 j_2} \cdots \partial a_{i_m j_m}}$ for: **a)** $m = 1,\ 2$; **b)** any m.

HINT. Use Laplace relations: write A_m for $\binom{n}{m} \times \binom{n}{m}$ matrix whose entries are $m \times m$-minors of A and $\widehat{A}_m$ for the matrix of their algebraic complements; then $\det A = \binom{n}{m}^{-1} \operatorname{tr}\left(A_m \cdot {}^t\widehat{A}_m\right)$.

3◇10*. Prove the following Taylor expansion for the polynomial function $\det(A)$ on the space of $n \times n$ - matrices:

$$\det(\lambda A + \mu B) = \sum_{p+q=n} \cdot \lambda^p \mu^q \cdot \sum_{\substack{I J: \\ \#I = \#J = p}} (-1)^{|I|+|J|} a_{IJ} b_{\widehat{I}\widehat{J}} \tag{2}$$

where[2] (a_{IJ}) is $p \times p$-minor of $A = (a_{ij})$ situated in rows $I = (i_1, i_2, \dots, i_p)$ and columns $J = (j_1, j_2, \dots, j_p)$ and $(b_{\widehat{I}\widehat{J}})$ is $q \times q$-minor of $B = (b_{ij})$ situated in the complementary rows $\widehat{I} = \{1,\ 2,\ \dots\ ,\ n\} \setminus I$ and the complementary columns $\widehat{J} = \{1,\ 2,\ \dots\ ,\ n\} \setminus J$.

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[1] i. e. is equal to a skew product of d linear forms

[2] in the notations from the hint to the previous problem, the last right sum in (2) is nothing but $\operatorname{tr}\left(A_p \cdot {}^t\widehat{B}_q\right)$