ALAG

(Abelian Lagrangian Algebraic Geometry)

that is how to quantize a classical mechanical system (symplectic manifold (M, ω)) prequantized by $U(1)$-bundle ('quantum phase' $L \to M$) in algebraic geometrical framework, or an interaction between symplectic geometry of Lagrangian cycles *inside* M and algebraic geometry of Abelian connections *over* M.

Published in:

1. A. L. Gorodentsev, A. N. Tyurin. *ALAG*. Prep. MPI-2000-7.

2. A. N. Tyurin. *On Bohr-Sommerfeld bases*. Warwick prep. Sept.1999/Publication 216.

3. A. N. Tyurin. *Geometric quantization and mirror symmetry*. Prep. arXiv: math.AG/9902027.

4. A. N. Tyurin. *Complexification of Bohr - Sommerfeld conditions*. Prep. arXiv: math.AG/9909094.

In these lectures we will discuss :

- what is 'the quantization', especially the geometric one

- prequantization equipment $(M, \omega, P_k, \alpha_k, L^{\otimes k})$, of level k

- Kähler - Berezin - Töplitz quantization

- spaces $\mathfrak{L}$, $\mathcal{B}_k$, and $\mathcal{P}_k$, of Lagrangian, Bohr - Sommerfeld, and Planckian cycles

- half weighted Bohr - Sommerfeld and Planckian cycles (spaces $\mathcal{B}_k^{\mathrm{hw}}$ and $\mathcal{P}_k^{\mathrm{hw}}$)

- integrable Kähler structure on the space $\mathcal{B}_{k,t}^{\mathrm{hw}}$, of half weighted Bohr - Sommerfeld cycles of a fixed volume t

- the BPU-map $\mathcal{B}_{k,t}^{\mathrm{hw}} \xrightarrow{\gamma} \mathbb{P}H^0(M, L^{\otimes k})$ induced by a holomorphic structure on M

‘Quantum’ introduction to ALAG

<u>**Classical mechanical system**</u> is presented by a smooth (compact or not) $2n$-dimensional C^∞-manifold M (called *a phase space*) equipped with a closed 2-form ω that has to induce a skew symmetric isomorphism $\omega : TM \xrightarrow{\sim} T^*M$. Then, each function $f \in C^\infty(M)$ produces the Hamiltonian vector field $H_f \overset{\text{def}}{=} \omega^{-1}(df)$ and there is a Lie algebra structure on $C^\infty(M)$ given by the Poisson bracket $\{f, g\} \overset{\text{def}}{=} \omega(H_f, H_g)$.

<u>**Dirac’s quantization concept**</u> supposes to construct an *irreducible* representation of complex smooth functions on M by linear endomorphisms of some complex Hilbert space $\mathcal{H}$:

$$C^\infty(M, \mathbb{C}) \xrightarrow{Q} \mathit{End}_{\mathbb{C}}(\mathcal{H})$$

such that $Q(1) = \mathrm{Id}$, real functions go to the self adjoint operators, and *the Dirac commutator relations* $Q(\{f, g\}) = [Q(f), Q(g)]/ih$ hold. Unfortunately (or fortunately, maybe), this is impossible by the Van Hove theorem. Mathematical attempts to budge this problem are concentrated in enlarging a classical area and, simultaneously, relaxing some of the previous ‘quantization rules’.

The 'quantum mathematics' looks today like a radioactive decay via *algebraic*, *analytic*, and *geometric* emanations shown on fig. A.

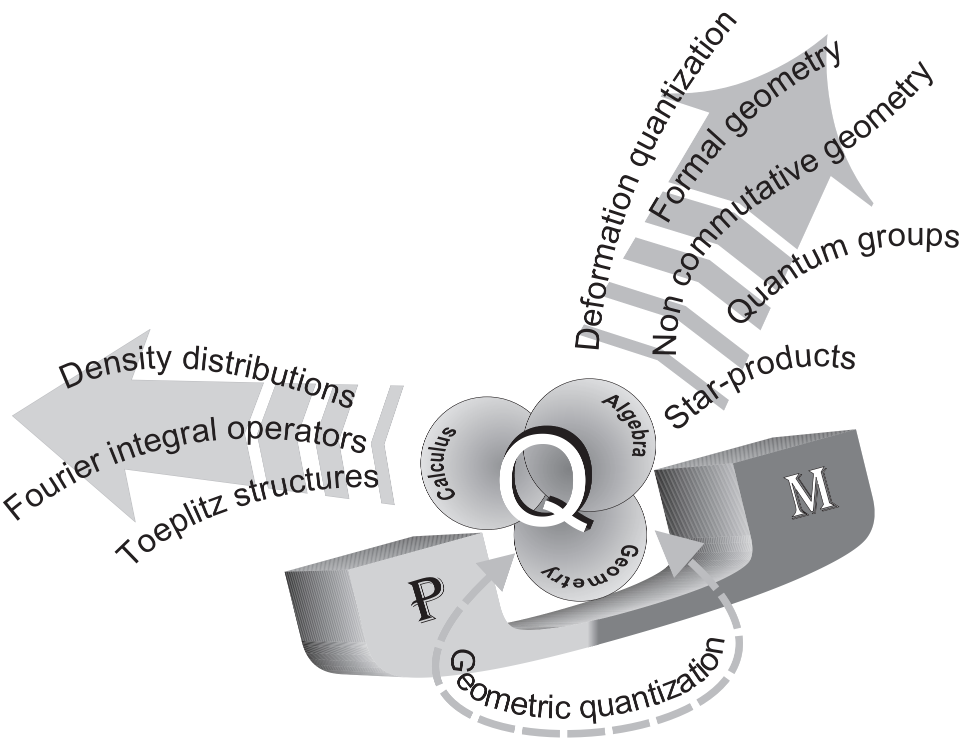

Fig. A. 'Quantum decay'.

Geometrically, most of the modified quantization rules aim at a choice of a suitable function class that can be quantized in a consistent fashion. This choice is nothing but a choice of some auxiliary structure on the manifold.

Geometrical prequantization equipment of a classical mechanical system (M, ω) consists of:

- a complex Hermitian line bundle $L \longrightarrow M$ (its sections are the 'quantum states'; the Hermitian structure allows to define their probabilities)

- a series of principal $U(1)$-bundles $P_k \xrightarrow{p_k} M$ tautologically associated with the tensor powers $L^{\otimes k}$ (the number $1/k$ plays a role of the 'Planck constant': we go to the 'quasiclassical limit' as $k \to \infty$)

- a Hermitian connection a on L, which induces the Hermitian connections a_k on $L^{\otimes k}$ such that the curvature form $F_a \stackrel{\text{def}}{=} da$ satisfies *compatibility condition*

$$\boxed{F_a = p_1^* \omega} \tag{1}$$

(we always consider the connection a_k as a real $U(1)$-invariant 1-form on P_k. In particular, 'the vertical projection' $TP_k \longrightarrow \mathfrak{u}(1) = i\mathbb{R}$ is given by $v \mapsto 2\pi i\, a_k(v)$ and 'the horizontal hyperplanes' in TP_k are the kernels of a_k)

Remark. A prequantization equipment exists iff the cohomology class of ω is integer:

$$[\omega] \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R})\,.$$

In this case the compatibility condition (1) implies that $[\omega] = c_1(L)$ and a_k presents a contact structure on P_k, that is the top form

$$a_k \wedge (d\alpha_k)^{\wedge n} = a_k \wedge (p_k^* \omega)^{\wedge n} \tag{2}$$

nowhere vanishes and produces a volume element on P_k.

Geometric quantization takes the Hilbert space $\mathcal{H}$ being the space of smooth sections $C^\infty(M, L)$ and lets $f \in C^\infty(M)$ act on $\Gamma^\infty(M, L)$ as:

$$Q(f) \stackrel{\text{def}}{=} \nabla_{H_f} + f \ : s \ \mapsto \nabla_{H_f}(s) + fs \tag{3}$$

where ∇_{H_f} is a covariant derivative along the Hamiltonian vector field H_f induced by the connection a.

Claim. The operators $Q(f)$ satisfy all the Dirac conditions except for irreducibility:

Unfortunately, the space $\mathcal{H}$ is 'two times larger' than it should be. Geometrically speaking, to take just 'a half of $\Gamma^\infty(M, L)$' we have to fix some *polarization* on M.

Polarization of M is a choice of complex n-dimensional tangent planes distribution $T' \subset TM \otimes \mathbb{C}$ closed w.r.t. the Lie brackets of vector fields. It allows to reduce $\mathcal{H}$ to a subspace of *polarized* sections $\mathcal{H}' \subset \Gamma^\infty(M, L)$, which a covariantly constant along all the vector fields from T'.

Typically, this gives the irreducibility of the geometric quantization procedure but leads to an other problem: only a function $f \in C^\infty(M)$ whose Hamiltonian field H_f preserves T' may be represented as an operator on $\mathcal{H}'$ via (3). This kills a lot of physically interesting Hamiltonians.

Kähler polarization. Let M be equipped with an integrable complex structure I compatible with ω in the sense that M_I is a complex Kähler manifold with the Kähler form ω. Then, the antiholomorphic tangent subbundle $T'' \subset TM \otimes \mathbb{C}$ gives *a polarization* on M called *the Kähler polarization*. In this case, L turns to a holomorphic positive line bundle on M_I (with the $\overline{\partial}$-operator induced by the Hermitian connection a) and the space of polarized sections turns to the space of global *holomorphic* sections $H^0(M_I, L)$.

But, only the holomorphic Hamiltonians may be represented via (3).

Wave functions and states. The space

$$\mathcal{H}_k \overset{\text{def}}{=} H^0(M_I, L^{\otimes k})$$

is called *a wave function* space of level k. Its projectivization $\mathbb{P}\mathcal{H}_k$ is called a space of *states*.

$\mathcal{H}_k$ is a Hilbert space w.r.t. scalar product

$$\langle\, s_1\,,\, s_2 \,\rangle_k \overset{\text{def}}{=} \int_M (s_1, s_2)_{L^{\otimes k}}\, \omega^{\wedge n}$$

where $(\,*\,,\,*\,)_{L^{\otimes k}}$ is the Hermitian form on $L^{\otimes k}$.

For $k \gg 0$ there is the canonical holomorphic projective embedding

$$\varphi_k : M_I \xhookrightarrow{x \mapsto Ann\,(x)} \mathbb{P}H^0(M_I, L^{\otimes k})^* \qquad (4)$$

The Hermitian structure on $L^{\otimes k}$ gives an antiholomorphic isomorphism

$$H^0(M_I, L^{\otimes k})^* \xrightarrow{\sim} H^0(M_I, L^{\otimes k})$$

whose composition with the above projective map gives an antiholomorphic embedding of M_I into the space of states. The states sitting in its image are called *coherent*.

The Hardy space. A subbundle of unit balls $D \subset L^*$ is a strictly pseudoconvex domain with the boundary P^*. Hence, the space of holomorphic functions on D is a closed subspace in the Hilbert space $L^2(P)$ (w.r.t. the volume element induced by the contact form a^*). This subspace is called *the Hardy space*. It comes with the natural $U(1)$-action, that is admits the orthogonal weight decomposition.

Claim: The Hardy space component of weight k coincides with the space $\mathcal{H}_k = H^0(M_I, L^{\otimes k})$, of level k wave functions, and Hermitian structure induced on $\mathcal{H}_k$ from $L^2(P^*)$ coincides with the initial Hermitian structure on $\mathcal{H}_k$. So, the Hardy space is nothing but $\bigoplus\limits_k \mathcal{H}_k$.

Töplitz operators. Write $L^2(P^*) \xrightarrow{\Pi_k} \mathcal{H}_k$ for the canonical orthogonal projector (called *the k-th Szegö projector*) and let $f \in C^\infty(M)$. Each $\psi \in H^0(M_I, L^{\otimes k}) \subset L^2(P^*)$ can be multiplied by $p^*_{-1} f$ in $L^2(P^*)$ and then sent back to $H^0(M_I, L^{\otimes k})$ by Π_k. The resulting map

$$T_f^{(k)} : H^0(M_I, L^{\otimes k}) \xrightarrow{\psi \mapsto \Pi_k(p^*_{-1} f \cdot \psi)} H^0(M_I, L^{\otimes k})$$

is called *k-th Töplitz operator* of f.

Berezin - Töplitz quantization represents each $f \in C^\infty(M)$ by the operators $T_f^{(k)} \in End_{\mathbb{C}}(\mathcal{H}_k)$ instead of the differential operators (3) used in the standard geometric quantization method. This representation

$$C^\infty(M) \xrightarrow{T^{(k)}} End_{\mathbb{C}}(\mathcal{H}_k)$$

satisfies all the Dirac quantization principles except for the commutator relation:

$$ih\, T^{(k)}_{\{f,g\}} \neq \left[T_f^{(k)}, T_g^{(k)}\right] .$$

Claim. Although for each level k it does preserve neither Lie nor associative algebra structures the discrepancy has 'good quasi-classical behavior':

$$\left\| ihT^{(k)}_{\{f,g\}} - \left[T_f^{(k)}, T_g^{(k)}\right] \right\| \sim O(1/k)$$

as $k \to \infty$.

Observation. Although an output of the Berezin quantization (wave function spaces, e.t.c.) *a priori* depends on the choice of a polarized complex structure I on M, in several examples (elliptic curves, say) the moduli space of the polarized complex structures on M admits a *flat* projective bundle with fibers $\mathbb{P}\left(H^0(M_I, L^{\otimes k}) \right)$.

ALAG constructs an *universal* Kähler manifold structure on appropriate space of Lagrangian cycles on M. This structure depends only on the (smooth Lagrangian) prequantization equipment and induces, in particular, the projective geometrical properties of any particular Berezin - Töplitz quantization procedure. So, the 'geometry of quantization' we hope to develop should look like

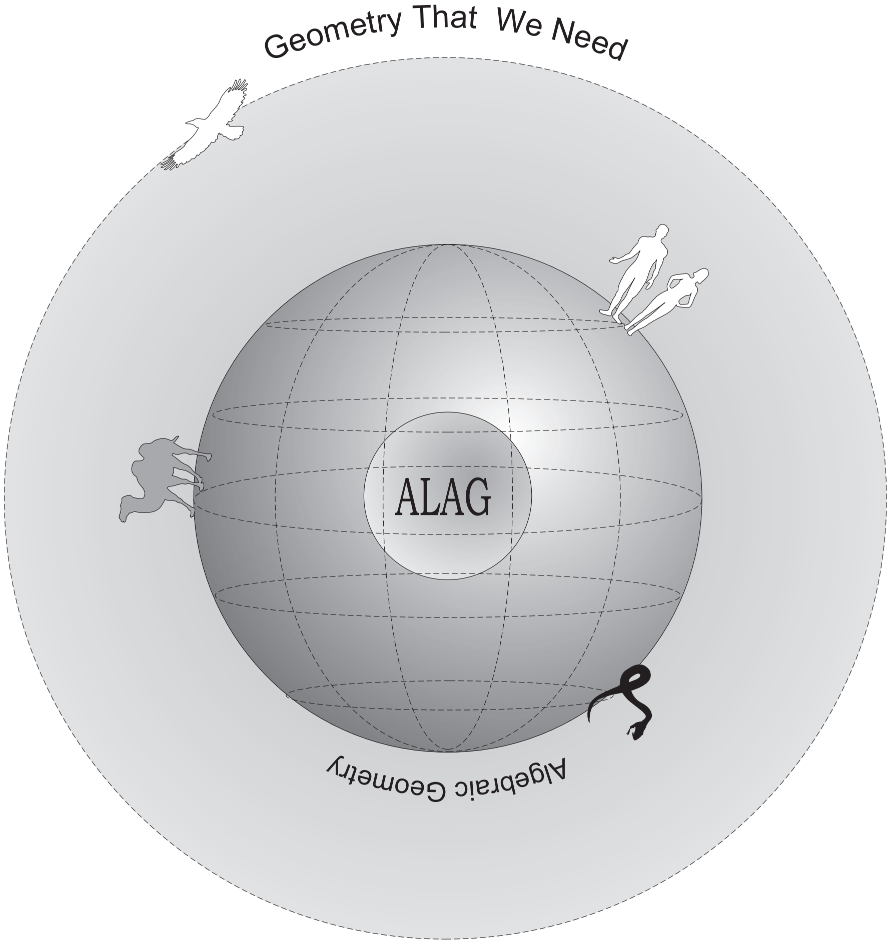

Fig. B. The Geometrical World.

Lagrangian setup for ALAG

We fix a smooth (maybe non-compact) $2n$-dimensional symplectic C^∞-manifold M, a smooth compact oriented n-dimensional C^∞-manifold S, and some smooth-homotopy-class $\mathit{Map}_{\mathrm{LA}}(S, M)$, of smooth maps $S \xrightarrow{\varphi} M$ such that $\varphi^*(\omega) = 0$ (such the maps are called *Lagrangian*).

<u>**Lagrangian cycles space**</u> is the orbit space

$$\mathfrak{L} \overset{\mathrm{def}}{=} \mathit{Map}_{\mathrm{LA}}(S, M)/\mathit{Diff}_0(S)$$

<u>**Crucial property of Lagrangian cycles.**</u> If φ is a smooth immersion, i. e. there is *an inclusion* of tangent bundles $TS \hookrightarrow \varphi^*(TM)$, a Lagrangian cycle $S \xrightarrow{f} M$ is called *smoothly immersed*. In this case the normal bundle $N_{M/S} \simeq \varphi^*(TM)/TS$ is naturally identified with the cotangent bundle

$$T^*S \simeq \varphi^*\left(T^*M\right)/\mathit{Ann}(TS) \overset{\omega}{\simeq} \varphi^*(TM)/TS$$

Main 'local' example: let $M = T^*S$ be a cotangent bundle over any smooth manifold S with the canonical projection $M \xrightarrow{p} S$. The 1-form $\eta \in C^\infty(M, T^*M)$ sending $w \in T_{(\alpha,x)}T^*S$ (with $x \in S$, $\alpha \in T_x^*S$) to $\eta(w) \stackrel{\text{def}}{=} p^*\alpha(w)$ is called *an action form*. Its differential $\omega \stackrel{\text{def}}{=} d\eta$ gives a symplectic structure on M (called *canonical*).

In local coordinates $(\, p_1, p_2, \dots, p_n\,,\ q_1, q_2, \dots, q_n\,)$, where q_i coordinate the base S and p_i are linear coordinates along dq_i in the fibers,

$$\eta(p,q) = \sum p_j dq_j\ , \quad \omega(p,q) = \sum dp_j \wedge dq_j\ .$$

A section $S \xhookrightarrow{s} M$, i. e. a graph of 1-form $\alpha_s = s^*(\eta) \in C^\infty(S, T^*S)$, is Lagrangian iff α_s is closed, because of:

$$0 = s^*(\omega) = s^*(d\eta) = ds^*(\eta) = d\alpha_s\ .$$

So, small deformations of S as a Lagrangian cycle in T^*S presented by the zero section $S \hookrightarrow T^*S$ may be identified with a small neighborhood of zero in the space $Z^1_{\mathrm{DR}}(S, \mathbb{R})$, of smooth real closed 1-forms on S.

The tangent space $T_{(S,\varphi)}\mathfrak{L}$ at a smoothly immersed Lagrangian cycle (S,φ) is naturally identified with $Z^1_{\mathrm{DR}}(S,\mathbb{R})$. Indeed, $T_{(S,\varphi)}\mathfrak{L}$ consists of ω-preserving local normal vector fields near S on M modulo the fields tangent to S. We send such a vector field v to a 1-form $\alpha = \varphi^*(i_v\omega) \in C^\infty(S, T^*S)$. It is closed, because of

$$d\alpha = d\varphi^*(i_v\omega) = \varphi^*(di_v\omega) = \\ = \varphi^*(\mathcal{L}_v\omega + i_v d\omega) = \varphi^*(\mathcal{L}_v\omega) = 0\,.$$

The Darboux - Weinstein uniformization.

Any smooth Lagrangian immersion $S \overset{\varphi}{\hookrightarrow} M$ can be extended to a smooth immersion $U \overset{\widehat{\varphi}}{\hookrightarrow} M$ of some tube neighborhood $U \subset T^*S$ around the zero section in such a way that $\widehat{\varphi}^*\omega = d\eta$ turns to the canonical symplectic form on T^*S. This continuation induces a smooth map from some neighborhood of zero in $Z^1_{\mathrm{DR}}(S,\mathbb{R})$ to $\mathfrak{L}$: a small enough closed 1-form goes to the $\widehat{\varphi}$-image of its graph. The image of this map contains an open (in suitable topology) neighborhood of S in $\mathfrak{L}$.

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Abelian setup for ALAG

A complex Hermitian line bundle $L \longrightarrow X$ on a C^∞-manifold X has a form $L = P \times_{U(1)} \mathbb{C}$, where P is the bundle of unit circles in L (called *a principal U*(1)*-bundle of* L). Topologically, L is uniquely defined by its *first Chern class*

$$c_1(L) \in H^1(X, \mathcal{C}^\infty(X, \mathbb{C}^*)) \simeq H^2(X, \mathbb{Z})$$

formed by its transition functions.

A Hermitian connection on P is a 1-form a on P such that $\mathcal{L}ie_u a = 0$ and $a(u) = 1$ for the standard $U(1)$-invariant 'vertical' vector field u dual to the 1-form u^* such that $\int_{U(1)} u^* = 1$. The kernel hyperplanes of α in TP are called *horizontal*.

A covariant derivation induced by a is a differential operator $\mathcal{C}^\infty(X, L) \xrightarrow{\nabla_a} \mathcal{C}^\infty(X, T^*X \otimes L)$ sending a local unitary section σ over $U \subset X$ to $\nabla_a \sigma = 2\pi i\, \alpha_\sigma \otimes \sigma$, where $\alpha_\sigma = \sigma^* a$ is called *a local connection 1-form* of a. An arbitrary local section $f\sigma$ is then differentiated as

$$\nabla_a(f\sigma) = (df + 2\pi i\, f\alpha_\sigma) \otimes \sigma\ .$$

An unitary base change (*gauge transform*) acting as $\sigma \longmapsto \sigma' = h\sigma$ with $h \in C^\infty(U, U(1))$ takes

$$\alpha_\sigma \longmapsto \alpha_{\sigma'} = \alpha_\sigma - \frac{i}{2\pi} h^{-1} dh \ .$$

Similarly, a *covariantly constant* (or *horisontal*) local sections $f\sigma$ have to satisfy the logarithmic differential equation $2\pi i\, f^{-1} df = \alpha_\sigma$.

The differentials $d\alpha_\sigma$, of local connection forms, do not depend on a choice of local unitary bases σ and can be glued into a global closed 2-form $\omega_a \in C^\infty(X, \Lambda^2 T^* X)$ called *a curvature* of a. Its cohomology class $[\omega_a] \in H^2(X, \mathbb{R})$ is integer and coincides with $c_1(L) \in H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$.

In terms of covariant derivative, the 'double differentiation' $C^\infty(X, L) \xrightarrow{\nabla_a^2} C^\infty(X, \Lambda^2 T^* X \otimes L)$ induced by ∇_a via Leibnitz rule is *a $C^\infty(X)$-linear operator* taking $s \longmapsto 2\pi i\, \omega_a \otimes s$.

In terms of the principal $U(1)$-bundle $P \xrightarrow{p} X$, we have $p^* \omega_a = da$ and this 2-form sends a pair of $U(1)$-invariant vector fields v_1, v_2 on P to

$$\begin{aligned} da(v_1, v_2) = \Big(\mathcal{L}ie_{v_1} a - d\big(a(v_1)\big)\Big)(v_2) = \\ = \mathcal{L}ie_{v_1} a(v_2) = -a([v_1, v_2]) \ . \end{aligned}$$

Flat Hermitian connections are defined by the condition $\omega_a \equiv 0$. If it holds, L admits a global never vanishing unitary (usually, non horizontal) section $X \overset{\sigma}{\hookrightarrow} L$. Horizontal sections $f\sigma$ exist only locally over simply connected open sets and are given by $f = c\, e^{-2\pi i g}$ with an arbitrary constant $c \in \mathbb{C}$ and any smooth local function g such that $dg = \alpha_\sigma$.

Gauge equivalence classes of flat Hermitian connections are in 1–1 correspondence with unitary characters

$$\pi_1(X) \xrightarrow{\chi} U(1)\ ,$$

of the fundamental group of X. A character χ_a corresponding to a connection a presents *the holonomy* (or *the periods*) of a, i. e. it sends a loop $\gamma \in \pi_1(X)$ to a fiber shift under the horizontal transport along γ.

Considering the unitary characters as cohomology classes in $H^1(X, U(1)) \simeq H^1(X,\mathbb{R})/H^1(X,\mathbb{Z})$, we parameterize the gauge classes of the flat Hermitian connections on L by the points of real torus

$$J_X \overset{\text{def}}{=} H^1(X,\mathbb{R})/H^1(X,\mathbb{Z})$$

called *a Jacobian* of X.

References on gauges and connections.

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Basic ALAG notions

Let as fix a *geometric prequantization data*

$$(M, \omega, L, a)\ ,$$

that is $2n$-dimensional C^∞-manifold M equipped with a symplectic form ω, Hermitian line bundle $L \longrightarrow M$, and Hermitian connection a whose curvature $\omega_a = da$ coincides with $p_1^*\omega$.

We write $P_k \xrightarrow{p_k} M$ for the principal $U(1)$-bundles associated with $L^{\otimes k}$ and write a_k for their Hermitian connections (induced by $a = a_1$).

Recall, that the cohomology class $[\omega]$ is integer. For any k the curvature $da_k = k\,\omega$, and a_k provides P_k with *a contact structure* whose volume element equals

$$a_k \wedge (d\alpha_k)^{\wedge n} = k^n a_k \wedge (p_k^*\omega)^{\wedge n}\ .$$

Bohr - Sommerfeld cycles. For any Lagrangian cycle $S \xrightarrow{\varphi} M$ a pull back $\varphi^* L^{\otimes k}$ is a topologically trivial Hermitian line bundle on S equipped with the flat Hermitian connection $\varphi^* a_k$, because its curvature $\omega_{\varphi^* a_k} = k\,\varphi^*(\omega) = 0$. A Lagrangian cycle (S, φ) is called *Bohr - Sommerfeld of level* k (or *a BS$_k$-cycle* for shortness), if $\varphi^* L^{\otimes k}$ admits a global horizontal section over S. The Bohr - Sommerfeld cycles of level k form a subvariety $\mathcal{B}_k \subset \mathfrak{L}$. For any integer k, m there are natural inclusions $\mathcal{B}_k \overset{\beta_{k,m}}{\hookrightarrow} \mathcal{B}_{mk}$.

Planckian cycles. Let $\mathit{Map}_{\mathrm{LE}}(S, P_k)$ be the space of smooth maps $S \xrightarrow{\widetilde{\varphi}} P_k$ such that $\widetilde{\varphi}^*(\alpha_k) = 0$ (such the maps are called *Legendrian*). The orbit space $\mathcal{P}_k \overset{\text{def}}{=} \mathit{Map}_{\mathrm{LE}}(S, P_k)/\mathit{Diff}_0(S)$ is called a space of *level k Planckian cycles*. For each Planckian cycle $S \xrightarrow{\widetilde{\varphi}} P_k$ the composition $\varphi : S \xrightarrow{p_k \circ \widetilde{\varphi}} M$ is Lagrangian, because of

$$\varphi^* \omega = d\widetilde{\varphi}^* \alpha_k / k = 0\ .$$

So, a Planckian cycle $(S, \widetilde{\varphi})$ is nothing but a global horizontal unitary section of $\varphi^* L^{\otimes k}$ over the Bohr - Sommerfeld cycle (S, φ).

Remark 1. There is a principal $U(1)$-bundle

$$\mathcal{P}_k \xrightarrow{\pi_k} \mathcal{B}_k$$

and the corresponding Hermitian line bundle

$$\mathcal{L}_k \longrightarrow \mathcal{B}_k$$

called *the level k Berry bundles*.

Remark 2. Taking m-th fiberwise power of a horizontal unitary section, we get degree m covering

$$\mathcal{P}_k \xrightarrow{\widetilde{\mu}_{m,k}} \beta^*_{m,k}\mathcal{P}_{mk} \ .$$

In terms of the line bundles, this means that

$$\beta^*_{k,m}\mathcal{L}_{mk} = \mathcal{L}_k^{\otimes m}$$

under the inclusions $\mathcal{B}_k \overset{\beta_{k,m}}{\hookrightarrow} \mathcal{B}_{mk}$.

Example: the Legendrian knots in S^3. Let us take the Riemannian sphere $S^2 = \mathbb{P}(\mathbb{C}^2)$ as M and the line bundle $\mathcal{O}(1)$, of homogeneous degree 1 forms, as L. The corresponding principal $U(1)$-bundle P may be identified with the unit sphere $S^3 \subset \mathbb{C}^2$ fibered over $\mathbb{P}(\mathbb{C}^2)$ via Hopf. The standard Hermitian connection on P is obtained by restricting the $U(2)$-invariant 1-form

$$2\pi i\, a \stackrel{\text{def}}{=} (\partial - \overline{\partial}) \log |z|$$

on $\mathbb{C}^2 \setminus \{0\}$. Its curvature $da = \dfrac{i}{2\pi} \partial\overline{\partial} \log |z|$ provides $\mathbb{P}_1$ with the standard symplectic structure ω such that $p^*\omega = da$. Taking $S = S^1$ as Lagrangian cycles source, we get *any* smooth immersion $S^1 \xrightarrow{\gamma} S^2$ being Lagrangian. The period of a_k along γ is $\oint_\gamma \alpha_k = k \iint_\gamma \omega = \dfrac{k}{2} A_\gamma$, where A_γ is *an oriented area* bounded by $\gamma(S) \subset S^2$. The Bohr - Sommerfeld conditions mean that it is integer multiple of 2π. So, the Planckian cycles are the Legendrian knots in S^3 laying over unicursal curves cutting a *rational* part of the whole sphere area.

Infinitesimal description of $\mathcal{B}_k$ and $\mathcal{P}_k$.

A tangent vector to $\mathcal{P}_k$ at $(S, \tilde{\varphi})$ is a section

$$v \in C^\infty\Big(S \, , \, \tilde{\varphi}^*(TP_k)/TS \Big)$$

such that $\mathcal{L}ie_v \alpha_k = 0$. Write its vertical component (w. r. t. α_k) as fu, where $f = \alpha_k(v) \in C^\infty(S)$ and u is the standard vertical field (as above). Let us identify the subspace of horizontal sections in $C^\infty(S, TP_k)$ with $C^\infty(S, \varphi^*TM)$ via dp_k. Then the horizontal component $v - fu$, of v, contracted with da_k and any horizontal w is

$$\begin{aligned} d\alpha_k(v - fu, w) &= d\alpha_k(v, w) = \\ &= \Big[\mathcal{L}ie_v(\alpha_k) - d\Big(\alpha_k(v)\Big) \Big] (w) = -df\,(w) \, . \end{aligned}$$

Hence, the both vertical and horizontal components of v are uniquely defined by $f \in C^\infty(S)$ and infinitesimal deformations preserving the Bohr - Sommerfeld conditions go along the subspace

$$B^1_{\mathrm{DR}}(S, \mathbb{R}) \subset Z^1_{\mathrm{DR}}(S, \mathbb{R}) = T_{(S,\varphi)}\mathfrak{L} \, .$$

Roughly speaking, each $f \in C^\infty(S)$ deforms $(S, \tilde{\varphi})$ by moving the underlying Bohr - Sommerfeld cycle to the graph of df in T^*S and simultaneous gauging the global horizontal section $\tilde{\varphi}$ by $e^{-2k\pi i\, f}$.

Claim 1: a tangent space $T_{(S,\varphi)}\mathcal{B}_k$ to BS_k-cycles space at a smoothly immersed cycle $S \overset{\varphi}{\hookrightarrow} M$ is naturally identified with the space of exact 1-forms $B^1_{\mathrm{DR}}(S)$. Under this identification, the differential of the inclusion

$$\mathcal{B}_k \overset{\beta_{k,m}}{\hookrightarrow} \mathcal{B}_{mk}$$

acts as the scalar multiplication by m.

Claim 2: a tangent space $T_{(S,\widetilde{\varphi})}\mathcal{P}_k$ at a smoothly embedded Planckian cycle $S \overset{\widetilde{\varphi}}{\hookrightarrow} P_k$ is naturally identified with $C^\infty(S)$. Under this identification, the differential of the covering

$$\mathcal{P}_k \xrightarrow{\widetilde{\mu}_{m,k}} \beta^*_{m,k}\mathcal{P}_{mk}$$

acts on $C^\infty(S)$ as the scalar multiplication by m.

Claim 3: under the above identifications, the differential of the Berry bundle $\mathcal{P}_k \xrightarrow{\pi} \mathcal{B}_k$ coincides with the external differentiation

$$C^\infty(S,\mathbb{R}) \xrightarrow{d} B^1_{\mathrm{DR}}(S,\mathbb{R}) \ .$$

Darboux - Weinstein uniformization.

Let $S \overset{\widetilde{\varphi}}{\hookrightarrow} P_k$ be a Planckian cycle laying over a smoothly immersed Bohr - Sommerfeld cycle $S \overset{\varphi}{\hookrightarrow} M$. Then $\mathcal{P}_k$ is smooth near $(S, \widetilde{\varphi})$ and is locally modeled by some neighborhood of zero in $C^\infty(S, \mathbb{R})$. Similarly, $\mathcal{B}_k$ is modeled near (S, φ) by a neighborhood of zero in $B^1_{\mathrm{DR}}(S, \mathbb{R})$.

Skethed proof. Let us continue (via Darboux - Weinstein) our Bohr - Sommerfeld cycle to a symplectic immersion $U \overset{\widehat{\varphi}}{\hookrightarrow} M$, where $U \subset T^*S$ is some neighborhood of the zero section. The Hermitian line bundle $\widehat{\varphi}^* L^{\otimes k}$ admits (perhaps, over smaller U) an unitary section $U \overset{\sigma}{\hookrightarrow} \widehat{\varphi}^* L^{\otimes k}$ which coincides with our Planckian cycle over the zero section $S \subset U$ and such that the covariant differentiation w. r. t. the connection a_k is given in the base σ by the canonical action form η on T^*S, i. e. $\nabla_a \sigma = 2\pi i\, \eta \otimes \sigma$ over U.

The *DW-coordinatization* of $\mathcal{P}_k$ by some neighborhood of zero in $C^\infty(S, \mathbb{R})$ sends a small enough function f to a BS_k-cycle $\Gamma_f \subset U$, which is the graph of df, equipped with the global unitary horizontal section $e^{-i\,f} \cdot \sigma|_{\Gamma_f}$.

Complex structure on $T\mathcal{P}_k$.

The tangent bundle $T\mathcal{P}_k$ admits a natural integrable complex structure. Namely, *the complex DW-coordinatization* of $T\mathcal{P}_k$ near a pair

$$S \overset{\widetilde{\varphi}}{\hookrightarrow} P_k\ , \quad g \in C^\infty(S,\mathbb{R}) = T_{(S,\widetilde{\varphi})}\mathcal{P}_k$$

takes a small enough $\psi = \psi_1 + i\psi_2 \in C^\infty(S,\mathbb{C})$ and treats its real part ψ_1 as a linear coordinate along the fiber of the tangent bundle and the imaginary part ψ_2 — as the DW-coordinate on $\mathcal{P}_k$.

So, in terms of splitting

$$T_{(\widetilde{\varphi},g)}(\,T\mathcal{P}_k\,) = T\mathcal{P}_k \oplus T\mathcal{P}_k = C^\infty(S,\mathbb{R}) \oplus C^\infty(S,\mathbb{R})$$

coming from the (real) DW-coordinatization of $\mathcal{P}_k$, the multiplication by i on sends

$$(\psi_1,\psi_2) \longmapsto (\psi_2, -\psi_1)\ ,$$

i. e. coincides with the usual complex structure on $C^\infty(S,\mathbb{C})$.

<u>Idea:</u> we are about to combine this (alrady integrated) complex structure with the canonical symplectic structure on $T^*\mathcal{P}_k$ in order to get an integrable Kähler structure on a suitable vector bundle over $\mathcal{P}_k$.

Measures. The equivalence $\lambda \sim \mu$ of two measures λ, μ on S means that $\lambda(B) = 0 \Leftrightarrow \mu(B) = 0$ for any Borel subset $B \subset S$. By the Radon - Nikodym theorem, $\lambda \sim \mu$ iff there are two real integrable positive functions

$$\varrho_{\mu\lambda} = \frac{d\mu}{d\lambda} \in L^1(S, \lambda) \; , \quad \varrho_{\lambda\mu} = \frac{d\lambda}{d\mu} \in L^1(S, \mu)$$

(called the Radon - Nikodym derivatives) such that

$$\int_B d\mu = \int_B \varrho_{\mu\lambda} \, d\lambda \; , \quad \int_B d\lambda = \int_B \varrho_{\lambda\mu} \, d\mu$$

for any Borel subset $B \subset S$. One writes usually $\lambda = \varrho_{\lambda\mu}\mu$ for two equivalent measures.

We will write $\mathfrak{M} = \mathfrak{M}(S)$ for the equivalence class of measures on S induced by its manifold structure and $\mathfrak{M}_0 \subset \mathfrak{M}$ for *smooth* measures, whose pairwise Radon - Nikodym derivatives are smooth. Certainly, $\varrho_{\mu\lambda}\varrho_{\lambda\zeta} = \varrho_{\mu\zeta}$ and $\varrho_{\mu\lambda}\varrho_{\lambda\mu} = 1$ almost everywhere w. r. t. any measure from $\mathfrak{M}$ for all $\lambda, \varrho, \zeta \in \mathfrak{M}$.

Half weighings. Consider the following equivalence of pairs (f, μ), where $\mu \in \mathfrak{M}$, $f \in L^2_{\mathbb{R}}(S, \mu)$:

$$(f_1, \mu_1) \sim (f_2, \mu_2) \;\Leftrightarrow\; f_1 \sqrt{\varrho_{\mu_1 \lambda}} = f_2 \sqrt{\varrho_{\mu_2 \lambda}}$$

almost everywhere w. r. t. any $\lambda \in \mathfrak{M}$. The corresponding equivalence classes usually are referred as *half weights* and denoted as $\vartheta = f\sqrt{\mu}$. A half weight $f\sqrt{\mu}$ is called *smooth*, if $f \in C^\infty(S)$ and $\mu \in \mathfrak{M}_0$.

The half weights form the real closed vector space $\mathfrak{H}(S)$ equipped with the scalar product

$$\Big(f_1 \sqrt{\mu_1}\, , f_2 \sqrt{\mu_2} \Big) \stackrel{\text{def}}{=} \int_S f_1 f_2 \sqrt{\varrho_{\mu_1 \lambda} \varrho_{\mu_2 \lambda}}\, d\lambda$$

(the linear combinations defined as

$$t_1 \cdot f_1 \sqrt{\mu_1} + t_2 \cdot f_2 \sqrt{\mu_2} \stackrel{\text{def}}{=} \stackrel{\text{def}}{=} \left(t_1 f_1 \sqrt{\varrho_{\mu_1 \lambda}} + t_2 f_2 \sqrt{\varrho_{\mu_2 \lambda}} \right) \sqrt{\lambda}$$

where λ in the both formulas is an arbitrary measure from $\mathfrak{M}$. The smooth half weights form a dense subspace $\overset{\circ}{\mathfrak{H}}(S) \subset \mathfrak{H}(S)$.

Half-weighted cycles. *Diff*$_0(S)$ acts on $\mathfrak{H}(S)$ via $(f,\mu) \xmapsto{h} \left(f \circ h^{-1}, \left(h^{-1}\right)^* \mu \right)$ and sends $\mathring{\mathfrak{H}}$ to itself. The orbit spaces of the corresponding diagonal actions:

$$\mathfrak{L}^{\mathsf{hw}} \stackrel{\mathsf{def}}{=} \mathit{Map}_{\mathsf{LA}}(S,M) \times \mathfrak{H}(S)/\mathit{Diff}_0(S)$$
$$\mathcal{P}_k^{\mathsf{hw}} \stackrel{\mathsf{def}}{=} \mathit{Map}_{\mathsf{LE}}(S,P_k) \times \mathfrak{H}(S)/\mathit{Diff}_0(S)\ .$$

are called spaces of *half weighted* Lagrangian and Planckian cycles of level k. Replace $\mathfrak{H}$ by $\mathring{\mathfrak{H}}$ in these definitions, we obtain the dense subspaces $\mathring{\mathfrak{L}}{}_k^{\mathsf{hw}} \subset \mathfrak{L}_k^{\mathsf{hw}}$ and $\mathring{\mathcal{P}}{}_k^{\mathsf{hw}} \subset \mathcal{P}_k^{\mathsf{hw}}$, of *smoothly* half weighted cycles.

The forgetful maps $\mathfrak{L}^{\mathsf{hw}} \xrightarrow{F} \mathfrak{L}$, $\mathcal{P}_k^{\mathsf{hw}} \xrightarrow{\widetilde{F}} \mathcal{P}_k$ (and their smooth versions) provide these spaces with the vector bundle structures. All these bundles are arranged in the commutative diagram:

$$\begin{tikzcd}
\mathcal{P}_k^{\mathsf{hw}} \arrow[r] \arrow[d, "\widetilde{F}=\pi^*F|_{\mathcal{B}_k^{\mathsf{hw}}}"'] & \mathcal{B}_k^{\mathsf{hw}} \arrow[r, hook] \arrow[d, "F|_{\mathcal{B}_k^{\mathsf{hw}}}"] & \mathfrak{L}^{\mathsf{hw}} \arrow[d, "F"] \\
\mathcal{P}_k \arrow[r, "\pi"] & \mathcal{B}_k \arrow[r, hook] & \mathfrak{L}\ .
\end{tikzcd}$$

Complex structure. For any fixed half weight $f\sqrt{\mu}$ the tangent space $T_{\widetilde{S}}\mathcal{P}_k = C^\infty(S)$ is included into $T_{f\sqrt{\mu}}\mathfrak{H} \simeq \mathfrak{H}$ by $f_1 \longmapsto f f_1\sqrt{\mu}$. This gives an inclusion $T\mathcal{P}_k \hookrightarrow \mathcal{P}_k^{\text{hw}}$ as a dense set. Restricting ourselves by the smooth half weights, we get the identification $T\mathcal{P}_k \xrightarrow{\sim} \mathring{\mathcal{P}}_k^{\text{hw}}$.

To push down the canonical integrable complex structure from $T\mathcal{P}_k$ To $\mathring{\mathcal{P}}_k^{\text{hw}}$, consider a smooth Planckian cycle $S \xhookrightarrow{\widetilde{\varphi}} P$ laying over a BS_k-cycle $S \xhookrightarrow{\varphi} M$ and half weighted by $\vartheta = f\sqrt{\mu}$. Using a local splitting $T_{(\widetilde{\varphi},\vartheta)}\mathring{\mathcal{P}}_k^{\text{hw}} \simeq T_\vartheta\mathring{\mathfrak{H}}(S) \oplus T_{\widetilde{\varphi}}\mathring{\mathcal{P}}_k^{\text{hw}} \simeq C^\infty(S) \oplus C^\infty(S)$ we coordinate a small neighborhood of $(\widetilde{\varphi},\vartheta)$ in $\mathring{\mathcal{P}}_k^{\text{hw}}$ by smooth complex valued functions $\psi = \psi_1 + i\psi_2 \in C^\infty(S,\mathbb{C})$ as follows. The imaginary part ψ_2 produces some Planckian cycle supported by $\Gamma_{\psi_2} \subset U \subset T^*S$ and having the DW-coordinate ψ_2. The real part ψ_1 provides it with the half weight

$$pr^*\vartheta = (pr \circ f)\cdot\sqrt{pr^*\mu}\,,$$

where $\Gamma_{\psi_2} \xrightarrow{pr} S$ is the restriction of the canonical projection $T^*S \longrightarrow S$.

Towards symplectic structure. The product of two half weights $\vartheta_1 = f_1\sqrt{\mu_1}$ and $\vartheta_2 = f_2\sqrt{\mu_2}$ is a complete weight on S, that is

$$\vartheta_1\vartheta_2 \stackrel{\text{def}}{=} f_1 f_2\sqrt{\varrho_{\mu_1}\varrho_{\mu_2\lambda}}\,\lambda\,,$$

does not depend on the choice of $\lambda \in \mathfrak{M}$ and produces a well defined integration

$$f \longmapsto \int_S f\, d\vartheta_1\vartheta_2 \stackrel{\text{def}}{=} \int_S f f_1 f_2 \sqrt{\varrho_{\mu_1\lambda}\varrho_{\mu_2\lambda}}\, d\lambda$$

for $f \in C^\infty(S)$.

So, each half weight ϑ produces a linear form

$$\varkappa(\vartheta) \ : \quad f \longmapsto \int_S f\, d\vartheta^2$$

on the tangent space $T_{\widetilde{S}}\mathcal{P}_k \simeq C^\infty(S)$. In other words, $\mathfrak{H}(S)$ double covers $T^*_{(S,\widetilde{\varphi})}\mathcal{P}_k$, because $\varkappa$ is quadratic in ϑ. The differential of this quadratic covering sends $\vartheta_1 \in \mathfrak{H}(S)$ to the linear form

$$d\varkappa_\vartheta(\vartheta_1) \ : \quad f \longmapsto 2\int_S f\, d\vartheta\vartheta_1\,.$$

on $C^\infty(S) = T_{\widetilde{S}}\mathcal{P}_k$. So, $d\varkappa$ identifies $T(\mathcal{P}_k^{\text{hw}})$ with $T(T^*\mathcal{P}_k)$ an may be used to pull back the canonical symplectic data from $T^*\mathcal{P}_k$ to $\mathcal{P}_k^{\text{hw}}$.

Kähler structure. Let us provide $\mathcal{P}_k^{\mathsf{hw}}$ with an action form $\mathsf{H} \stackrel{\mathsf{def}}{=} \varkappa^*\eta$ and the corresponding symplectic structure $\Omega = d\mathsf{H}$, where η is the canonical action 1-form on the cotangent bundle $T^*\mathcal{P}_k$. In terms of the local splitting $T_{(S,\vartheta_0)}\mathcal{P}^{\mathsf{hw}} \simeq \mathfrak{H}(S) \oplus C^\infty(S)$ induced by Darboux - Weinstein uniformization these two forms act on $v_1 = (\vartheta_1, f_1)$ and $v_2 = (\vartheta_2, f_2)$ as $\mathsf{H}(v_1) = 2\int_S f_1\, d\vartheta_0\vartheta_1$ and

$$\Omega\Big(\,(\,\vartheta_1, f_1\,)\;,\,(\,\vartheta_2, f_2\,)\Big) = \\ = \int_S f_2\, d\vartheta_0\vartheta_1 - \int_S f_1\, d\vartheta_0\vartheta_2\,.$$

In terms of the complex DW-coordinates on $\mathcal{P}_k^{\mathsf{hw}}$, the real tangent vector, along $(\psi_1, 0)$, and an imaginary tangent vector $(0, \psi_2)$ are paired by Ω as $\Omega\Big(\,(\,\psi_1, 0\,)\;,\,(\,0, \psi_2\,)\Big) = \int_S \psi_1\psi_2\, d\vartheta_0^2$ that coincides with L^2-product of ψ_1 and ψ_2 w. r. t. the measure ϑ_0^2. So, Ω is compatible with the complex structure on $\mathring{\mathcal{P}}_k^{\mathsf{hw}}$ and produces the Kähler triple whose Riemannian metric G takes

$$G\Big(\,(\vartheta_1, f_1)\,,\,(\vartheta_2, f_2)\Big) = \\ = \int_S f_1\, d\vartheta_1\vartheta_0 + \int_S f_2\, d\vartheta_2\vartheta_0\,.$$

Claim 1. The level k half weighted Planckian cycles space $\mathcal{P}_k^{\mathsf{hw}}$ has the natural structure of an infinite dimensional Kähler variety.

Half weighted Bohr - Sommerfeld cycles.

The natural $U(1)$-action on $\mathcal{P}_k$, which changes a covariant trivialization of L over a given $S \subset M$, has the standard lifting on $T\mathcal{P}_k$ and on $\mathcal{P}_k^{\mathsf{hw}}$ as well. This action preserves the Kähler structure, because $z \in U(1) \subset \mathbb{C}$ multiplies the complex DW - coordinates by z . The volume function $\mathcal{P}_k^{\mathsf{hw}} \xrightarrow{vol} \mathbb{R}$, which sends $(S, \tilde{\varphi}, \vartheta)$ to $\int_S \vartheta^2$, is $U(1)$-invariant. Each its level hypersurface $\mathcal{P}_{k,t}^{\mathsf{hw}}$ consists of weighted Planckian cycles whose underlying BS_k-cycle has the fixed volume tS. The orbit space $\mathcal{B}_{k,t}^{\mathsf{hw}} \stackrel{\mathsf{def}}{=} \mathcal{P}_{k,t}^{\mathsf{hw}}/U(1)$ is called a space of *half weighted* BS_k-cycles of volume t.

Claim 2. The space $\mathcal{B}_{k,t}^{\mathsf{hw}}$, of half weighted BS_k-cycles of volume t, also has a natural structure of an infinite dimensional Kähler variety induced from $\mathcal{P}_k^{\mathsf{hw}}$.

Skethed proof. In terms of Darboux - Weinstein coordinates, the tangent space $T_{(\widetilde{S},\vartheta_0)}\mathcal{P}^{\text{hw}}_{k,t}$ consists of $v = (f, \vartheta) \in T_{(S,\vartheta_0)}\mathcal{P}^{\text{hw}} \simeq \mathfrak{H}(S) \oplus C^\infty(S)$ such that $\int_S d\vartheta\vartheta_0 = 0$, because the infinitesimal volume increment along v is measured exactly by this integral. The $U(1)$-factorization takes (f, ϑ) to $(df, \vartheta) \in T_{(S,\vartheta_0)}\mathcal{B}^{\text{hw}}_{k,t}$. So, the both symplectic and Riemannian structures

$$\Omega\Big(\,(\,\vartheta_1, f_1\,)\;,\;(\,\vartheta_2, f_2\,)\Big) = \\ = \int_S f_2\, d\vartheta_0\vartheta_1 - \int_S f_1\, d\vartheta_0\vartheta_2$$

$$G\Big(\,(\vartheta_1, f_1)\,,\,(\vartheta_2, f_2)\Big) = \\ = \int_S f_1\, d\vartheta_1\vartheta_0 + \int_S f_2\, d\vartheta_2\vartheta_0$$

remain to be well defined on $\mathcal{B}^{\text{hw}}_{k,t}$, since the conditions

$$\int_S d\vartheta_1\vartheta_0 = \int_S d\vartheta_2\vartheta_0 = 0$$

annihilate an arbitrariness in lifting df_i to f_i (measured by a constant additive terms).

Return to Kähler quantization.

Let M be equipped with a Kähler structure whose Kähler form coincides with the symplectic form ω. Then the compatibility conditions on the prequantization data provide L with the canonical structure of holomorphic ample line bundle and we can consider the wave function spaces, i. e. the spaces of global holomorphic sections

$$\mathcal{H}_k = H^0(M, L^{\otimes k}) \; .$$

For any half weighted Planckian cycle $(S, \widetilde{\varphi}, \vartheta)$ supported by a BS$_k$-cycle $S \overset{\varphi}{\hookrightarrow} M$ with a horizontal unitary section $S \overset{\sigma}{\hookrightarrow} \varphi^* L^{\otimes k}$ and for any holomorphic section $s \in H^0(M_I, L^{\otimes k})$ consider a smooth complex valued function $\psi_{\sigma,\varphi} \in C^\infty(S, \mathbb{C})$ such that $\varphi^* s = \psi_{\sigma,\varphi} \cdot \sigma$ in $C^\infty(S, \varphi^* L^{\otimes k})$.

<u>**Claim:**</u> the linear map

$$H^0(M, L^{\otimes k}) \xhookrightarrow{s \mapsto \psi_{\sigma,\varphi}} C^\infty(S, \mathbb{C})$$

is *injective* and *holomorphic*.

<u>**Remark:**</u> $C^\infty(S, \mathbb{C})$ is a fiber of the *holomorphic* tangent bundle $T_{(S,\widetilde{\varphi},\vartheta)} \overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$ over $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$ considered with the intrinsic holomorphic structure defined above.

Towards the Borthwick - Paul - Uribe map.

Let us define a map

$$\overset{\circ}{\mathcal{P}}{}^{\mathsf{hw}} \xrightarrow{\ \mathfrak{m}_k\ } H^0(M, L^{\otimes k})^* \tag{5}$$

by sending a half weighted Planckian cycle $(\widetilde{\varphi}, \vartheta)$ to $\mathbb{C}$-linear form that takes $s \in H^0(M, L^{\otimes k})$ to

$$\int_S \psi_{\sigma,\varphi}\, d\vartheta^2 = \int_S \psi_{\sigma,\varphi}\, f^2\, d\mu \, \in \, \mathbb{C}$$

In terms of complex DW-coordinates on $\overset{\circ}{\mathcal{P}}{}^{\mathsf{hw}}_k$ near $(S, \widetilde{\varphi}, \vartheta)$ it sends $\psi_1 + i\psi_2$ to a linear form on $H^0(M, L^{\otimes k})$ that takes s to

$$\int_S \psi_{\sigma,\varphi} \psi_1 e^{i\psi_2} \psi_1^2\, d\vartheta^2$$

Its differential sends $\psi_1 + i\psi_2$ to a functional

$$s \longmapsto \int_S \psi_{\sigma,\varphi}(2\psi_1 + i\psi_2)\, d\vartheta^2 \ ,$$

Although this map is not $\mathbb{C}$-linear, the defect is measured by a constant factor. That is the map (5) has a *constant Kähler angle*, i. e. is almost holomorphic. In fact, using the metaplectic formalism, the definition (5) can be modified in order to get a holomorphic map whose differential would sesquilinearly dual to the restriction map $H^0(M, L^{\otimes k}) \xhookrightarrow{s \mapsto \psi_{\sigma,\varphi}} C^\infty(S, \mathbb{C})$.