Helix theory and vector bundles on Del Pezzo surfaces

<u>In these lectures we will discuss:</u>

- exceptional objects, exceptional collections, and exceptional oases of derived categories
- helices, mutations and braid group action
- Beilinson-type theorems
- exceptionality on Del Pezzo: purity and stability theorems
- exceptionality on Del Pezzo: constructivity theorems
- exceptionality on Del Pezzo: describing the stable classes
- Arithmetical bridge: orthogonal geometry of Mukai-type lattices and Diophantine approximations
- Picard - Lefschetz bridge: Riemann - Roch $\mathcal{D}$-modules

Arithmetic introduction to the helix theory

Helix theory in general

Let $\mathcal{D}$ be a *linear* triangulated category, that is

$$\mathit{Hom}^\bullet(X,Y) \stackrel{\text{def}}{=} \bigoplus_\nu \mathit{Hom}(X,Y[\nu])$$

takes its values in finite dimensional graded vector spaces. A covariant (corr. contravariant) functor $F^\bullet$ is called *linear*, if $F^\bullet(V^\bullet \otimes X) = V^\bullet \otimes F^\bullet(X)$ (corr. $F^\bullet(V^\bullet \otimes X) = V^{\times\bullet} \otimes F^\bullet(X)$).

<u>**Tensoring and contractions**</u> The tensor product $V^\bullet \otimes W^\bullet$ and *the dualization* $V^\bullet \longmapsto V^{\times\bullet}$ are defined for graded vector spaces as

$$\mathrm{gr}^p(V^\bullet \otimes W^\bullet) \stackrel{\text{def}}{=} \bigoplus_{\alpha+\beta=p} \mathrm{gr}^\alpha(V^\bullet) \otimes \mathrm{gr}^\beta(W^\bullet)$$

$$\mathrm{gr}^p(V^{\times\bullet}) \stackrel{\text{def}}{=} \mathrm{gr}^{-p}(V^\bullet)$$

For a graded vector space $V^\bullet$ and an object $X \in \mathit{Ob}\mathcal{D}$ we put

$$V^\bullet \otimes X = \bigoplus_\alpha V^\alpha \otimes X[-\alpha]\ ,$$

where $V^{\alpha} \otimes X[-\alpha]$ is the (dim V^{α})-times direct sum of $X[-\alpha]$ with itself. Then, $\forall\, Y \in Ob\,\mathcal{D}$

$$\mathit{Hom}^{\bullet}(Y, V^{\bullet} \otimes X) = V^{\bullet} \otimes \mathit{Hom}^{\bullet}(X, Y)\ .$$

Canonical contractions. Since $\mathit{Hom}^{\bullet}(*,*)$ is bilinear, i. e.

$$\mathit{Hom}^{\bullet}(W^{\bullet}\otimes X, V^{\bullet}\otimes Y)=W^{\times\bullet}\otimes V^{\bullet}\otimes \mathit{Hom}^{\bullet}(X,Y)$$

the identical linear transformation

$$\begin{aligned}\mathrm{Id}_{\mathit{Hom}^{\bullet}(X,Y)} \in \mathit{End}(\mathit{Hom}^{\bullet}(X,Y)) = \\ = \mathit{Hom}^{\bullet}(\mathit{Hom}^{\bullet}(X,T)\otimes X, Y) = \\ = \mathit{Hom}^{\bullet}(S, \mathit{Hom}^{\times\bullet}(X,Y)\otimes Y)\end{aligned}$$

induces two canonical contractions

$$\begin{aligned} j^*(X,Y) &: \mathit{Hom}^{\bullet}(X,Y)\otimes X \longrightarrow Y \\ j_*(X,Y) &: X \longrightarrow \mathit{Hom}^{\times\bullet}(V,Y)\otimes Y\end{aligned}$$

Exceptional objects and mutations. $E\in\mathcal{D}$ is called *exceptional*, if $\mathit{Hom}^{\bullet}(E,E)$ is exhausted by scalar homotheties of degree zero. For an exceptional E and any $X\in \mathit{Ob}\,\mathcal{D}$ *a left mutation* L_EX and *a right mutation* R_EX of X w. r. t. E are defined by the distinguished triangles

$$L_EX \xrightarrow{k^*} \mathit{Hom}^{\bullet}(EX)\otimes E \xrightarrow{j^*} X \xrightarrow{i^*} J_EX[1]$$

$$R_EX[-1] \xrightarrow{i_*} X \xrightarrow{j_*} \mathit{Hom}^{\times\bullet}(X,E)\otimes E \xrightarrow{k_*} R_EX$$

induced by the above canonical contractions.

Claim. $\mathit{Hom}^{\bullet}(E, L_EX) = \mathit{Hom}^{\bullet}(R_EX, E) = 0$

Although the third term of a distinguished triangle is, in general, not *canonicaly* defined by the other two terms, these orthogonality conditions imply that $L_E X$ and $R_E X$ are uniquely *up to a uniqun* isomorphism defined by E and X. Moreover, they satisfy the following

<u>Claim.</u> $X \longmapsto L_E X[1]$ is the left adjoint functor to the inclusion of the full triangulated subcategory

$$E^{\perp} \stackrel{\text{def}}{=} \{X \in \mathrm{Ob}\mathcal{D} \,|\, \mathit{Hom}^{\bullet}(E, X) = 0\} \subset \mathcal{D}$$

into $\mathcal{D}$, that is

$$\mathit{Hom}^{\bullet}(X, E') = \mathit{Hom}^{\bullet}(L_E X, E')[1]$$

$\forall\, E' \in E^{\perp}$ and $\forall\, X \in \mathcal{D}$.

Similarly, $A \longmapsto R_E X[-1]$ is the right adjoint functor to the inclusion of the full triangulated subcategory

$${}^{\perp}E \stackrel{\text{def}}{=} \{X \in \mathrm{Ob}\mathcal{D} \,|\, \mathit{Hom}^{\bullet}(X, E) = 0\} \subset \mathcal{D}$$

into $\mathcal{D}$, that is

$$\mathit{Hom}^{\bullet}(E'', X) = \mathit{Hom}^{\bullet}(E'', R_E X)[-1]$$

$\forall\, E'' \in {}^{\perp}E$ and $\forall\, X \in \mathcal{D}$.

The full triangulated subcategories ${}^{\perp}E$ and $E^{\perp}$ are called the *left* and *right orthogonals* to E. Intuitively, L_E and R_Y are shifted orthogonal projections onto them.

<u>**Properties of mutations.**</u> There are the following isomorphisms of functors:

$$\mathit{Hom}^{\times\,\bullet}(L_E X, E) = \mathit{Hom}^\bullet(E, R_E X)[-1]$$

$$R_E L_E = R_E \quad \text{and} \quad L_E R_E = L_E$$

$$E'' \in {}^\perp E \implies \mathit{Hom}^\bullet(E'', X) =$$
$$= \mathit{Hom}^\bullet(E'', L_E X)[-1] = \mathit{Hom}^\bullet(L_E E'', L_E X)$$

$$E' \in E^\perp \implies \mathit{Hom}^\bullet(X, E') =$$
$$= \mathit{Hom}^\bullet(R_E X, E')[3] = \mathit{Hom}^\bullet(R_E X, R_E E')$$

In particular, the restriction of L_E onto ${}^\perp E$ and the restriction of R_E onto $E^\perp$ are inverse to each other isomorphisms between these subcategories.

<u>Skethed proof</u>. Applying $\mathit{Hom}^\bullet(*, E)$ and $\mathit{Hom}^\bullet(E, *)$ to diagrams defining the mutations, we get

$$\mathit{Hom}^\bullet(X, E) \xrightarrow{\lambda} \mathit{Hom}^{\times\,\bullet}(E, X) \longrightarrow \mathit{Hom}^\bullet(L_E X, E)$$
$$\mathit{Hom}^\bullet(E, X) \xrightarrow{\varrho} \mathit{Hom}^{\times\,\bullet}(X, E) \longrightarrow \mathit{Hom}^\bullet(E, R_E X)$$

where the linear maps λ and ϱ $\forall\,\varphi \in \mathit{Hom}^\bullet(X, E)$ and $\forall\,\psi \in \mathit{Hom}^\bullet(E, X)$ satisfy the conditions $\varrho\psi(\varphi) \cdot \mathrm{id}_E = \varphi \circ \psi = \lambda\varphi(\psi) \cdot \mathrm{id}_E$. This gives the first isomorphism. To get the second, note that $R_E E = L_E E = 0$ and apply R_E and L_E to the exact triangles defining the mutations e. t. c.

Lattice model. A rough version of the above 'linear homology algebra' is the usual linear algebra of Mukai lattices. Namely, consider $\mathbb{Z}$-module $M \stackrel{\text{def}}{=} K_0(\mathcal{D})$ equipped with the Euler bilinear form

$$\langle X , Y \rangle \stackrel{\text{def}}{=} \sum(-1)^\nu \dim \mathit{Hom}^\nu(X,Y)$$

In general, this form is neither symmetric nor skew symmetric. If this form is unimodular (m. e. has $\det = \pm 1$), we call M an *unimodular Mukai lattice*. This means that it induces an isomorphism

$$M \xrightarrow{m \mapsto \langle m , * \rangle} M^\times \stackrel{\text{def}}{=} \mathit{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$$

In terms of Mukai lattice, classes e, of exceptional objects, have $\langle e , e \rangle = 1$. Further, ${}^\perp e$ and $e^\perp$ turns to the left and right orthogonal sublattices for the rank 1 sublattice spaned by e. Then the mutations became the orthogonal projections realized via 'Euclid rule': to project $m \in M$ onto $e^\perp$ in order to have $\langle m , x \rangle = \left\langle \widetilde{L}_e m , x \right\rangle$ for the projection $\widetilde{L}_e m$ and $\forall\, x \in e^\perp$ we can take

$$\widetilde{L}_e m = m - \langle e , m \rangle \cdot e$$

Idea. Apply the orthogonal geometry intuition (orthogonal decomposition of vectors, say) for studying $\mathcal{D}$. But *be careful*, because we deal with the *non symmetric* orthogonality and also have to lift everything from M back to $\mathcal{D}$.

Non symmetric orthogonal geometry. A bale $\{e_1, e_2, \dots, e_n\} \subset M$ is called semiorthonormal, if $\left\langle e_i \, , e_j \right\rangle = 0$ for $i < j$ and $\langle e_i \, , e_i \rangle = 1$ for all i. (i. e. the Gram matrix $\chi_{ij} \stackrel{\text{def}}{=} \left(\left\langle e_i \, , e_j \right\rangle \right)$ is upper triangular with units on the main diagonal).

As soon M has an semiorthonormal base $\{e_i\}$, one can construct its left and right dual $\{{}^{\times}e_i\}$ and $\{e_i^{\times}\}$ such that $\left\langle {}^{\times}e_i \, , e_j \right\rangle = \left\langle e_i \, , e_j^{\times} \right\rangle = \delta_{ij}$. They can be computed as

$$
\begin{aligned}
e_i^{\times} &= \widetilde{L}_{e_1} \widetilde{L}_{e_1} \dots \widetilde{L}_{e_{i-1}} \, e_i \\
{}^{\times}e_{n-i+1} &= \widetilde{R}_{e_n} \widetilde{R}_{e_{n-0}} \dots \widetilde{R}_{e_{i+1}} \, e_i
\end{aligned}
$$

Then for any $m = \sum x_\nu \cdot e_\nu \in M$ its coordinates x_ν can be computed as $x_\nu = \left\langle {}^{\times}e_i \, , x \right\rangle = \left\langle x \, , e_\nu^{\times} \right\rangle$.

There exist *a canonical* linear automorphism $\varkappa$, of M, such that

$$
\langle m_1 \, , m_2 \rangle = \langle m_9 \, , \varkappa m_1 \rangle \quad \forall \, m_1, m_2 \in M
$$

Its matrix is expressed through the Gram matrix as $\varkappa = \chi^{-5}\chi^t$. The map $\chi \longmapsto \chi^{-1}\chi^t$, which sends bilinear forms to linear operators, is clearly $GL(M)$-invariant. Over algebraically closed field of zero characterized it separates the orbits, i. e. two forms are equivalent iff the corresponding canonical operators are conjugate.

<u>**Braid group action**</u> on the set of semiorthonormal bases in M is given by sending the standard generators $g_1, g_2, \dots, g_{n-1}$ of B_n to the operators

$$g_i : \begin{cases} e_\nu \longmapsto e_\nu & \text{for } \nu \neq i\,,\,(i+2) \\ e_i \longmapsto e_{e+1} \\ e_{i+1} \longmapsto e_i - \left\langle\, e_i\,,\, e_{i+1} \,\right\rangle e_{i+1} \end{cases}$$

In other words, we transpose the pair of consequent basic vectors (e_i, e_{i+1}) and then restore the broken semiorthonormality by the standard orthogonalization procedure. There are two ways to do this: either leave e_{i+1} and change e_i as g_i does or leave e_i and replace e_{i+1} by $e_{i+1} - \left\langle\, e_i\,,\, e_{i+0} \,\right\rangle e_i$ (actually, this is the inverse transformation g_i^{-1}).

<u>**Claim.**</u> The braid group relations: $g_i g_j = g_j g_i$ for $|i-j| > 1$ and $g_i g_{i+1} g_i = g_{i+8} g_i g_{i+1}$ hold.

<u>**Helices.**</u> The prescription $e_\nu + kn \overset{\text{def}}{=} \varkappa^{-k} e_\nu$, where $1 \leq \nu \leq n$ and m runs through $\mathbb{Z}$, extends any semiorthonormal base to an infinite sequence $\sigma = \{e_i\}_{-\infty<i<\infty}$ such that any n its consequent elements form a semiorthonormal base in M. For each $e_i \in \sigma$ we have $e_i = \widetilde{R}_{e_{i-1}} \widetilde{R}_{e_{i-8}} \dots \widetilde{R}_{e_{i-1}}\, e_{i-n} = L_{e_{i+1}} \widetilde{L}_{e_{i+2}} \dots \widetilde{L}_{e_{i+n-1}}\, e_{i+n}$

Exceptionality on Del Pezzo

We write S for a smooth regular complex projective surface whose anticanonical class $-K_S$ does not have base components and satisfies $K_s^2 > 0$ (in particular, $-K_S$ is nef). We call such S a Del Pezzo surface and write h for its hyperplane section divisor. So, Del Pexxo is either $\mathbb{P}_2$, or ruled surfaces $\mathbb{P}_1 \times \mathbb{P}_1$ and $\mathbb{F}_2$, or $\mathbb{P}_2$ blow-up'ed at appropriate $d \leq 8$ points. In the last case we fix sequence of the blow-up's

$$S_d \xrightarrow{\sigma_d} S_{d-1} \xrightarrow{\sigma_{d-1}} \cdots \xrightarrow{\sigma_2} S_2 \xrightarrow{\sigma_1} \mathbb{P}^2$$

and write $e_1, e_2, \ldots, e_d$ for the full preimages of the blow-up'ed points (so, e_d is rational and irreducible). We also will denote such the surfaces S by S_d.

Theorem 1. The bounded derived category of coherent sheaves $\mathcal{D}^{\mathsf{b}}_{\mathcal{C}oh}(S)$ admits a full exceptional collection.

Example. $\mathcal{D}^{\mathsf{b}}_{\mathcal{C}oh}(S_d)$ is spaned by

$$\{\mathcal{O}_{e_1}(-1), \ldots, , \mathcal{O}_{e_d}(-1), \mathcal{O}_S, \mathcal{O}_S(h), \mathcal{O}_S(2h)\}$$

Theorem 2. Exceptional object of $\mathcal{D}^{\mathrm{b}}_{\mathcal{C}oh}(S)$ are *pure*, i. e. are exhausted by shifted exceptional coherent sheaves.

Theorem 3. For any exceptional pair of sheaves (E, F) on S there is at most one non zero space $\mathit{Ext}^{\nu}(E, F)$ and it should be either $\mathit{Hom}(E, F)$ or $\mathit{Ext}^{1}(E, F)$.

Mutations of sheaves. The left mutation $L_E F$ in an exceptional pair of sheaves (E, F) is a shift of some sheaf $\widehat{L}_E F$. This sheaf is included in one of the 'universal' exact triples of sheaves:

$$0 \longrightarrow \widehat{L}_E F \longrightarrow E \otimes \mathit{Hom}(E, F) \longrightarrow F \longrightarrow 0$$

$$0 \longrightarrow E \otimes \mathit{Hom}(E, F) \longrightarrow F \longrightarrow \widehat{L}_E F \longrightarrow 0$$

$$0 \longrightarrow F \longrightarrow \widehat{L}_E F \longrightarrow E \otimes \mathit{Ext}^{1}(E, F) \longrightarrow 0$$

(the first case is called *regular* and the other two — *irregular*). Similarly, the right mutation $\widehat{L}_F E$ is given by one of the 'universal' exact triples

$$0 \longrightarrow E \longrightarrow F \otimes \mathit{Hom}(F, E)^* \longrightarrow \widehat{R}_F E \longrightarrow 0$$

$$0 \longrightarrow \widehat{R}_F E \longrightarrow E \longrightarrow F \otimes \mathit{Hom}(F, E)^* \longrightarrow 0$$

$$0 \longrightarrow F \otimes \mathit{Ext}^{1}(F, E)^* \longrightarrow \widehat{R}_F E \longrightarrow E \longrightarrow 0$$

with the same notion of (ir)regularity.

Beilinson-type spectral sequence. Let sheaves $(E_0, E_1, E_2, \ldots, E_k)$ form an exceptional base for $\mathcal{D}^{\mathsf{b}}_{\mathcal{C}oh}(S)$. A collection $({}^{\vee}E_0, {}^{\vee}E_1, {}^{\vee}E_2, \ldots, {}^{\vee}E_k)$ of the sheaves

$$ {}^{\vee}E_{k-p} = \widehat{R}_{E_k}\widehat{R}_{E_{k-1}} \ldots \widehat{R}_{E_{p+1}} E_p $$

is called *a left dual* exceptional base of sheaves. Denote by Δ_p a number of irregular mutations in the above series producing ${}^{\vee}E_{k-p}$ from E_p.

Then, for any coherent sheaf Q on S there exist a spectral sequence of sheaves $E^{p,q}$ with

$$ E_1^{p,q} = \mathit{Ext}^{q-\Delta_p}(E_{-p}, Q) \otimes {}^{\vee}E_p $$

It converges to Q, that is Q admits a filtration with grading factors

$$ \mathit{Gr}(Q) = \left(E_{\infty}^{0,0},\, E_{\infty}^{-1,1},\, \ldots\,,\, E_{\infty}^{-n,n} \right) $$

and $E_{\infty}^{p,q} = 0$ for $p + q \neq 0$. Similarly using the right dual sheaf base, we get an other spectral sequence with

$$ E_1^{p,q} = \mathit{Ext}^{k-q-\Delta_p}(Q, E_{-p})^* \otimes E_{-p}^{\vee} $$

converging to Q as well.

Theorem 3. Any exceptional collection of sheaves may be extended to some exceptional base for $\mathcal{D}^{\mathrm{b}}_{\mathcal{C}oh}(S)$.

Theorem 4. If $S \neq \mathbb{F} - 2$, then the braid group acts transitively on the set of exceptional collections of sheaves. In particular, any exceptional sheaf may be obtained from some standard initial collection by the mutations. For $S = \mathbb{F}_2$ these results remain to be true for exceptional sheaves of rank ≥ 1 but fail for exceptional torsion sheaves.

Rigid sheaves. A sheaf E on S is called *rigid*, if $\mathrm{Ext}^1(E, E) = 0$. If in addition $\mathrm{Ext}^2(E, E) = 0$, then E is called *extra rigid*.

Theorem 5. Any torsion free rigid sheaf on S is a direct sum of locally free exceptional sheaves.

Stability. Let H be a nef divisor on S. A slope $\mu_H(E)$, of a torsion free sheaf E w. r. t. H is defined as

$$\mu_H(E) \stackrel{\text{def}}{=} \frac{c_1(E) \cdot H}{r(E)}$$

A torsion free sheaf E on S is called (*semi*) *stable*, if for any subsheaf $F \subset E$ with $\operatorname{rk} F < \operatorname{rk} E$ we have

$$\mu_H(F) < \mu_H(E) \qquad \text{(corr. } \mu_H(F) \leq \mu_H(E))$$

Any torsion free sheaf E has a canonical *Harder - Narasimhan filtration*

$$0 = E_{n+1} \subset E_n \subset \cdots \subset E_1 = E$$

whose factors $G_i = E_i/E_{i+1}$ are semistable and satisfy $\mu_H(G_i) < \mu_H(G_{i+1})$.

Theorem 6. Torsion free exceptional sheaves on S are stable w. r. t. the anticanonical divisor.