Conference «Science of the Future», Mathematical section
St.Petersburg , September 17 – 20 , 2013

ALAG:
ABELIAN LAGRANGIAN ALGEBRAIC GEOMETRY

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Contents

Space of Lagrangian cycles $\mathfrak{L}$

We write (M,ω) for a smooth real $2n$-dimensional oriented manifold M equipped with a closed 2-form $\omega \in \Lambda^2 T^*M$ that provides an isomorphism

$$\omega : TM \xrightarrow{\sim} T^*M \,.$$

We fix a real n-dimensional compact oriented manifold S and a homotopy class $\mathrm{Map}_{\mathrm{LA}}(S,M)$ of smooth maps $\varphi : S \to M$ such that $\varphi^*(\omega) = 0$.

Definition

Factor space $\mathfrak{L} \stackrel{\text{def}}{=} \mathrm{Map}_{\mathrm{LA}}(S,M)/\mathrm{Diff}_0(S)$ is called *a space of Lagrangian cycles* (of fixed topological type).

Here $\mathrm{Diff}_0(S)$ stays for the identity component of the group of diffeomorphisms $S \xrightarrow{\sim} S$.

Basic example: cotangent bundle

Cotangent bundle $M = T^*S$ of a smooth compact oriented manifold S possesses canonical *universal 1-form* η such that for any 1-form α on S one has

$$\alpha = s_\alpha^* \eta \,,$$

where $s_\alpha : S \hookrightarrow T^*S$ is the section provided by α. Locally, in coordinates q_i on S and linear coordinates p_i along dq_i in the fibers of T^*S

$$\eta(p,q) = \sum_{j=1}^{n} p_j \, dq_j \,, \quad \omega(p,q) \stackrel{\text{def}}{=} d\eta = \sum_{j=1}^{n} dp_j \wedge dq_j \,.$$

Thus, $\omega = d\eta$ provides M with the canonical symplectic structure. The fibers of the projection $p : T^*S \twoheadrightarrow S$ and the zero section $s_0 : S \hookrightarrow M$ are obviously Lagrangian cycles. An arbitrary section $s_\alpha : S \hookrightarrow M$ given by 1-form $\alpha = s^*\eta$ is Lagrangian iff α is closed, because $s_\alpha^*\omega = s_\alpha^* d\eta = d s_\alpha^* \eta = d\alpha$.

Tangent space $T_\varphi\mathfrak{L}$

Claim

Let $\varphi : S \hookrightarrow M$ be a smooth Lagrangian immersion. Then the tangent space $T_\varphi\mathfrak{L}$ is the space $Z^1_{\mathrm{DR}}(S)$ of closed 1-forms on S.

Since the condition $\varphi^*\omega = 0$ is preserved under an infinitesimal deformation of φ along a vector field v iff $\varphi^*\,\mathcal{L}ie_v\,\omega = 0$,

$$T_\varphi\left(\mathrm{Map}(S,M)/\mathrm{Diff}_0(S)\right) = \{v \in \mathcal{C}^\infty(S, N_\varphi) \mid \varphi^*\,\mathcal{L}ie_v\,\omega = 0\}.$$

If φ is a smooth immersion, the isomorphism $\omega : TM \xrightarrow{\sim} T^*M$ identifies the normal bundle of φ with the cotangent one

$$N_\varphi = \varphi^*(TM)/TS \xrightarrow[\sim]{\ \omega\ } \varphi^*(T^*M)/\mathrm{Ann}(TS) \simeq T^*S$$

by sending a local normal field v to 1-form $\xi = \varphi^* v \llcorner \omega \in \mathcal{C}^\infty(S, T^*S)$. Under this identification $v \in T_\varphi\mathfrak{L}$ go precisely to closed 1-forms

$$d\xi = d\varphi^*(v \llcorner \omega) = \varphi^* d(v \llcorner \omega) = \varphi^*(\mathcal{L}ie_v\,\omega + v \llcorner d\omega) = \varphi^*\,\mathcal{L}ie_v\,\omega = 0.$$

Darboux – Weinstein uniformisation of $\mathfrak{L}$

Claim

Near a smooth immersion $\varphi : S \hookrightarrow M$ the space of Lagrangian cycles $\mathfrak{L}$ is an (infinite dimensional) smooth manifold locally modeled by some open neighborhood $\mathfrak{U}$ of the zero in a vector space $Z^1_{\mathrm{DR}}(S)$ of closed 1-forms.

Classical results of A. Weinstein show that

- any smooth immersion $\varphi : S \hookrightarrow M$ can be extended to a smooth immersion $\widehat{\varphi} : \mathcal{U} \hookrightarrow M$ of some open tube neighbor $\mathcal{U} \subset T^*S$ about the zero section of the cotangent bundle in such a way that $\widehat{\varphi}^*\omega = d\eta$ is the standard symplectic form on T^*S
- let $\mathfrak{U} \subset Z^1_{\mathrm{DR}}(S) = T_S\mathfrak{L}$ be the set 1-forms α on S whose graphs $\Gamma_\alpha \subset T^*S$ lie in $\mathcal{U}$; then the previous extension leads to a smooth *exponential mapping* $\exp : \mathfrak{U} \to \mathfrak{L}$ sending such a form α to Lagrangian cycle $\widehat{\varphi}(\Gamma_\alpha) \subset M$ laying near $\varphi(S)$
- the image of the exponential mapping contains an open neighborhood of φ (in appropriate topology on $\mathfrak{L}$).

Hermitian line bundles

Let X be a smooth manifold, $p : P \twoheadrightarrow X$ be a principal U_1-bundle, $L \twoheadrightarrow X$ be associated Hermitian line bundle , for which $P \subset L$ is the bundle of unit circles. Write $u = 2\pi\,\partial/\partial t$ for the vertical U_1-invariant vector field on P whose flow turns round about P in the unit time.

A 1-form $\alpha \in T^*P$ such that $\mathcal{L}ie_u\,\alpha = 0$ and $\alpha(u) = 1$ is called *Hermitian connection* on P. Associated with α is *covariant differentiation* of local sections $\nabla_\alpha : \mathcal{C}^\infty(\mathcal{U}, L) \to \mathcal{C}^\infty(\mathcal{U}, T^*X \otimes L)$ over open $\mathcal{U} \subset X$. It takes an unitary section $\sigma : \mathcal{U} \hookrightarrow P \subset L$ to $\nabla_\alpha \sigma \overset{\text{def}}{=} i\xi_\sigma \otimes \sigma$, where $\xi_\sigma \overset{\text{def}}{=} \sigma^*\alpha \in \mathcal{C}^\infty(\mathcal{U}, T^*X)$ is called the *connection 1-form* of unitary trivialization σ. An arbitrary section $s = f\sigma$ with $f \in \mathcal{C}^\infty(\mathcal{U}, \mathbb{C})$ is differentiated by the Leibniz rule:

$$\nabla_\alpha s = \nabla_\alpha(f\sigma) \overset{\text{def}}{=} (df + if\xi_\sigma) \otimes \sigma \,.$$

Under a change of unitary trivialization $\sigma \mapsto \tau = h\sigma$ with $h \in \mathcal{C}^\infty(\mathcal{U}, U_1)$ the connection form ξ_σ turns to $\xi_\tau = \xi_\sigma - i\,d\ln h$. Thus, over simply connected $\mathcal{U}$, any form in the class $\xi_\sigma + i d\big(\mathcal{C}^\infty(\mathcal{U}, \mathbb{R})\big)$ can be achieved as the connection 1-form of appropriate unitary trivialization.

Curvature, flatness, horizontality...

Since differentials $d\xi_\sigma$ do not depend on σ, there is global closed 2-form $\omega_a \stackrel{\text{def}}{=} d\xi_\sigma \in \mathcal{C}^\infty(X, \Lambda^2 T^* X)$ called (real, normalized) *curvature form* of α.

Claim

The De Rahm cohomology class $[\omega_a] = 2\pi \cdot c_1(L) \in 2\pi \cdot H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$.

Here $c_1(L)$ is *first Chern class* of L represented by (additive) $\mathbb{Z}$-valued Čech's 2-cocycle $\left(\ln(g_{ij}) + \ln(g_{jk}) + \ln(g_{ki})\right)/(2\pi i)$, where g_{ij} is U_1-valued (multiplicative) Čech's 2-cocycle formed by transition functions of L.

A section $s = f\sigma \in \mathcal{C}^\infty(\mathcal{U}, L)$, where $\sigma : \mathcal{U} \to P$ is unitary, is called *horizontal* if $\nabla_\alpha s = 0$ or, equivalently, $d \ln f = -i\xi_\sigma$. Over simply connected $\mathcal{U}$ horizontal sections exist iff $\omega_\alpha|_\mathcal{U} \equiv 0$. In this case connection α is called *flat*, and unitary horizontal trivialization stay in bijection with constants $c \in U_1 \subset \mathbb{C}$ and are given as $\sigma_c = c\, e^{-ig}\sigma$, where $\sigma \in \mathcal{C}^\infty(\mathcal{U}, P)$ and $g \in \mathcal{C}^\infty(\mathcal{U}, \mathbb{C})$ with $dg = \xi_\sigma$ are arbitrarily fixed. Such σ_c provide fibers of L with *parallel displacement* along paths in $\mathcal{U}$.

Flat hermitian connections

Associated with globally flat Hermitian connection α are

- *conjugation class of character* $\chi_\alpha : \pi_1(X) \to U_1$ that sends a loop γ to the rotation of fiber of L at a base point under the horizontal displacement along γ (horizontality of $f\sigma \iff d\ln f = -i\xi_\sigma$)
- *cohomology class* $[\chi_\alpha] \in H^1(X, U_1)$ represented by a Čech cocycle provided by transition functions between local horizontal unitary trivializations of L; it is annihilated by the red arrow in

$$H^1(X,\mathbb{Z}) \hookrightarrow H^1(X,\mathbb{R}) \to H^1(X,U_1) \to H^2(X,\mathbb{Z})$$

  which is the long exact cohomology sequence coming from the exponential exact triple of coefficients $0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{t \mapsto \exp it} U_1 \to 1$.

Equivalence classes of flat hermitian connections on L w.r.t. the action of gauge group $\mathcal{C}^\infty(X, U_1)$ stay in bijection with the conjugation classes of unitary characters $\pi_1(X) \to U_1$ and with the points of the *real Jacobian torus*

$$J_X \overset{\text{def}}{=} H^1(X,\mathbb{R})/2\pi \cdot H^1(X,\mathbb{Z}).$$

Pre-quantum equipment

Let the cohomology class of the symplectic form ω on M satisfy

$$[\omega] \in 2\pi \cdot H^2(M,\mathbb{Z}) \subset H^2(M,\mathbb{R}) .$$

Then M admits a unique (up $\mathcal{C}^\infty(M,U_1)$-isomorphism) Hermitian line bundle $L \twoheadrightarrow M$ with hermitian connection α whose (real, normalized) curvature 2-form $\omega_\alpha = \omega$. Such L is called a *pre-quantization line bundle*.

Definition

A *pre-quantum equipment* of level $k \in \mathbb{N}$ on M is a quadruple $(M,\omega,L^{\otimes k},\alpha_k)$, where L is pre-quantization line bundle and α_k is the hermitian connection on $L^{\otimes k}$ that has $\omega_{\alpha_k} = k\,\omega$.

Note that α_k provides the principal U_1-bundle of unit circles $P_k \subset L^{\otimes k}$ with the *contact structure*.

Bohr – Sommerfeld cycles

A pull back $\varphi^* L^{\otimes k}$ of the level k pre-quantization bundle along any Lagrangian immersion $\varphi : S \hookrightarrow M$ is a flat hermitian line bundle on S, because

$$\omega_{\varphi^*\alpha_k} = k \cdot \varphi^* \omega = 0 \,.$$

Definition

An immersed Lagrangian cycle $\varphi : S \hookrightarrow M$ is called *Bohr – Sommerfeld* of level k if $\varphi^* L^{\otimes k}$ allows a *global horizontal trivialization*, i.e. when the class $[\chi_{\varphi^*\alpha_k}] \in J_S = H^1(S,\mathbb{R})/H^1(S,\mathbb{Z})$ vanishes.

We write $\mathfrak{B}_k \subset \mathfrak{L}$ for the locus of Bohr – Sommerfeld cycles of level k.

Thus, $\mathfrak{B}_k \subset \mathfrak{L}$ is the zero set of section $\lambda_k : \varphi \mapsto [\chi_{\varphi^*\alpha_k}]$ of the trivial *Jacobian bundle* $\mathfrak{L} \times J_S$ over $\mathfrak{L}$ with fiber J_S. In particular, it has finite «expected codimensinon» $\mathrm{ex.codim}_{\mathfrak{L}} \mathfrak{B} = b_1(S) = \dim H^1(X,\mathbb{R})$.

For example, we expect a finite number of Bohr – Sommerfeld cycles in a generic n-dimensional family of n-dimensional Lagrangian tori T^n.

Planckian cycles and the Berry bundle

Write $\mathsf{Map}_{\mathsf{LE}}(S, P_k) \overset{\text{def}}{=} \{\widetilde{\varphi} : S \hookrightarrow P_k \mid \widetilde{\varphi}^* \alpha_k = 0\}$ for the space of smooth *Legendrian embeddings*. Its factor space

$$\mathfrak{P}_k \overset{\text{def}}{=} \mathsf{Map}_{\mathsf{LE}}(S, P_k)/\mathsf{Diff}_0(S)$$

is called a *space of Planckian cycles* of level k. In other words, a Planckian cycle $\widetilde{\varphi} : S \hookrightarrow P_k$ is a smoothly immersed Bohr – Sommerfeld cycle $\varphi : S \hookrightarrow M$ together with a horizontal unitary section $\widetilde{\varphi} : S \hookrightarrow \varphi^* L^{\otimes k}$ fitting into commutative diagram

$$\begin{tikzcd} & P_k \subset L^{\otimes k} \arrow[d, "p_k"] \\ S \arrow[ur, dashed, "\widetilde{\varphi}"] \arrow[r, "\varphi"'] & M . \end{tikzcd}$$

Thus, Planckian cycles form a principal U_1-bundle $\pi_k : \mathfrak{P}_k \twoheadrightarrow \mathfrak{B}_k$ over Bohr – Sommerfeld cycles (so called *principal Berry bundle* of level k).

Darboux – Weinstein uniformizations of $\mathfrak{P}_k$ and $\mathfrak{B}_k$

Extend underlying Bohr – Sommerfeld cycle $\varphi = p \circ \widetilde{\varphi} : S \to M$ to symplectic immersion $\widehat{\varphi} : \mathcal{U} \to M$ of a tube neighborhood $\mathcal{U} \subset T^*S$ of the zero section and write $\sigma : \mathcal{U} \hookrightarrow \widehat{\varphi}^* L^{\otimes k}$ for the unique horizontal unitary trivialization that coincides with $\widetilde{\varphi}$ over zero section and has connection 1-form $\xi_\sigma = \eta^{\otimes k}$, where η is the universal action form on T^*S. Restriction of $h\sigma \in \mathcal{C}^\infty(\mathcal{U}, \widehat{\varphi}^* P_k)$ onto Lagrangian cycle $s_\zeta : S \hookrightarrow \mathcal{U}$ (graph Γ_ζ of a closed 1-form $\zeta \in Z^1_{\mathrm{DR}}(S, \mathbb{R})$) is a horizontal section of $s_\zeta^* \widehat{\varphi}^* L^{\otimes k}$ iff $d \ln(s_z^* h) = -i s_z^* \xi_\sigma = -i s_z^* \eta^{\otimes k} = -i \zeta^{\otimes k}$. We write $s_z^* h = e^{-ikg}$ with $g \in \mathcal{C}^\infty(S, \mathbb{R})$ and get $\zeta = k dg$.

Darboux – Weinstein's coordinate on $\mathfrak{P}_k$ near $\widetilde{\varphi}$ takes smooth function $g \in \mathcal{C}^\infty(S, \mathbb{R})$ such that graph Γ_{dg} of exact 1-form dg lies in $\mathcal{U}$ to Bohr – Sommerfeld cycle $\widehat{\varphi}\left(\Gamma_{dg}\right)$ equipped with a horizontal unitary section $\sigma_g = e^{-ikg} \sigma|_{\Gamma_{dg}}$.
Near a smooth immersion $\varphi : S \hookrightarrow M$ the space $\mathfrak{B}_k$, of Bohr – Sommerfeld cycles, is a smooth manifold locally modeled by open neighborhood of zero in the vector space $B^1_{\mathrm{DR}}(S)$, of exact 1-forms on S, which is canonically identified with $T_\varphi \mathfrak{B}$. Differential of the Berry bundle $\pi : \mathfrak{P}_k \twoheadrightarrow \mathfrak{B}_k$ takes $g \mapsto dg$.

Pseudo Kähler structures

We call a symplectic manifold (M,ω) *pseudo Kähler*, if a complex structure

$$I : TM \to TM\,, \quad I^2 = -\mathrm{Id} \qquad \text{(possibly non-integrable)}$$

is fixed such that $\omega(Iv,Iw) = \omega(v,w)$ and $g(v,w) \stackrel{\text{def}}{=} \omega(Iv,w)$ is positive. Then

$$\mathbb{C} \otimes TM = T_+M \oplus T_-M\,, \quad \text{where} \quad T_\pm M \stackrel{\text{def}}{=} \{v \in TM \mid Iv = \pm v\}$$

and we can speak about pseudo holomorphic functions $f : M \to \mathbb{C}$, whose differentials $df : \mathbb{C} \otimes TM \to \mathbb{C}$ annihilate T_-M, and about pseudo holomorphic subvarieties in M (possibly singular), which are locally defined as the zero sets of finite systems of pseudo holomorphic functions.

Claim

The restriction of symplectic form ω onto the tangent cone C_xV to any pseudo holomorphic subvariety $V \subset M$ at any point $x \in V$ is non-degenerated, i.e. $\forall$ non-zero $v \in C_xV\ \exists u \in C_xV : \omega(v,u) \neq 0$.

Pseudo-holomorphic line bundles and the canonical class

Each Hermitian line bundle L on pseudo Kähler M inherits U_1-invariant pseudo-holomorphic structure whose $T_+P \subset TP \otimes \mathbb{C}$ are formed by the kernels $\ker \alpha^{\mathbb{C}}$ of complexified connection form $\alpha^{\mathbb{C}} \in \mathcal{C}^{\infty}(P, T*P \otimes \mathbb{C})$. Thus, we get a vector space $H^0_I(M, L)$ of *pseudo holomorphic sections* annihilated by covariant differentiation along vector fields from T_M.

Claim

A non-zero pseudo holomorphic section of any pseudo holomorphic line bundle on M can not vanish identically along a Lagrangian cycle.

In particular, there is *canonical* pseudo-holomorphic line bundle $K \stackrel{\text{def}}{=} \Lambda^n T_+ M$. Note that the *canonical class* $c_1(K) \in H^2(M, \mathbb{Z})$ does not depend on a choice of pseudo Kähler structure, because such structures compatible with given ω form a contractible bundle over M whose fibers are the Siegel half-spaces.

Half-weighted Plankian cycles

Assume that $c_1(L) \sim c_1(K)$ are proportional and there exists a pseudo holomorphic line bundle $\mathcal{W} = \sqrt{L \otimes K}$ such that $\mathcal{W}^{\otimes 2} = L \otimes K$. Then $\varphi^*\mathcal{W}$ is topologically trivial for any Lagrangian immersion $\varphi : S \hookrightarrow M$ and each Planckian lift (S, σ) of φ produces non-degenerate pairing

$$\Omega_\sigma : \mathcal{C}^\infty(S, \varphi^*\mathcal{W}) \times \mathcal{C}^\infty(S, \varphi^*\mathcal{W}) \to \mathcal{C}^\infty(S, \varphi^* K)$$

by prescription $\mathcal{W} \otimes \mathcal{W} \ni \zeta_1 \otimes \zeta_2 = \sigma \otimes \Omega_s(\zeta_1, \zeta_2) \in L \otimes K$.

A *half-weight* on (S, σ) is a smooth global section $\varkappa : S \hookrightarrow \varphi^*\mathcal{W}$ such that $\mu(U) = \int_U \Omega_s(\varkappa, \varkappa)$ provides M with a measure Radon-Nikodym equivalent to the one coming from a smooth manifold structure.

Space $\mathfrak{P}^{\mathrm{hw}}$ of half-weighted Planckian cycles is locally modeled by a neighborhood of zero in the space $\mathcal{C}^\infty(S, \mathbb{C})$ of smooth functions $\psi = \psi_1 + i\psi_2$, $\psi_{1,2} \in \mathcal{C}^\infty(S, \mathbb{R})$. *Complex Darboux – Weinstein coordinate* near $(S, \sigma, \varkappa)$ takes ψ to the graph $\Gamma_{d\psi_2}$ of exact 1-form $d\psi_2$ equipped with a horizontal unitary section $\sigma_{\psi_2} \overset{\text{def}}{=} e^{-i\psi_2}\sigma|_{\Gamma_{\psi_2}}$ and half-weighted by $\varkappa_{\psi_1} \overset{\text{def}}{=} e^{\psi_1}\varkappa$.

Bortwick – Paul – Uribe map

Let us send half-weighted Planckian cycle $(S,\varkappa,\sigma)$ to $\mathbb{C}$-linear functional

$$\beta_{S,\varkappa,\sigma}: H^0(M,\mathcal{W}) \to \mathbb{C}, \quad \varrho \mapsto \int_S \Omega_\sigma(\varkappa,\varrho)$$

and compose the resulting map $\mathfrak{P}^{\mathrm{hw}} \to H^0(M,\mathcal{W})^*$ with $\mathbb{C}$-anti-linear isomorphism $H^0(M,\mathcal{W})^* \mathrel{\tilde{\to}} H^0(M,\mathcal{W})$ provided by the Hermitian form on $\mathcal{W}$. We get *the Bortwick – Paul – Uribe map* $\beta: \mathfrak{P}^{\mathrm{hw}} \to H^0(M,\mathcal{W})$.

- $d\beta: \mathcal{C}^\infty(S,\mathbb{C}) \to H^0(M,\mathcal{W})$ is $\mathbb{C}$-linear being computed in Darboux – Weinstein coordinates
- Hamiltonian reduction of β w.r.t. U_1-action produces a map

$$\{(S,\varkappa,\sigma)\}/\!/U_1 = \mathfrak{B}_t^{\mathrm{hw}} \to \mathbb{P}(H^0(M,\mathcal{W})^*) = H^0(M,\mathcal{W})^*/\!/U_1$$

from the space $\mathfrak{B}_t^{\mathrm{hw}}$ of half-weighted Bohr – Sommerfeld cycles of fixed volume t to the space of *conformal blocks* in the Kähler quantization of (M,ω).

Moduli space of rank 2 bundles on a Riemann surface

Write Σ_J for a Riemann surface Σ of genus $g \geqslant 2$ equipped with complex structure J. Let $E \twoheadrightarrow \Sigma$ be topologically trivial complex vector bundle, $\operatorname{rk} E = 2$. By Narasimhan – Seshadri – Donaldson theorem, moduli space of

- structures of a stable holomorphic vector bundle on Σ
- flat Hermitian connections on E compatible with J (up to $\mathcal{C}^\infty(\Sigma, \mathrm{SU}_2)$-gauge)
- irreducible representations $\varrho : \pi_1(\Sigma) \to \mathrm{SU}_2$ (up to isomorfism)

is the same space $\mathfrak{M} = \mathfrak{M}\left(2, 0, \Sigma_J\right)$, which has

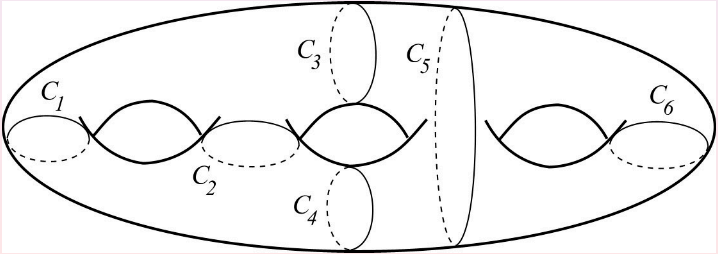
- integrable Kähler structure I (depending on J) and $\dim_{\mathbb{C}} \mathfrak{M} = 3g - 3$
- holomorphic bundles $\mathcal{T}_E \mathfrak{M} = H^1(\Sigma, \mathrm{Ad}(E))$, $\mathcal{T}^*_E \mathfrak{M} = \mathrm{Ad}(E, E \otimes K_\Sigma)$
- $\mathrm{Pic}(\mathfrak{M}) = \mathbb{Z} \cdot \mathcal{L}$, where $\mathcal{L} \overset{\text{def}}{=} \mathcal{O}_{\mathfrak{M}}(\Theta)$ is associated with *non-abelian theta-divisor* $\Theta \overset{\text{def}}{=} \left\{ E \in \mathfrak{M} \mid H^0\left(\Sigma, E(g-1)\right) \neq 0 \right\}$
- symplectic structure $\omega_{\mathfrak{M}}(\xi_1, \xi_2) = \int_\Sigma \mathrm{tr}(\xi_1 \wedge \xi_2)$, where $\xi_{1,2} \in \mathcal{C}^\infty(\Sigma, T^*\Sigma \otimes \mathfrak{su}_2)$ and $\mathrm{tr} : \mathfrak{su}_2 \otimes \mathfrak{su}_2 \to \mathbb{R}$ is contraction with the Killing form.

Goldman's polarization

Each simple loop $\gamma : S^1 \to \Sigma$ produces a Hamiltonian

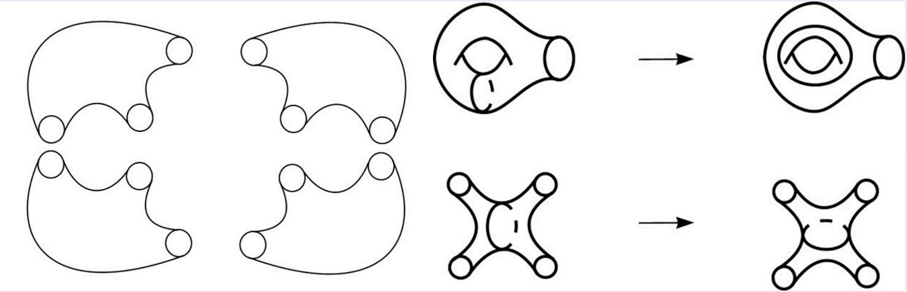
$$H_\gamma : \mathfrak{M} \to [0,1], \quad \varrho \mapsto arccos\big(\mathrm{tr}(\varrho(\gamma))/2\big)$$

where $\varrho \in \mathfrak{M}$ is considered as a representation $\varrho : \pi_1(\Sigma) \to \mathrm{SU}_2$. Goldman has shown that $\{H_{\gamma_1}, H_{\gamma_2}\}_{\omega_{\mathfrak{M}}} = 0$ for non-isotopic non-intersecting simple loops $\gamma_{1,2} \in \pi_1(\Sigma)$. Thus, $3g-3$ such loops $\gamma_1, \gamma_2, \dots, \gamma_{3g-3}$ lead to lagrangian fibration $\Gamma : \mathfrak{M} \twoheadrightarrow \Delta \subset [0,1]^{3g-3}$, where the Delzant polyhedron Δ is given by the triangle inequalities written for all triples of loops that bounding pants.



Graph of pant decompositions

Complete systems of Goldman's Hamiltonians stay in bijection with pant decompositions and form the vertexes of graph whose edges are elementary regluings in pairs of adjacent pants



Generalized Knizhnik – Zamolodchikov correspondence

Bohr – Sommerfeld fibers of Goldman's fibration $\Gamma : \mathfrak{M} \twoheadrightarrow \Delta \subset [0,1]^{3g-3}$ w.r.t. pre – quantization bundle $L = \mathcal{L}^{\otimes k}$ are those lying over $\frac{1}{k}$-integer points $\Delta \cap (\mathbb{Z}/k)^{3g-3}$. The Bortwick – Paul – Uribe map allows to attach a holomorphic section of $\mathcal{L}^{\otimes k}$ defined up to a constant factor. These sections form a basis in projective space $\mathbb{P}H^0(\mathfrak{M}, \mathcal{L})$.

When J runs through the moduli space $\mathcal{M}_g$ of algebraic curves of genus g conformal blocks $\mathbb{P}H^0(\mathfrak{M}, \mathcal{L})$ fill projective bundle $\mathcal{P} \twoheadrightarrow \mathcal{M}_g$. Each vertex of the graph of pants provides projective $\mathcal{P}$ with flat connection whose horizontal sections are those coming from Bohr – Sommerfeld fibers of Goldman's fibration.

Each $J \in \mathcal{M}_g$ equips the edges of the graph of pants with transition matrices between Bohr – Sommerfeld bases in $\mathbb{P}H^0(\mathfrak{M}, \mathcal{L})$ coming from Goldman's fibrations staying at the joined vertexes. These matrices produce «discrete field theory» of Wess – Zumino – Witten type.

THANKS FOR YOUR ATTENTION !