cohomology. Roughly speaking, the homological algebra is a calculus of cohomologies: it reformulates incoming problems in terms of cohomologies[3] and develops a series of tools for analyzing and computing them.

Example 5.1: tensor product of complexes. Given two complexes of modules (U, d_U), (V, d_V), their tensor product $U \otimes V$ carries a natural differential, which is a canonical extension of d_U, d_V by the Leibnitz rule w.r.t. the tensor product. Namely,

$$d = d_{U\otimes V} \overset{\text{def}}{=} d_U \otimes 1 + 1 \otimes d_V$$

satisfies $d^2 = d_U^2 \otimes 1 + d_U \otimes d_V - d_U \otimes d_V + 1 \otimes d_V^2 = 0$ (the minus sign appeared because of Koszul sign rule, which forces $(1 \otimes d_V)\circ(d_U \otimes 1) = -d_U \otimes d_V$, since $|d_U| = |d_V| = 1$). Note that, being applied to elements, this differential takes

$$d(u \otimes v) = (d_U \otimes 1)(u \otimes v) + (1 \otimes d_V)(u \otimes v) = (d_U u) \otimes v + (-1)^{|v|} v \otimes (d_V v)$$

(again by Koszul sign rule). So, we get the graded version of Leibnitz formula. Of course, this construction has a straightforward extension on products $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ with arbitrary number of multipliers.

Example 5.2: Hom-complex. Given two complexes of modules (U, d_U), (V, d_V), the graded module $\mathrm{Homgr}(U, V)$ also carries a natural differential. It is defined by the graded commutator $\psi \longmapsto [d, \psi]$, which, in expanded form, takes homogeneous ψ to:

$$[d, \psi] = d_V \circ\psi - (-1)^{|\psi|}\psi\circ d_U$$

Cohomologies of this complex are called *homotopy classes of graded morphisms of complexes*. For example, the zero cohomology group $H^0(\mathrm{Homgr}(U, V))$ consists of zero degree maps $U \xrightarrow{\psi} V$ commuting with the differentials: $d_V\psi = \psi d_U$ and considered up to equivalence: $\varphi \sim \psi$ if $\varphi - \psi = d_V\circ\gamma + \gamma\circ d_U$ for some map $U \xrightarrow{g} V$ of degree -1. In expanded form, the condition $\varphi \sim 0$ is visualized by the diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "d_U"] & U^{i-1} \arrow[r, "d_U"] \arrow[d, "\varphi"] & U^i \arrow[r, "d_U"] \arrow[d, "\varphi"] \arrow[dl, "\gamma"'] & U^{i+1} \arrow[r, "d_U"] \arrow[d, "\varphi"] \arrow[dl, "\gamma"'] & U^{i+2} \arrow[r, "d_U"] \arrow[d, "\varphi"] \arrow[dl, "\gamma"'] & \cdots \\
\cdots \arrow[r, "d_V"] & V^{i-1} \arrow[r, "d_V"] & V^i \arrow[r, "d_V"] & V^{i+1} \arrow[r, "d_V"] & V^{i+2} \arrow[r, "d_V"] & \cdots
\end{tikzcd}$$

where $\varphi = d_V\circ\gamma + \gamma\circ d_U$; in this case γ is called *a contracting homotopy* for φ. In general case, when $\varphi - \psi = d_V\circ\gamma + \gamma\circ d_U$, γ is called *a homotopy between* φ and ψ.

Example 5.3: dual complex. Given a complex of vector spaces (U, d), the dual complex (U^*, d^*) is defined as $\mathrm{Homgr}(U, k)$, where k is a complex whose unique non zero component is the ground field k situated in degree zero. In accordance with Example 5.2, the ν-th degree component of U^* is $(U^*)^\nu = (U^{-\nu})^*$, i.e. the dual space to U_ν, and the differential $(U^*)^\nu \xrightarrow{d^*} (U^*)^{\nu+1}$ coincides with the map $(U^{-\nu})^* \longrightarrow (U^{-\nu-1})^*$, which is, up to a sign, dual to $U^{-\nu-1} \xrightarrow{d} U^{-\nu}$ and takes $\psi \in (U^{-\nu})^*$ to $(-1)^\nu\psi\circ d \in (U^*)^{\nu+1}$.

CATEGORY OF COMPLEXES $\mathscr{C}om\,(\mathscr{A})$. Given an Abelian category $\mathscr{A}$, one can construct another Abelian category $\mathscr{C}om\,(\mathscr{A})$ whose objects are complexes and morphisms are degree zero elements $\varphi \in \mathrm{Homgr}(U, V)$ commuting with the differentials in U and in V, i.e. satisfying $[d, \varphi] = 0$. This category is called *category of complexes* over $\mathscr{A}$.

Exercice 5.4. Show that category $\mathscr{C}om\,(\mathscr{A})$ is Abelian; in particular, verify that for any $\psi \in \mathrm{Hom}_{\mathscr{C}om\,(\mathscr{A})}(U, V)$

a) the kernels $\ker\psi \subset U^\nu$ form a well defined complex (w.r.t. the restriction of the differential d_U), which is the kernel in the category $\mathscr{C}om\,(\mathscr{A})$;

b) the cokernels $\mathrm{coker}\,(\psi) = V^\nu/(\mathrm{im}\,\psi)$ form a well defined complex (w.r.t. the differential induced by d_V), which is the cokernel in the category $\mathscr{C}om\,(\mathscr{A})$.

Exercice 5.5. Show that for any morphism of complexes $U \xrightarrow{\psi} V$ there is a well defined morphism $H(U) \xrightarrow{\widehat{\varphi}} H(V)$ induced by φ.

An important remark is that any zero homotopic morphism $\varphi \in \mathrm{Hom}_{\mathscr{C}om\,(\mathscr{A})}(U, V)$, i.e.

$$U \xrightarrow{\varphi} V \quad \text{such that} \quad \varphi = d\gamma + \gamma d \quad \text{for some} \quad \gamma \in \mathrm{Homgr}^{-1}(U, V)\ ,$$

induces the zero map $H(U) \xrightarrow{\widehat{\varphi}} H(V)$, because $du = 0$ implies $\varphi u = d\gamma u \equiv 0$ in $H(V)$. So, $\widehat{\varphi} = \widehat{\psi}$ as soon $\varphi \sim \psi$ are homotopy equivalent.

[3]in fact, this is the most wide mathematical treatment of the 'linearization concept' — the central idea of the classical calculus; indeed, each cohomology business really deals with (sometimes complicated) aggregates of linear equation systems

Example 5.6: a long exact sequence of cohomologies. Let $0 \longrightarrow U \xrightarrow{\varphi} V \xrightarrow{\psi} W \longrightarrow 0$ be an exact triple in the category of complexes $\mathscr{C}om\,(\mathscr{A})$. In expanded form, we have a commutative diagram with exact columns

$$\begin{tikzcd}
\cdots \arrow[r, "d_U"] & U^{i-1} \arrow[r, "d_U"] \arrow[d, hook, "\varphi"] & U^{i} \arrow[r, "d_U"] \arrow[d, hook, "\varphi"] & U^{i+1} \arrow[r, "d_U"] \arrow[d, hook, "\varphi"] & U^{i+2} \arrow[r, "d_U"] \arrow[d, hook, "\varphi"] & \cdots \\
\cdots \arrow[r, "d_V"] & V^{i-1} \arrow[r, "d_V"] \arrow[d, two heads, "\psi"] & V^{i} \arrow[r, "d_V"] \arrow[d, two heads, "\psi"] & V^{i+1} \arrow[r, "d_V"] \arrow[d, two heads, "\psi"] & V^{i+2} \arrow[r, "d_V"] \arrow[d, two heads, "\psi"] & \cdots \\
\cdots \arrow[r, "d_W"] & W^{i-1} \arrow[r, "d_W"] & W^{i} \arrow[r, "d_W"] & W^{i+1} \arrow[r, "d_W"] & W^{i+2} \arrow[r, "d_W"] & \cdots
\end{tikzcd}$$

Then we have a *snake morphism* $H(W) \xrightarrow{\delta} H(U)$ of degree 1 defined as follows. For each cycle $w \in W^i$, pick up some preimage $v \in V^i$, $\psi(v) = w$. Then $d_W w = 0$ implies $d_V v \in \ker\psi \cap V^{i+1} = U^{i+1}$. We put $\delta(w) \overset{\text{def}}{=} d_V v$. This gives a well defined map of cohomology classes $H^i(W) \xrightarrow{\delta} H^{i+1}(U)$, because taking another preimage $v' \in V^i$, $\psi(v') = w$, we change $\delta(w)$ by the boundary $d_U(v - v')$ and $\delta\big(d_W(W^{i-1})\big) = 0$.

Exercice 5.7. Show that there is a long exact sequence of cohomologies

$$\cdots \xrightarrow{\delta} H^i(U) \xrightarrow{\widehat{\varphi}} H^i(V) \xrightarrow{\widehat{\psi}} H^i(W) \xrightarrow{\delta} H^{i+1}(U) \xrightarrow{\widehat{\varphi}} \cdots$$

Example 5.8: Koszul complexes of a vector space. Let W be d-dimensional vector space over a field k of zero characteristic, $\Lambda = \oplus \Lambda^i W$ be its exterior algebra, $S = \oplus S^i W^*$ be the symmetric algebra of its dual, that is the algebra of polynomial functions on W. Consider the tensor product $V = \Lambda \underset{k}{\otimes} S$ as a graded S-module:

$$V = \bigoplus_{m=0}^{d} \Lambda^m W \otimes S$$

whose grading is given by powers of skew symmetric monomials. Fixing a pair of dual basises $e_i \in W$, $x_i \in W^*$ we can write degree m homogeneous elements of V as sums $v = \sum e_I \otimes f_I(x)$, where e_I are the standard basic Grassmannian monomials of the same degree $\sharp I = m$ and $f_I(x) \in S$ are some (not necessary homogeneous) symmetric polynomials in x_i. The Koszul differential $V \xrightarrow{\partial} V$ is defined as multiplication by the identity operator (that does not depend on the choice of dual bases) $\mathrm{Id}_W = \sum_{i=1}^{d} e_i \otimes x_i \in W^* \otimes W = \mathrm{Hom}(W,W)$ and takes

$$e_I \otimes f_I \overset{\partial}{\longmapsto} \sum_{i=1}^{d} (e_i \wedge e_I) \otimes (x_i \cdot f_I)\ .$$

In expanded form, Koszul complex looks as diagram

$$0 \longrightarrow \Lambda^0 W \otimes S \xrightarrow{\partial} \Lambda^1 W \otimes S \xrightarrow{\partial} \Lambda^2 W \otimes S \xrightarrow{\partial} \Lambda^3 W \otimes S \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Lambda^d W \otimes S \longrightarrow 0 \tag{26}$$

Exercice 5.9. Check that ∂ is S-module homomorphism and satisfies $\partial^2 = 0$.

To compute its cohomologies, consider a homogeneous degree -1 k-linear map $V \xrightarrow{\gamma} V$ defined by De Rham type differential operator

$$\gamma = \sum_{\nu=1}^{d} \frac{\partial}{\partial_{e_\nu}} \otimes \frac{\partial}{\partial_{x_\nu}} \ :\ e_I \wedge f_I \longmapsto \sum_{\nu=1}^{d} \frac{\partial e_I}{\partial e_\nu} \otimes \frac{\partial f_I(x)}{\partial x_\nu}$$

Exercice 5.10. Check that $[\partial, \gamma] = \partial\gamma + \gamma\partial$ sends each direct summand $\Lambda^m \otimes S^n \subset V^m = \bigoplus_n \Lambda^m W \otimes S^n W^*$ to itself and acts there as multiplication by $(n + d - m)$ (this is a version of Euler's theorem about homogeneous functions).

HINT. $\partial\gamma$ and $\gamma\partial$ take a basic monomial $e_I \otimes x^N = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m} \otimes x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d}$ (where $\sum n_\nu = n$) to

$$\left(\sum_{i \in I} n_i\right) \cdot e_I \otimes x^N + \sum_{j \notin I} \sum_{\nu=1}^{m} (-1)^{\nu-1} n_{i_\nu} \cdot e_j \wedge e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_\nu} \wedge \cdots \wedge e_{i_m} \otimes x_1^{n_1} \cdots x_{i_\nu}^{n_{i_\nu}-1} \cdots x_j^{n_j+1} \cdots x_d^{n_d}$$

$$\left(\sum_{j \notin I} (n_j + 1)\right) \cdot e_I \otimes x^N + \sum_{j \notin I} \sum_{\nu=1}^{m} (-1)^{\nu} n_{i_\nu} \cdot e_j \wedge e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_\nu} \wedge \cdots \wedge e_{i_m} \otimes x_1^{n_1} \cdots x_{i_\nu}^{n_{i_\nu}-1} \cdots x_j^{n_j+1} \cdots x_d^{n_d}$$

respectively, where «hat» over e_{i_ν} means that this multiplier should be removed from the wedge product.

This exercise shows that there is k-linear zero degree map $V \xrightarrow{\varepsilon} V$, of the Koszul to itself, which takes

$$e_I \otimes x^N \overset{\varepsilon}{\longmapsto} (n + d - m) \cdot e_I \otimes x^M$$

(in particular, it commutes with the Koszul differential d) and is zero homotopic: $\varepsilon = [\partial, \gamma]$. The last means that the induced map $H(V) \xrightarrow{\widehat{\varepsilon}} H(V)$ annihilates the homologies $H(V)$. But there is precisely one monomial annihilated by φ, namely $e_1 \wedge e_2 \wedge \cdots \wedge e_d \otimes 1 \in V^m$, which also lies in $\ker \partial$. Since $\partial\left(V^{d-1}\right) \subset \bigoplus_{n\geqslant 1} \Lambda^d \otimes \Lambda^n \subset V^m$, this monomial actually does represent non trivial cohomology class. So, the Koszul complex (26) is acyclic in degrees $0 \leqslant m \leqslant (d-1)$ and $H^d(V) \simeq k$ is isomorphic to $\Lambda^d W \otimes 1$.

Another (partially dual) version of the Koszul complex is constructed from the algebra $\widetilde{V} = \Lambda W^* \otimes SW^*$, where the exterior algebra of W^* is used now instead of the previous ΛW (but the symmetric algebra SW^* is the same as before). Again, we will consider $\widetilde{V}$ as graded S-module but now we put $\Lambda^m \otimes S$ in degree $-m$, i.e. write $\widetilde{V} = \bigoplus_{m=-d}^{0} \widetilde{V}^m$, where $\widetilde{V}^m = \Lambda^{-m} \otimes S$ for $-d \leqslant m \leqslant 0$. A differential $\widetilde{V} \xrightarrow{\widetilde{\partial}} \widetilde{V}$ or, in expanded form, a diagram

$$0 \longrightarrow \Lambda^d W^* \otimes S \xrightarrow{\widetilde{\partial}} \Lambda^{d-1} W^* \otimes S \xrightarrow{\widetilde{\partial}} \Lambda^{d-2} W^* \otimes S \xrightarrow{\widetilde{\partial}} \cdots \xrightarrow{\widetilde{\partial}} \Lambda^1 W^* \otimes S \xrightarrow{\widetilde{\partial}} \Lambda^0 W^* \otimes S \longrightarrow 0 \tag{27}$$

is obtained as a direct sum of maps $\Lambda^m W^* \otimes S^n W^* \longrightarrow \Lambda^{m-1} W^* \otimes S^{n+1} W^*$ defined by compositions

$$\Lambda^m W^* \otimes S^n W^* \xrightarrow{\omega^* \otimes 1} \Lambda^{m-1} W^* \otimes W^* \otimes S^n W^* \xrightarrow{1 \otimes \mu} \Lambda^{m-1} W^* \otimes S^{n+1} W^*$$

where $\Lambda^m W^* \xrightarrow{\omega^*} \Lambda^{m-1} W^* \otimes W^*$ is the dual map[4] to the skew product $\omega : \Lambda^{m-1} W \otimes W \xrightarrow{\psi \otimes w \mapsto w \wedge \psi} \Lambda^m W$ and $\mu : W^* \otimes S^n W^* \xrightarrow{x \otimes f \mapsto x \cdot f} S^{n+1} W^*$ is the symmetric product.

Exercice 5.11. Check that $\widetilde{\partial}$ is S-linear, satisfies $\widetilde{\partial}^2 = 0$, and is written in coordinates as

$$\widetilde{\partial} = \sum_{\nu=1}^{d} \frac{\partial}{\partial_{x_\nu}} \otimes x_\nu \; : \; x_I \otimes f(x) \longmapsto \sum_{\nu=1}^{d} \frac{\partial x_I}{\partial_{x_\nu}} \otimes x_\nu \cdot f(x)$$

In fact, there is an isomorphism of complexes $\widetilde{V} \xrightarrow{\sim} V[m]$ induced by isomorphisms $\Lambda^m W^* \xrightarrow{\sigma} \Lambda^{d-m} W$, which appear as soon as some basic vector $\Omega \in \Lambda^d W$ is fixed. The map σ takes $\xi \in \Lambda^m W^*$ to unique $\sigma(\xi) \in \Lambda^{d-m} W$ such that

$$\zeta \wedge \sigma(\xi) = \langle\, \zeta\, ,\, \xi \,\rangle \cdot \Omega \quad \forall\, \zeta \in \Lambda^m W$$

where we write $\langle *\, , * \rangle$ for the pairing between $\Lambda^m W$ and $\Lambda^m W^*$ defined (if $\operatorname{char} k = 0$) by the full contraction of complete polarizations[5].

Exercice 5.12. Check that σ are well defined isomorphisms for all m and direct sum of maps

$$\Lambda^m W^* \otimes S^n W^* \xrightarrow{\sigma \otimes 1} \Lambda^{d-m} W \otimes S^n W^*$$

really gives an isomorphism of complexes $\widetilde{V} \xrightarrow{\sim} V[m]$.

In particular, the cohomologies $H(\widetilde{V})$ are exhausted by 1-dimensional space situated in degree zero and spanned by the constant monomial $1 \otimes 1$.

Exercice 5.13. Compute cohomologies $H(\widetilde{V})$ directly by arguments similar to ones used above for computation $H(V)$.

Example 5.14: Koszul complex of a regular sequence. Let K be a commutative ring, $f_1, f_2, \dots, f_m \in K$ be a sequence of elements. Write L_ν for a complex of free K-modules of rank 1

$$\begin{tikzcd} 0 \arrow[r] & L_\nu^0 \arrow[r, "d_\nu"] \arrow[d, equal] & L_\nu^1 \arrow[r] \arrow[d, equal] & 0 \\ & K \arrow[r, "x \mapsto f_\nu \cdot x"] & K & \end{tikzcd}$$

which has just two non zero graded components $L_\nu^0 = L_\nu^1 = K$ (situated in degrees 0, 1) and the differential defined as multiplication by f_ν. The tensor product

$$K_{(f_1, f_2, \dots, f_m)} = \bigotimes_{\nu=1}^{m} L_\nu$$

of these short complexes is called *Koszul complex* of a sequence $f_1, f_2, \dots, f_m$. Its non zero homogeneous components are sitting in degrees $0 \leqslant p \leqslant m$ and p-th component is obtained by direct summation

$$K^p_{(f_1, f_2, \dots, f_m)} = \bigoplus L_1^{\varepsilon_1} \otimes L_2^{\varepsilon_2} \otimes L_3^{\varepsilon_3} \otimes \cdots \otimes L_m^{\varepsilon_m} \tag{28}$$

[4] w. r. t. the pairing between $\Lambda^m W$ and $\Lambda^m W^*$ defined (if $\operatorname{char} k = 0$) by the full contraction of their complete polarizations

[5] recall that $\langle\, e_I\, ,\, x_J \,\rangle = m! \cdot \delta_{IJ}$ under this pairing

of all tensor products with precisely p out of ε_ν's equal to 1 and the rest $(m-p)$ out of ε_ν's equal to 0. Since each product in (28) is isomorphic to K, their sum $K^p_{(f_1,f_2,\dots,f_m)}$ is a free K-module of rank $\binom{m}{p}$ that can be identified with p-th exterior power $\Lambda^p K^m$ of the free K-module K^m of rank m by sending the summand

$$L_1^{\varepsilon_1} \otimes L_2^{\varepsilon_2} \otimes L_3^{\varepsilon_3} \otimes \cdots \otimes L_m^{\varepsilon_m} \simeq K$$

containing $\varepsilon_\nu = 1$ at positions $\nu = i_1, i_2, \dots, i_p$ to K-linear span of the monomial $e_I \in \Lambda^p K^m$, where $I = (i_1, i_2, \dots, i_p)$, coming from the standard base $e_1, e_2, \dots, e_m$ for K^m. So, we can rewrite the Koszul complex as

$$0 \longrightarrow \Lambda^0 K^m \longrightarrow \Lambda^1 K^m \longrightarrow \Lambda^2 K^m \longrightarrow \cdots \longrightarrow \Lambda^m K^m \longrightarrow 0$$

Exercice 5.15. Check that the differential here takes $e_I \longmapsto \sum\limits_{\nu=1}^{m} f_\nu \cdot e_\nu \wedge e_I$, i.e. coincides with exterior multiplication by vector $\sum f_\nu e_\nu = (f_1, f_2, \dots, f_m) \in K^m$.

Exercice 5.16. Check that if we take $K = S^\bullet(V^*)$, $m = \dim V$, and $f_\nu = x_\nu$, then we get precisely the first of two complexes considered in Example 5.8.

Now, as a generalization of Example 5.8, let us prove that

$$H^p(K_{(f_1,f_2,\dots,f_m)}) = \begin{cases} 0 & , \quad \text{if } 0 \leqslant p \leqslant (m-1) \\ K/(f_1, f_2, \dots, f_m) & , \quad \text{if } p = m \end{cases}$$

if f_ν is not a zero divisor in $K/(f_1, f_2, \dots, f_{\nu-1})$ for all $1 \leqslant \nu \leqslant m$ (sequences $f_1, f_2, \dots, f_m$ satisfying this property are called *regular*).

We use induction on m. For $m = 1$ we have the short two term complex

$$0 \longrightarrow K \xrightarrow{x \mapsto f_1 \cdot x} K \longrightarrow 0$$

whose differential is injective as soon as f_1 does not divide zero in K. So, $H^0 = 0$, $H^1 = K/(f_1)$. Now, suppose that $K_{(f_1,f_2,\dots,f_{m-1})}$ has prescribed cohomologies. It follows from Lemma 5.17 below that the cohomologies of $K_{(f_1,f_2,\dots,f_m)} = K_{(f_1,f_2,\dots,f_{m-1})} \otimes L_m$ are fitted in the following long exact sequence of cohomologies:

$$\begin{gathered} \cdots \longrightarrow H^{i-1}\left(K_{(f_1,f_2,\dots,f_{m-1})}\right) \xrightarrow{\xi \mapsto (-1)^{i-1} f_m \xi} H^{i-1}\left(K_{(f_1,f_2,\dots,f_{m-1})}\right) \longrightarrow H^i\left(K_{(f_1,f_2,\dots,f_m)}\right) \longrightarrow \\ \longrightarrow H^i\left(K_{(f_1,f_2,\dots,f_{m-1})}\right) \xrightarrow{\xi \mapsto (-1)^{i} f_m \xi} H^i\left(K_{(f_1,f_2,\dots,f_{m-1})}\right) \longrightarrow \cdots \end{gathered} \tag{29}$$

By inductive hypothesis, $H^p\left(K_{(f_1,f_2,\dots,f_m)}\right)$ whith $0 \leqslant p \leqslant (m-2)$ stays in this sequence between two zero modules. So, it should be zero as well. For $p = (m-1)$, m

$$H^{m-1}\left(K_{(f_1,f_2,\dots,f_{m-1})}\right) = \ker\left(K/(f_1, f_2, \dots, f_{m-1}) \xrightarrow{\xi \mapsto f_m \xi} K/(f_1, f_2, \dots, f_{m-1})\right) = 0$$

$$H^{m}\left(K_{(f_1,f_2,\dots,f_{m-1})}\right) = \operatorname{coker}\left(K/(f_1, f_2, \dots, f_{m-1}) \xrightarrow{\xi \mapsto f_m \xi} K/(f_1, f_2, \dots, f_{m-1})\right) = K/(f_1, f_2, \dots, f_m)$$

<u>Lemma 5.17</u>. *Let C be any complex of K-modules and L be a short complex $K \xrightarrow{x \mapsto fx} K$ situated in degrees 0, 1 as before. Then there is an exact triple of complexes*

$$0 \longrightarrow C[-1] \hookrightarrow C \otimes L \twoheadrightarrow C \longrightarrow 0 \tag{30}$$

which produces the long exact sequence of cohomologies

$$\cdots \longrightarrow H^{i-1}(C) \xrightarrow{\xi \mapsto (-1)^{i-1} f \xi} H^{i-1}(C) \longrightarrow H^i(C \otimes L) \longrightarrow H^i(C) \xrightarrow{\xi \mapsto (-1)^{i} f \xi} H^i(C) \longrightarrow \cdots$$

whose snake homomorphisms $H^i(C) \longrightarrow H^{i+1}(C[-1]) = H^i(C)$ are induced, up to a sign, by multiplication by f.

Proof. Clearly, $(C \otimes L)^i = C^i \otimes L^0 \oplus C^{i-1} \otimes L^1 \simeq C^i \oplus C^{i-1} = (C \oplus C[-1])^i$. In terms of this direct decomposition, the differential in $C \otimes L$ takes $(c^i, c^{i-1}) \in C^i \oplus C^{i-1}$ to

$$\begin{aligned} d_{C \otimes L}(c^i \otimes 1^{(0)} + c^{i-1} \otimes 1^{(1)}) &= \\ &= d_C c^i \otimes 1^{(0)} + (-1)^i c^i \otimes d_L 1^{(0)} + d_C c^{i-1} \otimes 1^{(1)} + (-1)^{i-1} c^{i-1} \otimes d_L 1^{(1)} = \\ &= d_C c^i \otimes 1^{(0)} + (-1)^i f \cdot c^i \otimes 1^{(1)} + d_C c^{i-1} \otimes 1^{(1)} \end{aligned}$$

which is $\left(d_C c^i,\, d_C c^{i-1} + (-1)^i f c^i\right) \in C^{i+1} \oplus C^i = (C \oplus C[-1])^{i+1}$. In other words, in terms of decomposition $(C \otimes L)^i = (C \oplus C[-1])^i$, the differential $d_{C \otimes L}$ can be presented via multiplication by matrix:

$$\begin{pmatrix} c^i \\ c^{i-1} \end{pmatrix} \longmapsto \begin{pmatrix} d_C & 0 \\ (-1)^i f & d_C \end{pmatrix} \cdot \begin{pmatrix} c^i \\ c^{i-1} \end{pmatrix}$$

This means that $C[-1]$ is not only a graded submodule but a *subcomplex* of $C\otimes L$ and the factor complex $C\otimes L/C[-1]$ is isomorphic to C. So, we get the required exact triple of complexes (30), which leads to the long exact sequence of cohomologies. To compute the snake morphism, we can lift a cycle $\xi\in C^i$ into $(C\otimes L)^i = C^i\oplus C^{i-1}$ as $(\xi,0)$, Then $d_{C\otimes L}$ sends it to $(0,(-1)^i f\xi)\in (C[-1])^{i+1}$ as required. $\square$

<u>FILTERED COMPLEXES AND SPECTRAL SEQUENCES.</u> This is the main technical tool for computing and/or comparing homologies of complexes. It generalizes the long exact sequence of cohomologies used above. A *filtered* complex $V=\bigcup\limits_{\nu\in\mathbb{Z}} F^\nu V$ is a complex equipped with decreasing sequence of subcomplexes: $F^\nu V\subset V$, $F^\nu V\supset F^{\nu+1}V$, $d_V(F^\nu V)\subset F^\nu V$. Then for each ν we there is an exact triple of complexes

$$0\longrightarrow F^{\nu+1}V\xhookrightarrow{\varphi_\nu} F^\nu V\xrightarrow{\psi_\nu} G^\nu V\longrightarrow 0$$

where $G^\nu V\overset{\text{def}}{=} F^\nu V/G^{\nu+1}V$. In the simplest case, when the filtration consist of two terms:

$$V=F^0V\supset F^1V\supset F^2V=0$$

we have just one exact triple of complexes

$$\begin{tikzcd} 0 \arrow[r] & F^1V \arrow[r, hook] \arrow[d, equal] & V \arrow[r, two heads] & G^0V \arrow[r] & 0 \\ & G^1V & & & \end{tikzcd}$$

which produces a long exact sequence of cohomologies

$$\cdots\longrightarrow H^{i-1}\left(G^0V\right)\longrightarrow H^i\left(G^1V\right)\longrightarrow H^i\left(V\right)\longrightarrow H^i\left(G^0V\right)\longrightarrow H^{i+1}\left(G^1V\right)\longrightarrow\cdots\tag{31}$$

This long exact sequence «computes» the cohomologies of V in the following sense: $H^i(V)$ is represented as an extension

$$0\longrightarrow \operatorname{coker}\Big(H^{i-1}\left(G^0V\right)\to H^i\left(G^1V\right)\Big)\hookrightarrow H^i(V)\twoheadrightarrow \ker\Big(H^i\left(G^0V\right)\to H^{i+1}\left(G^1V\right)\Big)\longrightarrow 0$$

i. e. there is a short filtration $H(V)\supset F^1H(V)\supset 0$ on $H(V)=\oplus H^i(V)$ whose two factors

$$G^0H(V)=H(V)/F^1H(V)=\bigoplus\ker\Big(H^i\left(G^0V\right)\to H^{i+1}\left(G^1V\right)\Big)$$
$$G^1H(V)=F^1H(V)=\bigoplus\operatorname{coker}\Big(H^{i-1}\left(G^0V\right)\to H^i\left(G^1V\right)\Big)$$

can be described as two cohomologies of a short complex

$$\bigoplus_i H^i\left(G^0(V)\right)\xrightarrow{D}\bigoplus_i H^{i+1}\left(G^1(V)\right)\tag{32}$$

whose differential D is a direct sum of snake morphisms in (31).

Although this description of $H(V)$ looks too prolix and complicated, in practice we often are given with a short filtration on V such that the complex (32) is quite easy: see, for example, the previous computation of the Koszul cohomologies of a regular sequence.

Now we are going to extend this description of $H(V)$ onto general case, when the filtration on V contains more than two terms. Namely, we will construct a sequence of tables E_r, $r\in\mathbb{N}$, called *a spectral sequence*. Cells $E_r^{p,q}$ of each table E_r will be numbered by integers $(p,q)\in\mathbb{Z}\times\mathbb{Z}$ (p varies along horizontals, q — along the verticals). These tables will have the following properties:

- each table E_r is equipped with a differential d_r bihomogeneous of bidegree $(r,1-r)$ in (p,q), that is sending $E_r^{p,q}\xrightarrow{d_r}E_r^{p+r,q+1-r}$ $\forall\, p,\, q$ and satisfying $d_r\circ d_r=0$;
- for each table E_r the next table E_{r+1} consist of the cohomologies of E_r w. r. t. the differential d_r, i. e.

$$E_{r+1}^{p,q}=\frac{\ker\left(E_r^{p,q}\xrightarrow{d_r}E_r^{p+r,q+1-r}\right)}{d_r\left(E_r^{p-r,q-1+r}\right)}\,;$$

- the first table $E_1^{p,q} = H^{p+q}\left(G^pV\right)$ is a straightforward generalization of (32).

It is convenient to imagine that the modules V^i, forming the initial complex V, are placed over these tables along the diagonals $p+q=i$ in such a way that the elements of m-th filtering submodule $F^mV^i \subset V^i$ fill all the cells with $p \geqslant m$ on the i-th diagonal. Then the original differential $V \xrightarrow{d} V$ moves the elements of i-th diagonal to $(i+1)$-th diagonal in such a way that p-coordinate of dv is greater than or equal to p-coordinate of v for all $v \in V^i$.

In these terms the first table E_1 can be considered as the first crude approximations to $H(V)$: its (p,q) cell

$$E_1^{p,q} = H^{p+q}\left(F^pV/F^{p+1}V\right)$$

consists of classes of elements $v \in F^pV^{p+q}$ which are mapped not to zero (i.e. they are not the cycles) but are mapped inside $F^{p+1}V^{p+q+1}$ (i.e. inside one step deeper filtering submodule); and these classes are considered not modulo $d(V^{p+q-1})$ but modulo $d(F^pV^{p+q-1}) + F^{p+1}V^{p+q}$.

Other tables refine this approximation: r-th table takes elements which are sent by d to r steps deeper filtering submodule and factorizes them through the image of r steps lager filtering submodule. More precisely, consider full preimages

$$Z_r^{p,q} = F^pV^{p+q} \cap d^{-1}\left(F^{p+r}V^{p+q+1}\right)$$

and put $E_r^{p,q} = \dfrac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + d\left(Z_{r-1}^{p-r+1,q+r-1}\right)}$.

Exercice 5.18. Check that d induces well defined differential $E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q+1-r}$ and show that the cohomology of this induced differential is isomorphic to $E_{r+1}^{p,q}$.

Our spectral sequence computes the cohomologies $H(V)$ in the following sense. Suppose, that the filtration on V is *bounded*, i.e. $\forall i\ \exists\ p_{\min}(i)$, $p_{\min}(i)$ such that $F^pV^i = V^i\ \forall\ p \leqslant p_{\min}(i)$ and $F^pV^i = 0\ \forall\ p \geqslant p_{\min}(i)$. Then the both differentials: coming in and going out from any (p,q)-cell in i-th diagonal of E_r, will be zero for all $r > \max\left(p_{\min}(i-1) - p_{\max}(i)\,,\, p_{\min}(i) - p_{\max}(i+1)\right)$. Hence, the contents of each cell becomes stable for $r \gg 0$. These stabilized modules are denoted by $E_\infty^{p,q}$ and called *limit terms* of the spectral sequence.

Exercice 5.19. Consider the natural filtration on cohomologies $H(V)$ by the submodules $H(F^pV)$ embedded into $H(V)$ via the inclusions $F^pV \hookrightarrow V$. Check that the factors of this filtration are canonically isomorphic to limit terms of the spectral sequence, i.e.

$$\frac{H^{p+q}(F^pV)}{H^{p+q}(F^{p+1}V)} \simeq E_\infty^{p,q}$$

In other words, there is a natural filtration on $H^i(V)$ whose factors appear along the i-th diagonal of limit table E_∞.

Usually, the situation considered in this section is described by a short (and not quite precise) sentence: *there is a spectral sequence with $E_1^{p,q} = H^{p+q}(G^pV)$ which converges to $H^{p+q}(V)$.*

DOUBLE COMPLEXES AND THEIR TOTAL COMPLEXES. A bigraded object $K = \bigoplus_{p,q} K^{p,q}$ equipped with two differentials

$$d_I\ :\ K^{p,q} \longrightarrow K^{(p+1),q}\ ,\quad d_{II}\ :\ K^{p,q} \longrightarrow K^{p,(q+1)}$$

is called *a double complex*, if

$$d_I^2 = d_{II}^2 = 0 \quad \text{and} \quad [d_I, d_{II}] = d_I \circ d_{II} + d_{II} \circ d_I = 0 \tag{33}$$

If we place objects $K^{p,g}$ in the cells of plane table whose rows are numbered by q and columns are numbered by p, then we get a complex with the differential d_I in each row and a complex with the differential d_{II} in each column.

Exercice 5.20. Let $H_I^{p,q}$ be p-th cohomology of the horizontal complex staying in q-th row w.r.t. the differential d_I. Show that d_{II} induces well defined vertical maps $H_I^{p,q} \xrightarrow{\widetilde{d_{II}}} H_I^{p,q+1}$, which satisfy $\widetilde{d_{II}}^2 = 0$. Similarly, let $H_{II}^{p,q}$ be q-th cohomology of the vertical complex staying in p-th column w.r.t. the differential d_{II}. Show that d_I induces well defined horizontal differentials $H_I^{p,q} \xrightarrow{\widetilde{d_I}} H_I^{p,q+1}$.

Taking direct sums along diagonals we get the modules

$$Tot^i(K) = \bigoplus_{p+q=i} K^{p,q}$$

They form a complex w.r.t. differential $d = d_I + d_{II}$ (in fact, three conditions (33) are equivalent to one condition $d^2 = 0$). This complex is called *the total complex* of the initial double complex K.

The total complex has natural filtration by the subcomplexes

$$F^\nu \mathit{Tot}(K) = \bigoplus_{p \geqslant \nu} K^{p,q}$$

So, the cohomologies $H^{p+q}(\mathit{Tot}(K))$ can be computed using the spectral sequence constructed in the previous section. It starts with the table containing cohomologies

$$E_1^{p,q} == H^{p+q}\left(G^p \mathit{Tot}(K)\right)$$

w.r.t. the differential induced by d. But $G^p \mathit{Tot}(K)$ with the differential induced by $d_I + d_{II}$ coincides with the column complexes from Exercice 5.20 staying in p-th column of the double complex $K^{p,q}$. So, the first table E_1 is formed by *vertical homologies*

$$E_1^{p,q} = H^q_{d_{II}}(K^{p,\bullet})$$

i.e. the cohomologies of the column complexes from Exercice 5.20 w.r.t. the vertical differentials d_{II}. Further, the spectral differential $E_1^{p,q} \xrightarrow{d_1} E_1^{p+1,q}$ coincides with the map

$$H^q_{d_{II}}(K^{p,\bullet}) \xrightarrow{\widehat{d_I}} H^q_{d_{II}}(K^{p+1,\bullet})$$

induced on the vertical homologies by the horizontal differentials d_I. Hence, the second table has $E_2^{p,q} = H^p_{\widehat{d_I}} H^q_{d_{II}}(K)$. Changing the roles of p, q (i.e. flipping $K^{p,q}$ along the main diagonal $p = q$) we get the second spectral sequence, whose differentials go in upper left direction.

Claim 5.21. *Each double complex K produces two spectral sequences*

$${}^{I}E_2^{p,q} = H^p_{\widehat{d_I}} H^q_{d_{II}}(K) \quad , \qquad {}^{II}E_2^{p,q} = H^q_{\widehat{d_{II}}} H^p_{d_I}(K)$$

which compute the same homologies $H^{p+q}(\mathit{Tot}(K))$, of the total complex of K. □

HOMOTOPY CATEGORY $\mathscr{H}ot(\mathscr{A})$ of an Abelian category $\mathscr{A}$ has the same objects as $\mathscr{C}om(\mathscr{A})$ but

$$\mathrm{Hom}_{\mathscr{H}ot(\mathscr{A})}(U,V) \overset{\text{def}}{=} H^0(\mathrm{Homgr}(U,V)) \ .$$

In other words, the homotopy category is obtained from the category of complexes by gluing together all homotopy equivalent arrows. As a consequence, some morphisms that are not isomorphisms in $\mathscr{C}om(\mathscr{A})$ became those in $\mathscr{H}ot(\mathscr{A})$.

Exercice 5.22. Let $\mathscr{V}ec$ be category of finite dimensional vector spaces over a field. Show that $\mathscr{H}ot(\mathscr{V}ec)$ is equivalent to the category of graded vector spaces by proving that each complex V, of vector spaces, is homotopy equivalent to a complex $H(V)$, whose elements are homologies of V and all the differentials are zero maps.

HINT. Decompose each component V^ν as $D^\nu \oplus \ker d = D^\nu \oplus C^\nu \oplus \operatorname{im} d$, where D^ν is complement to the kernel of $V^\nu \xrightarrow{d} V^{\nu+1}$ and C^ν is complement to the image $V^{\nu-1} \xrightarrow{d} V^\nu$ inside this kernel; then C^ν form a subcomplex in V, which is isomorphic (in $\mathscr{C}om(\mathscr{A})$) to the complex $H(V)$ with zero differentials; further, inclusion $C \hookrightarrow V$ and projection $V \hookrightarrow C$ are two maps in $\mathscr{C}om(\mathscr{A})$ that became inverse isomorphisms in $\mathscr{H}ot(\mathscr{A})$.

On the other hand, some non trivial arrows in $\mathscr{C}om(\mathscr{A})$ became zero maps in $\mathscr{H}ot(\mathscr{A})$. In particular — and this stays in a contrast with the previous exercise — the category $\mathscr{H}ot(\mathscr{A})$ *is not* an Abelian category in general.

Exercice 5.23*. Let $\mathscr{A}$ be the category of finitely generated $\mathbb{Z}$-modules.

a) Construct an example of a map between two complexes $U \xrightarrow{\varphi} V$ in $\mathscr{C}om(\mathscr{A})$ that is not zero homotopic but gives an isomorphism $H(U) \xrightarrow{\widehat{\varphi}} H(V)$ on cohomologies.

b) Show that the homotopy category $\mathscr{H}ot(\mathscr{A})$ is not Abelian.