Polynomial ideals

Everywhere below k means an algebraically closed field (you can put $k = \mathbb{C}$, if you want)

2◇1. Give an example of proper non-principal ideal in **a)** $k[x, y]$ **b)** $\mathbb{Z}[x]$.

2◇2. Let $J = (xy, yz, zx) \subset k[x, y, z]$. Describe $V(J) \subset \mathbb{A}^3$ and $I(V(J)) \subset k[x, y, z]$. Is it possible to define the same variety by 2 polynomial equations?

2◇3. Find $f \in I(V(J)) \setminus J$ for $J = (x^2 + y^2 - 1, y - 1) \subset k[x, y]$.

2◇4. Describe $V(J) \subset \mathbb{A}^3$ and $I(V(J)) \subset k[x, y, z]$ for:
a) $J = (xy, (x - y)z)$ **b)** $J = (xy + yz + zx, x^2 + y^2 + z^2)$

2◇5. Let $I, J \in k[x_1, x_2, \dots, x_n]$ be two ideals, $K = \{ab \mid a \in I,\, b \in J\}$, and IJ be an ideal spanned by K. Construct counter examples or prove the following questionable coincidences:
a) $K = IJ$ is already an ideal as it is
b) $K = I \cap J$ (by the way, is $I \cap J$ an ideal?)
c) $V(I) \cup V(J) = V(IJ)$
d) $V(I) \cup V(J) = V(I \cap J)$
e) ([1]) $\sqrt{IJ} = \sqrt{I \cap J}$
f) $\sqrt{IJ} = \sqrt{I}\sqrt{J}$
g) $(I = \sqrt{I}\ \&\ J = \sqrt{J}) \quad \Rightarrow \quad IJ = \sqrt{IJ}$

2◇6. Let a polynomial f vanish along a hypersurface given in $\mathbb{A}^n$ by a polynomial equation $g = 0$. Prove that each irreducible factor of g divides f.

2◇7 (Zariski topology). Let $X = V(J) \subset \mathbb{A}^n$ be an affine algebraic variety, defined by an ideal $J \subset k[x_1, x_2, \dots, x_n]$. We write $k[X] \stackrel{\text{def}}{=} k[x_1, x_2, \dots, x_n]/J$ for algebra of polynomial functions restricted onto X.
a) Check that sets $V_X(I) \stackrel{\text{def}}{=} \{p \in X \mid f(p) = 0 \quad \forall f \in I\}$, where I runs through all the ideals of $k[X]$, satisfy the axioms for the closed sets of some topology[2] on X (it is called *Zariski topology*)
b) Show that regular morphisms[3] of affine algebraic varieties are continuous in Zariski topology.
c) Prove that any open covering of X in Zariski topology contains a finite sub-covering.
d) Show that *principal* open sets $D(f) = X \setminus V_X((f)) = \{p \in X \mid f(p) \neq 0\}$, where $f \in k[X]$, form a base of Zariski open sets[4]

2◇8. Let $B \supset A$ be an extension of commutative rings such that B is finitely generated as A-module. Prove that $\mathfrak{m}B \neq B$ for any maximal ideal $\mathfrak{m} \subset A$.

2◇9*. Show that any finitely generated[5] field is finite (as a set).

[1] given an ideal $I \subset A$ in a commutative ring A, we write $\sqrt{I}$ for its *radical*: $\sqrt{I} \stackrel{\text{def}}{=} \bigcup_{n \geqslant 0} \{a \in A \mid a^n \in I\}$

[2] i. e. find ideals I whose $V_X(I)$ are: $\varnothing$; X; $\bigcap V_X(I_\nu)$ (for any given family of ideals I_ν); and $V_X(I_1) \cup V_X(I_2)$ (for any two given ideals $I_{1,2}$)

[3] a set-theoretical map $X \xrightarrow{\varphi} Y$ of affine algebraic varieties is called *regular morphism*, if its *pull back homomorphism* φ^*, which takes k-valued functions on Y to its compositions with φ, sends polynomial functions to the polynomial ones, that is, induces a ring homomorphism $k[Y] \xrightarrow{\varphi^*} k[X]$

[4] i. e. each Zariski open set is a *finite* union of principal open sets

[5] in absolute sense, i. e. as $\mathbb{Z}$-algebra w. r. t. the action $m \cdot a \stackrel{\text{def}}{=} \underbrace{a + a + \cdots + a}_{m \text{ times}}$