FINAL WRITTEN EXAM
IN COMMUTATIVE AND HOMOLOGICAL ALGEBRA

to get the maximal exam mark it is enough to collect 40 points
by solving any problems you like from the list below

Problem 1 (10 points). Let $M \subset \mathbb{Z}^3$ be a sublattice spanned by vectors $(2,3,3)$, $(3,2,3)$ and $(3,3,2)$. Describe the factor group $\mathbb{Z}^3/M$, i. e. present it as $\mathbb{Z}^r \oplus \mathbb{Z}/p_1^{m_1} \oplus \cdots \oplus \mathbb{Z}/p_s^{m_s}$ with distinct prime p_i's and some integer $r \geqslant 0$.

Problem 2 (10 points). Let A, B be two $n\times n$ - matrices. Fix some positive integer $k < n$. Show that $\det A \cdot \det B = \sum\limits_J \det A'_J \cdot \det B'_J$, where the sum is running over all collections $J = (j_1, j_2, \dots, j_k)$ with $1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n$ and for each J the matrices A'_J, B'_J are obtained from A, B by order preserving exchange of the first k columns in A with the J-columns in B.

Problem 3. Let a Noetherian ring A have a unique proper maximal ideal $0 \neq \mathfrak{m} \subset A$. Show that:

a) (10 points) $\mathfrak{m}M \neq M$ for any non zero finitely generated A-module M;

b) (10 points) any finitely generated projective A-module is free.

Problem 4 (10 points). For any complex of modules $\cdots]; \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow 0$ construct a bicomplex of projective modules $P_{p,q}$ situated in degrees $p,q \geqslant 0$ with differentials of bidegrees $(-1,0)$ and $(0,-1)$ such that:

(1) p-th column of $P_{p,q}$ gives projective resolution for M_p;

(2) homologies of row-complexes in $P_{p,q}$ are projective modules and vertical differentials arrange them into projective resolutions for $H(M_\bullet)$.

Problem 5 (10 points). On the set $\mathbb{N}$, of positive integers, consider a partial ordering given by the divisibility relation: $n \preceq m \overset{\text{def}}{\iff} n|m$. In the category of Abelian groups compute the injective limit $\varinjlim \mathbb{Z}/n\mathbb{Z}$ w. r. t. a direct system of arrows $\mathbb{Z}/n\mathbb{Z} \xhookrightarrow{[1]\mapsto[m/n]} \mathbb{Z}/m\mathbb{Z}$ given for any ordered pair $n \preceq m$.

Problem 6 (10 points). Let principal open sets $\mathscr{D}(f_\nu)$, $\nu = 1, \dots, n$, cover an affine (maybe reducible) algebraic variety $X = \operatorname{Spec}_{\mathrm{m}} A$. Show that the Koszul complex $\bigotimes\limits_{\nu=1}^{n} \left(A \xrightarrow{x\mapsto f_\nu x} A\right)$ is exact.

Problem 7 (10 points). Show that any finite dimensional (as a vector space) algebra over an arbitrary field has only a finite set of prime ideals and all these ideals are maximal.

Hint. Use properties of integer ring extensions when one of rings is a field.

Problem 8 (10 points). Let $M = \mathbb{C}[x,y]/(x,y)$, $N = \mathbb{C}[x,y]/(x-1)$. In the category of $\mathbb{C}[x,y]$-modules compute all $\operatorname{Ext}^\nu(M,N)$.