Izvestiya: Mathematics **65**:3 ??–??

Izvestiya RAN: Ser. Mat. **65**:3 15–50 DOI 10.1070/IM2001v065n03ABEH000334

Abelian Lagrangian algebraic geometry

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Abstract. This paper begins a detailed exposition of a geometric approach to quantization, which is presented in a series of preprints ([23], [24], ...) and which combines the methods of algebraic and Lagrangian geometry. Given a prequantization $U(1)$-bundle L on a symplectic manifold M, we introduce an infinite-dimensional Kähler manifold $\mathcal{P}^{\mathrm{hw}}$ of half-weighted Planck cycles. With every Kähler polarization on M we canonically associate a map $\mathcal{P}^{\mathrm{hw}} \xrightarrow{\gamma} H^0(M,L)$ to the space of holomorphic sections of the prequantization bundle. We show that this map has a constant Kähler angle and its "twisting" to a holomorphic map is the Borthwick–Paul–Uribe map. The simplest non-trivial illustration of all these constructions is provided by the theory of Legendrian knots in S^3.

Introduction

0.1. Quantization. The natural mathematical framework for the classical Hamiltonian dynamics is symplectic geometry (see [1]). A smooth (possibly non-compact) C^∞-manifold M is called *symplectic* if it is endowed with a *symplectic structure*, that is, with a *closed* 2-form ω which defines a skew-symmetric *isomorphism* $\omega\colon TM \xrightarrow{\sim} T^*M$ between the tangent and cotangent bundles of M. (In particular, $\dim M$ is even.) The symplectic form provides the space $C^\infty(M)$ of smooth functions with the structure of Lie algebra with respect to the *Poisson bracket*

$$\{f_1,f_2\} \overset{\mathrm{def}}{=} \omega(H_{f_1},H_{f_2}).$$

Here $H_f \overset{\mathrm{def}}{=} \omega^{-1}(df)$ denotes the *Hamiltonian vector field* (or *symplectic gradient*) of the smooth function $f\in C^\infty(M)$. This field is uniquely determined by requiring that $\omega(H_f,v)=df(v)$ for any field v. The flows along Hamiltonian vector fields preserve the symplectic form ω and give a solution of the classical system of Hamilton equations on the phase trajectories of particles.

Ideally, a *quantization* of such a mechanical system is understood as a hypothetical procedure Q that transforms each symplectic manifold (M,ω) to a Hilbert space $Q(M,\omega)$ called the *wave function space* and takes each real-valued function $f\in C^\infty(M)$ to a self-adjoint operator $Q(M,\omega)\xrightarrow{Q(f)} Q(M,\omega)$ on this space such that the following conditions hold: $Q(f)$ depends linearly on f, constants are represented by scalar homotheties, and we have the *Dirac commutation relations*

$$[Q(f_1),Q(f_2)] = ih\,Q(\{f_1,f_2\}). \tag{0.1}$$

AMS 2000 *Mathematics Subject Classification.* 53D50, 53C15.

Moreover, one usually imposes some irreducibility conditions on the resulting representation. Unfortunately (or, may be, fortunately), van Howe's theorem says that it is impossible to satisfy these requirements (see [9]). Mathematical attempts to overcome this difficulty proceed by searching an appropriate extension of the domain of classical objects and/or an appropriate weakening of the "quantization rules" listed above. Of course, this can be done in many ways and, as a result, the development of modern "quantum mathematics" resembles radioactive decay running mainly along three directions that may be described as the *algebraic*, *analytic* and *geometric* approaches.

We do not discuss the first two directions, which presently are particularly well-developed. We only remark that algebraic quantization (quantum deformations, $*$-products, non-commutative geometry and so on) usually completely ignores the geometry of the initial classical mechanical system, while the analytic approach (Hardy spaces, Szegö projectors, spectral theory of Fourier operators and so on) is closely related to this geometry.

Geometric quantization starts with a complex line bundle L on M such that L possesses a Hermitian structure and a Hermitian connection a with curvature form $F_a = ih\omega$. The wave function space $Q(M,\omega)$ is defined to be the space $\Gamma(L)$ of smooth sections of L, and each function $f \in C^\infty(M)$ corresponds to the operator

$$Q(f) = ih\nabla^a_{H_f} + f$$

on this space. (In the right-hand side of this formula, the first term $\nabla^a_{H_f}$ denotes the covariant differentiation of sections of L with respect to the connection a along the Hamiltonian vector field H_f, and the second term is the operator that multiplies these sections by the function f.) One easily verifies that all these operators satisfy the Dirac commutation relations (0.1) and are self-adjoint with respect to the standard Hilbert space structure on $\Gamma(L)$, which is given by integrating the pointwise Hermitian scalar product of sections over the symplectic volume of M. But the result of this prequantization does not fit in the Schrödinger scheme described above, even in the simplest two-dimensional case: the space of all sections is too large.

The geometric modification of the quantization rules consists in fixing an additional geometric structure on M: a *polarization*. That is, we choose a Lagrangian subbundle $P \subset TM \otimes \mathbb{C}$ in the complexified tangent bundle such that P is closed under Lie brackets. For the wave function space, we take the subspace $\Gamma_P(L) \subset \Gamma(L)$ of sections that are covariantly constant along P. However, the subspace $\Gamma_P(L)$ is *a priori* not preserved by an arbitrary operator $Q(f)$, whence we must restrict ourselves by the space of those f whose Hamiltonian flow preserves the polarization P. Unfortunately, this class of Hamiltonians turns out to be too small, and we must sacrifice something again ...

Nowadays there are several theories that follow this direction, and many of them contain well-understood examples showing that the result of quantization is essentially *independent* of the choice of this additional geometric rigidity on M. This observation revives the hope that there is a universal "quantization geometry" that underlies all the "rules" listed above and includes all known physically meaningful examples. The present paper is an attempt to lay down the foundations of such a geometry. We see these foundations in Abelian Lagrangian algebraic geometry (ALAG), which originates from the "geometric approach" of quantum mathematics and brings the complicated analytic methods of Lagrangian geometry and the physically meaningful formal quantum algebra together in the customary framework of complex projective geometry. In this paper we restrict ourselves by the case of *Kähler* polarization. For the simplest example $M = \mathbb{C}$, this leads to the Bargmann–Segal–Fock space of holomorphic functions on $\mathbb{C}$ with summable square[1], which in particular contains the Hamiltonian of the harmonic oscillator.

0.2. From symplectic geometry to algebraic geometry. The algebro-geometric spirit is present in almost all basic analytic constructions. Namely, when one studies classical and quasiclassical mechanical systems by the methods of symplectic geometry, the main objects are *Lagrangian submanifolds* (see [25], p. 5 for details of this "symplectic creed"), and symplectic geometry may algebraically be treated as a category whose morphisms are defined in terms of Lagrangian submanifolds (see [25]). This is very close to the motivic ideology in algebraic geometry. Moreover, the spectral theory of Fourier integral operators, which forms the basis for quantization of contact manifolds after Boutet de Monvel and Guillemin, is completely parallel to the theory of Hilbert polynomials.

The most direct way from symplectic geometry to algebraic geometry is by fixing a *Kähler polarization* on M, that is, by endowing M with a complex structure I and with a Hermitian metric whose imaginary part equals ω. If we additionally assume that the cohomology class $[\omega]$ is integer, then there is a holomorphic complex Hermitian line bundle L on the Kähler manifold M_I with $c_1(L) = [\omega]$ and there is a Hermitian connection a on L with curvature form $2\pi i\omega$. For every $k \in \mathbb{N}$, a standard geometric construction associates with this bundle a map

$$\varphi_k\colon M_I \xhookrightarrow{x \mapsto \mathrm{Ann}(x)} \mathbb{P}H^0(M_I, L^{\otimes k})^* \tag{0.2}$$

of M to the complex projective space. (Here $H^0(M_I, L^{\otimes k})$ denotes the space of global holomorphic sections of the bundle $L^{\otimes k}$ with respect to the complex structure induced by I.) By Kodaira's theorem, the maps (0.2) are holomorphic embeddings for $k \gg 0$. This enables us to regard M_I as a projective algebraic variety.

On the other hand, the space of holomorphic sections may analytically be regarded as a subspace $H^0(M_I, L^{\otimes k}) \subset C^\infty(M, L^{\otimes k})$ in the infinite-dimensional space of all smooth sections. The space of smooth sections carries a scalar product and a norm induced by the Liouville volume form $\Omega = \omega^{\wedge n}$ on M and the Hermitian form $(\,*\,,\,*\,)_{L^{\otimes k}}$ on $L^{\otimes k}$ in the usual way:

$$\langle s_1, s_2\rangle_k \stackrel{\text{def}}{=} \int_M (s_1, s_2)_{L^{\otimes k}}\,\Omega, \qquad \|s\|_k \stackrel{\text{def}}{=} \sqrt{\langle s, s\rangle_k}, \qquad s_1, s_2, s \in C^\infty(M, L^{\otimes k}).$$

Thus $Q_k(M_I) \stackrel{\text{def}}{=} H^0(M_I, L^{\otimes k})$ is a closed subspace of the Hilbert space $L^2(M, L^{\otimes k})$ (the completion of $C^\infty(M, L^{\otimes k})$ with respect to this norm). The Hilbert space

[1] With respect to the Gaussian measure related to the Hermitian structure on the trivial line bundle.

$Q_k(M_I) \subset L^2(M, L^{\otimes k})$ is called the *wave function space of level* k, its projectivization is called the *state space*, and the orthogonal projector

$$L^2(M, L^{\otimes k}) \xrightarrow{\Pi_k} H^0(M_I, L^{\otimes k})$$

is called the *Szegö projector*. If $k \gg 0$, then the composite of the holomorphic projective embedding φ_k described above and the isomorphism $\mathbb{P}\big(H^0(M_I, L^{\otimes k})^*\big) \xrightarrow{\sim} \overline{\mathbb{P}\big(H^0(M_I, L^{\otimes k})\big)}$ (which originates from the Hermitian structure on $L^{\otimes k}$) is an anti-holomorphic embedding

$$M_I \xhookrightarrow{\overline{\varphi}_k} \mathbb{P}\big(H^0(M_I, L^{\otimes k})\big).$$

Elements of the image of this embedding are called *coherent states*. Furthermore, every function $f \in C^\infty(M)$ defines a *Toeplitz operator*

$$H^0(M_I, L^{\otimes k}) \xrightarrow{T_f^{(k)}} H^0(M_I, L^{\otimes k}): \quad s \longmapsto \Pi_k(f \cdot s).$$

The correspondence $f \longmapsto T_f^{(k)}$ determines a linear Berezin–Toeplitz map of the (commutative) algebra of functions to the (non-commutative) matrix algebra:

$$C^\infty(M) \xrightarrow{T_k} \operatorname{End}\big(H^0(M_I, L^{\otimes k})\big).$$

This map is by no means compatible with either multiplication or Poisson structure, but it has good asymptotic behaviour in k in the following sense. By the Bordemann–Meinrenken–Schlichenmaier theorem (see [12]), for every function $f \in C^\infty(M)$ there is a constant $K_f > 0$ such that

$$\|f\|_\infty + \frac{K_f}{k} \leqslant \|T_f^{(k)}\| \leqslant \|f\|_\infty,$$

and for any functions $f_1, f_2 \in C^\infty(M)$ we have the following asymptotic formula as $k \to \infty$:

$$\big\|k \cdot i\big[T_{f_1}^{(k)}, T_{f_2}^{(k)}\big] - T_{\{f_1,f_2\}}^{(k)}\big\| \sim \big\|T_{f_1}^{(k)} \cdot T_{f_2}^{(k)} - T_{f_1 \cdot f_2}^{(k)}\big\| \sim O(1/k),$$

where $\|f\|_\infty$ is the sup-norm on $C^\infty(M)$, and $\operatorname{End}\big(H^0(M_I, L^{\otimes k})\big)$ is endowed with the standard operator norm

$$\big\|T_f^{(k)}\big\| = \sup_{s \neq 0}\big(\|T_f^{(k)}(s)\| / \|s\|\big).$$

In other words, the Dirac conditions hold asymptotically as $k \to \infty$. (This is usually referred to as *good quasiclassical behaviour*.) Moreover, Schlichenmaier showed that Berezin–Toeplitz maps give a unique strict quantum deformation of the algebra $C^\infty(M)$ with good quasiclassical behaviour.

0.3. ALAG. As already mentioned, the examples for which the previous quantization procedure was calculated in detail show that the projective state spaces $\mathbb{P}(Q_k(M_I))$ are *post factum* independent of the choice of the complex structure, although this choice is *a priori* essential for the construction of the Hilbert spaces $Q_k(M_I)$. More precisely, there is a holomorphic vector bundle $Q_k \xrightarrow{\pi_k} \mathcal{M}$ over the moduli space of polarized complex structures on (M, ω) (the fibre of this bundle over a point $I \in \mathcal{M}$ is $Q_k(M_I)$), and this bundle carries a natural *flat projective connection* (see [19] for details). Geometrically, this phenomenon is explained by the fact that the space of Lagrangian cycles on M contains a natural finite-codimensional submanifold possessing an *internal* Kähler structure, which depends only on symplectic geometry of M. This enables us to construct a "universal embedding" of the symplectic geometry of M into algebraic geometry such that all special algebro-geometric properties of M (appearing when we fix additional data on M) can be controlled by the usual algebro-geometric means. The precise meaning of this will be explained in the main body of the paper.

The paper is organized as follows. In § 1 we recall the necessary (mostly standard) concepts of symplectic and differential geometry: Lagrangian cycles, Darboux–Weinstein uniformization, interaction between symplectic and almost Hermitian structures, Abelian connections and unitary characters of the fundamental group, Jacobians and so on. In § 2 we introduce the spaces of Bohr–Sommerfeld and Planck cycles along with additional structures: Berry bundles, half-weights and so on. We construct canonical integrable Kähler structures on the spaces $\mathcal{P}_k^{\mathrm{hw}}$ of half-weighted Planck cycles of level k and on the spaces $\mathcal{B}_{k,t}^{\mathrm{hw}}$ of Bohr–Sommerfeld cycles of level k with a fixed volume t. The central point of the paper is § 3, where we describe in detail the procedure of geometric quantization by Kähler polarization and give a geometric description of the Borthwick–Paul–Uribe holomorphic maps $\mathcal{B}_{k,t}^{\mathrm{hw}} \xrightarrow{\mathrm{BPU}_k} \mathbb{P}\big(H^0(L^{\otimes k})^*\big)$. In the concluding § 4 we review all our constructions for the simplest non-trivial example when the initial symplectic system (M, ω) is the Riemann sphere $S^2 \simeq \mathbb{CP}_1$ with the standard Kähler form. Then Lagrangian cycles become unicursal curves in S^2, and Planck cycles become Legendrian knots lying over such curves in the sphere S^3, the total space of the Hopf bundle over S^2. Thus, this "simplest" illustration of ALAG includes the theory of Legendrian knots.

§ 1. Preliminaries and notation

1.1. Lagrangian geometry. We always denote by (M, ω) a smooth $2n$-dimensional oriented real manifold M with a fixed closed skew-symmetric 2-form ω that defines an isomorphism between the tangent and cotangent bundles:

$$\omega\colon TM \xrightarrow{\sim} TM^*. \tag{1.1}$$

We fix a smooth n-dimensional oriented real manifold S and a homotopy class $\mathrm{Map}_{\mathrm{LA}}(S, M)$ of smooth maps

$$S \xrightarrow{\varphi} M$$

that satisfy $\varphi^*(\omega) = 0$. The space $\mathrm{Map}_{\mathrm{LA}}(S, M)$ is called the *space of Lagrangian maps* (of fixed topological type). It is an infinite-dimensional manifold naturally acted on by the group $\mathrm{Diff}_0(S)$ of all diffeomorphisms of S homotopic to the identity:

$$h\colon \varphi \mapsto \varphi \circ h^{-1},$$

where $h \in \mathrm{Diff}_0(S)$, $\varphi \in \mathrm{Map}_{\mathrm{LA}}(S, M)$. The corresponding orbit space

$$\mathfrak{L} \stackrel{\mathrm{def}}{=} \mathrm{Map}_{\mathrm{LA}}(S, M) / \mathrm{Diff}_0(S)$$

is called the space of *Lagrangian cycles* [2].

A Lagrangian cycle $S \xrightarrow{f} M$ is called *smoothly embedded* (resp. *smoothly immersed*) if φ is a smooth embedding (resp. immersion). In the first case, the image $f(S) \subset M$ is called a *Lagrangian submanifold* in M, and the condition $\varphi^*\omega = 0$ means that the restriction of ω to TS is identically zero form. In both cases, the normal bundle is identified with the cotangent bundle by the isomorphism (1.1) that comes from the symplectic structure of M:

$$N_\varphi \stackrel{\mathrm{def}}{=} \varphi^*(TM)/TS \simeq \varphi^*(T^*M)/\operatorname{Ann}(TS) \simeq T^*S.$$

1.1.1. *Example*: *the cotangent bundle.* Let $M = T^*S \xrightarrow{p} S$ be the cotangent bundle of a smooth oriented manifold S. Then M carries a canonical 1-form η, called the *action form*, which sends a tangent vector $w \in T_{(\xi,x)}T^*S$, $x \in S$, $\xi \in T_x^*S$ to the number

$$\eta(w) \stackrel{\mathrm{def}}{=} p^*\xi(w). \tag{1.2}$$

Its differential $\omega \stackrel{\mathrm{def}}{=} d\eta$ defines a symplectic structure on M. Given a local open chart $U \subset S$ with $T^*S|_U = U \times \mathbb{R}^n$, we introduce the coordinates

$$(q,p) = (q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n),$$

where q_i are the coordinates on U and p_i are the linear coordinates along dq_i on $\mathbb{R}^n$. Then the forms η and ω are described by

$$\eta(p,q) = \sum_{j=1}^n p_j\, dq_j, \qquad \omega(p,q) = \sum_{j=1}^n dp_j \wedge dq_j. \tag{1.3}$$

In particular, all fibres of the projection $M \twoheadrightarrow S$ and also the zero section $S \xhookrightarrow{s_0} M$ are Lagrangian submanifolds in M. Any section $S \xhookrightarrow{s} M$ may be regarded as the graph of a 1-form $\alpha_s = s^*(\eta)$ such that we locally have $s(x) = (x, \alpha_s(x)) \in U \times \mathbb{R}^n = T^*S|_U$. This graph is a Lagrangian submanifold in M if and only if the form α_s is closed. Indeed, $s^*(\omega) = s^*(d\eta) = ds^*(\eta) = d\alpha_s$.

[2] In this paper we completely ignore the problems related to the global geometry and topology of the space $\mathfrak{L}$. We deal with the local analysis near smooth points, where $\mathfrak{L}$ admits a rather simple description (see below).

1.1.2. *The tangent space to $\mathfrak{L}$ at a smoothly immersed cycle.* The tangent space $T_\varphi \operatorname{Map}(S, M)$ at a smooth immersion $S \xrightarrow{\varphi} M$ is the space $C^\infty(S, \varphi^*TM)$ of smooth sections of the lifted tangent bundle to M. Indeed, small 1-parameter deformations of φ are shifts along integral curves of vector fields v defined locally near $\varphi(S)$. Such a shift preserves the condition $\varphi^*(\omega) = 0$ if and only if $\varphi^*(\mathcal{L}_v\omega) = 0$. Finally, the tangent space to the orbit of the group $\operatorname{Diff}_0(S)$ at φ consists of $v \in C^\infty(S, TS) \subset C^\infty(S, \varphi^*TM)$. Thus,

$$T_\varphi\big(\operatorname{Map}(S, M)/\operatorname{Diff}_0(S)\big) = \big\{v \in C^\infty(S, N_\varphi) \bigm| \varphi^*(\mathcal{L}ie_v\omega) = 0\big\}.$$

Identifying the normal bundle N_φ and the cotangent bundle $T^*\varphi(S)$ as described in § 1.1, we see that the vector field v in the previous formula corresponds to a 1-form $\xi = \varphi^*(i_v\omega) \in C^\infty(S, T^*S)$. The fields $v \in T_\varphi\mathfrak{L}$ correspond to *closed* 1-forms because

$$d\xi = d\varphi^*(i_v\omega) = \varphi^*(di_v\omega) = \varphi^*(\mathcal{L}_v\omega + i_v d\omega) = \varphi^*(\mathcal{L}_v\omega) = 0.$$

We get the following proposition.

1.1.3. Proposition. *The tangent space $T_\varphi\mathfrak{L}$ at a smoothly immersed Lagrangian cycle $S \xrightarrow{\varphi} M$ is naturally identified with the space $Z^1_{\mathrm{DR}}(S)$ of closed 1-forms on S.*

1.1.4. *Darboux–Weinstein uniformization.* One of the many restatements of the famous Darboux–Weinstein theorem (see [18], Ch. 4, § 1, [27], § 6 and [28], § 4) says that for every smoothly immersed Lagrangian cycle $S \xhookrightarrow{\varphi} M$ one can find an open tube neighbourhood U of the zero section in the cotangent bundle T^*S and a smooth immersion $U \xhookrightarrow{\widehat{\varphi}} M$ such that $\widehat{\varphi}^*\omega = d\eta$ is the standard symplectic structure on T^*S, and the restriction of $\widehat{\varphi}$ to the zero section coincides with the initial immersion φ. If the initial immersion is an embedding, then one can take $U = T^*S$, and $\widehat{\varphi}$ is a symplectic isomorphism of the cotangent bundle T^*S (with the standard symplectic structure described in § 1.1.1) onto an open tube neighbourhood of the Lagrangian submanifold $f(S) \subset M$.

The Darboux–Weinstein extension $T^*S \supset U \xrightarrow{\widehat{\varphi}} M$ of the initial immersion $S \xrightarrow{\varphi} M$ is a smooth immersion that takes the graphs (contained in U) of closed 1-forms to Lagrangian cycles lying near $\varphi(S)$ in M. This yields the *exponential map*

$$T_S\mathfrak{L} \supset \mathfrak{U} \xrightarrow{\varepsilon} \mathfrak{L}, \tag{1.4}$$

where $\mathfrak{U} \subset Z^1_{\mathrm{DR}}(S)$ consists of all 1-forms ξ whose graphs are contained in U. It follows from Weinstein's results ([27], § 6, [28], § 4) that the image of the map (1.4) contains an open neighbourhood of the initial cycle φ in the appropriate topology on $\mathfrak{L}$. Thus $\mathfrak{L}$ is a smooth (infinite-dimensional) manifold near the smooth immersion $S \xrightarrow{\varphi} M$, and we can use a neighbourhood of zero in the space $Z^1_{\mathrm{DR}}(S)$ of closed 1-forms as a chart for this manifold. The subspaces $B^1_{\mathrm{DR}}(S) \subset Z^1_{\mathrm{DR}}(S) = T_S\mathfrak{L}$ of *exact* 1-forms define a smooth distribution on $\mathfrak{L}$ of codimension $b_1 = \dim H^1(S, \mathbb{R})$. The exponential map locally integrates this distribution to a foliation near each smoothly immersed cycle. Its leaves are of codimension b_1 and are called *isodrasts*[3]. Graphs of two closed 1-forms on S (which are close to each other) lie on one isodrast if and only if the forms are cohomologous.

1.2. Pseudo-Kählerian geometry. A smooth real $2n$-dimensional manifold M is called *pseudoholomorphic* (resp. *pseudo-Hermitian*) if TM is endowed with the structure of a complex (resp. Hermitian) bundle of rank n. The last condition means that TM is endowed with an $\mathbb{R}$-linear operator $TM \xrightarrow{I} TM$, a Riemannian metric $g\colon TM \xrightarrow{\sim} TM^*$ and a non-degenerate skew-symmetric form $\omega\colon TM \xrightarrow{\sim} TM^*$ such that $I^2 = -\operatorname{Id}$, $g(v,w) = \omega(Iv,w)$, $\omega(v,Iw) = -\omega(Iv,w)$ and $g(v,Iw) = -g(Iv,w)$. (These requirements actually imply that the triple (I,g,ω) is uniquely determined by any two of its elements.) Pseudoholomorphy requires only the existence of the complex structure operator I. A pseudo-Hermitian manifold M is called *pseudo-Kähler* if the skew form ω is closed, that is, defines a symplectic structure on M.

Pseudoholomorphic manifolds possess many "linear algebra" properties of complex-analytic spaces because the complexified tangent bundle of any pseudoholomorphic manifold M is decomposed into the direct sum $\mathbb{C}\otimes TM = T_+M \oplus T_-M$ of the $(\pm i)$-eigenspaces of the operator I. In particular, one can define the class of *pseudoholomorphic functions* $M \xrightarrow{f} \mathbb{C}$ (whose differential df annihilates the subspace T_-M) as well as pseudoholomorphic submanifolds (locally defined as the common zero sets of systems of pseudoholomorphic functions), pseudoholomorphic complex vector bundles and so on.

1.2.1. Lemma. *Let $V \subset M$ be a pseudoholomorphic submanifold (possibly singular) of a smooth pseudo-Hermitian manifold M. Then the restriction of the form ω to the tangent cone TC_xV of V at any point $x \in V$ is non-degenerate in the following sense: for every non-zero vector $v \in TC_xV$ there is $u \in TC_xV$ such that $\omega(v,u) \neq 0$.*

Proof. Since V is pseudoholomorphic, the operator I maps the tangent cone TC_xV to itself, and this map is bijective because $I^{-1} = -I$. Since the Riemannian metrics g is positive-definite, any non-zero vector $v \in TC_xV$ satisfies $\omega(v,Iv) = -g(v,v) < 0$, and we may take $u = Iv$.

1.2.2. Corollary. *A non-zero pseudoholomorphic section of a pseudoholomorphic line bundle on a pseudo-Hermitian manifold cannot vanish identically on a Lagrangian cycle.*

Proof. The zero set of a non-zero pseudoholomorphic section of a line bundle is a pseudoholomorphic submanifold Z of dimension $\leqslant (2n-2)$ in the initial $2n$-dimensional pseudo-Hermitian manifold M. Therefore, if Z contains an n-dimensional Lagrangian cycle S, then the restriction of ω to TZ must be degenerate along S, which is impossible by the previous lemma.

[3] Exhausting etymology of this (greek) term is contained in [26].

1.2.3. *Example*: *tangent bundles*. Let $M = TX$ be the tangent bundle of a real-analytic n-dimensional oriented manifold X. Then there is a canonical pseudoholomorphic structure on M defined by the splitting

$$TM = T(TX) = TX_{\uparrow} \oplus TX_{\rightarrow},$$

where the "vertical" subspace $TX_{\uparrow} \simeq TX$ is the kernel of the differential of the natural projection $TX \twoheadrightarrow X$, and the "horizontal" subspace $TX_{\rightarrow} \simeq TX$ is the image of the differential of the zero section $X \hookrightarrow TX$. In terms of this splitting, the operator of multiplication by $i \in \mathbb{C}$ at a point $(v, q) \in TX$ with $v \in T_x X$, $q \in X$ maps a tangent vector $w = (w_1, w_2) \in T_{(v,q)}TX$ to $(-w_2, w_1)$. Moreover, this complex structure is integrable[4] by the Bruhat–Whitney theorem (see [29]).

We recall (see § 1.1.1) that the cotangent bundle T^*X has the canonical symplectic structure $\omega = d\eta$, whose symplectic basis in the tangent space $T(T^*X)$ is of the form $(dy, \partial/\partial y)$. Hence each bundle isomorphism $TX \xrightarrow{\sim} T^*X$ defines a *weak* pseudo-Kähler structure on both manifolds TX, T^*X, that is, the triple (I, g, ω) on these manifolds satisfies all conditions stated in the beginning of § 1.2 except probably for the positiveness of the symmetric form $g(u_1, u_2) = \omega(u_1, Iu_2)$. In particular, if X is endowed with a Riemannian metric $TX \xrightarrow{G} T^*X$, then the cotangent bundle $M = T^*X$ admits an integrable complex structure

$$I\colon (\xi, v) \longmapsto (Gv, -G^{-1}\xi),$$

which is compatible with the canonical symplectic form $\omega = d\eta$ of (1.3). Moreover, then the symmetric form $g(u_1, u_2) = \omega(u_1, Iu_2)$ is positive definite, and $M = T^*X$ is a true (although non-compact) Kähler manifold.

1.3. Hermitian line bundles. In this paper we deal with complex Hermitian line bundles over smooth oriented real manifolds X. Such a bundle $L \twoheadrightarrow X$ is uniquely determined (up to isomorphism) by its first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$ and is tautologically associated with the principal $U(1)$-bundle $P \twoheadrightarrow X$ of unit circles in L, that is, $L = P \times_{U(1)} \mathbb{C}$.

We always identify the Lie algebra $\mathfrak{su}(1)$ with $\mathbb{R}$ such that the exponential map $\mathbb{R} = \mathfrak{su}(1) \twoheadrightarrow U(1)$ takes the form $t \longmapsto e^{2\pi i t}$ and the tautological action of $\mathbb{R} = \mathfrak{su}(1)$ on $\mathbb{C}$ is given by $z \overset{t}{\longmapsto} 2pitz$. We denote by ν the unique invariant 1-form on $U(1)$ with $\int_{U(1)} \nu = 1$, and let u be the dual invariant vector field such that $\nu(u) = 1$. We also denote by u the corresponding vertical vector field on any principal $U(1)$-bundle $P \twoheadrightarrow X$.

1.3.1. *Hermitian connections* on a principal $U(1)$-bundle $P \overset{p}{\twoheadrightarrow} X$ are given by 1-forms α such that $\mathcal{L}ie_u \alpha = 0$ and $\alpha(u) = 1$. Here the first condition means that the form α is $U(1)$-invariant, and the second one means that α projects the tangent space of P onto the vertical subspace (tangent to the fibre), that is, α splits the canonical exact triple of tangent spaces

$$0 \longrightarrow TU(1) \underset{\longrightarrow}{\overset{\alpha}{\longleftarrow}} TP \xrightarrow{dp} p^*TX \longrightarrow 0.$$

[4] *Translators' Note.* From here till the end of the paragraph, the original text is changed in the course of translation.

(The hyperplanes $\ker\alpha \subset TP$ are called *horizontal*.) Any vector field v on X may be lifted (in many ways) to a $U(1)$-invariant vector field w on P such that $dp(w) = v$. Among these lifts w, there is a unique horizontal vector field $\tilde{v} = w - \alpha(w)u$, which is called the *horizontal lift*.

Let L be the vector bundle associated with P. The connection α defines a differential operator

$$C^\infty(X, L) \xrightarrow{\nabla_\alpha} C^\infty(X, T^*X \otimes L),$$

which is called the *covariant differentiation* of sections. This operator acts on a local unitary section σ over a small open ball $U \subset X$ by $\nabla\sigma = 2\pi i\, \xi_\sigma \otimes \sigma$, where $\xi_\sigma = \sigma^*\alpha$ is the lift of the 1-form α from P to U along the local section $U \xhookrightarrow{\sigma} P$. The covariant derivative of arbitrary local section $f\sigma$ with $f \in C^\infty(U, \mathbb{C})$ is given by

$$\nabla(f\sigma) = (df + 2\pi i\, f\xi_\sigma) \otimes \sigma.$$

If we choose another local unitary basis $\sigma' = h\sigma$ with $h \in C^\infty(U, U(1))$, then the coefficients ξ_σ are replaced by

$$\xi_{\sigma'} = \xi_\sigma - \frac{i}{2\pi}\frac{dh}{h} = \xi_\sigma - \frac{i}{2\pi}\, d\log h. \tag{1.5}$$

In particular, a local section $f\sigma$ is covariantly constant if and only if the function f satisfies the differential equation

$$d\log f = -2\pi i\, \xi_\sigma. \tag{1.6}$$

This equation is (locally) soluble if and only if $d\xi_\sigma = 0$.

1.3.2. *Flat connections and the Jacobian.* Formula (1.5) shows that the differentials $d\xi_\sigma$ are independent of the choice of a local unitary basis. They glue together into a global closed 2-form $\omega_a \in C^\infty(X, \Lambda^2 T^*X)$, which will be called the *real curvature form*[5] of the connection a. Its lift $p^*\omega_a = d\alpha_a$ to the principal bundle $P \xrightarrow{p} X$ acts on a pair v_1, v_2 of $U(1)$-invariant vector fields on P by

$$d\alpha(v_1, v_2) = \big(\mathcal{L}ie_{v_1}\alpha - d\big(\alpha(v_1)\big)\big)(v_2) = \mathcal{L}ie_{v_1}\alpha(v_2) = -\alpha([v_1, v_2]).$$

Using exact sequences (1.8) and (1.7) below, we easily see that the cohomology class $[\omega_a] \in H^2(X, \mathbb{R})$ is integer-valued and coincides with the first Chern class of the bundle L:

$$[\omega_a] = c_1(L) \in H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R}).$$

A Hermitian connection a is called *flat* if its curvature is equal to zero: $\omega_a = 0$. Then the bundle L is topologically trivial, that is, L has a global smooth unitary section $X \xhookrightarrow{\sigma} L$. At the same time, the bundle L admits a *covariantly constant* trivialization over each simply connected open set U because equations (1.6) are

[5]The *curvature form* of a connection a is usually defined to be the 2-form $F_a \overset{\text{def}}{=} 2\pi i\, \omega_a$ with values in $\mathfrak{su}(1)$.

solved by $f = ce^{-2\pi i g}$, where g is an arbitrary (but fixed) function on U with $dg = \xi_\sigma$, and $c \in \mathbb{C}$ is an arbitrary constant. Covariantly constant *unitary* sections over U correspond to $|c| = 1$ and form a principal homogeneous space over $U(1)$.

In particular, the lift $\pi^* L \twoheadrightarrow \widetilde{X}$ of a flat bundle L to the universal covering $\widetilde{X} \xrightarrow{\pi} X$ always admits a global covariantly constant unitary section. The action of the fundamental group $\pi_1(X)$ on such sections is determined by a unitary character $\pi_1(X) \xrightarrow{\chi_\alpha} U(1)$, which is called the *holonomy* of the connection α. The value $\chi(\gamma)$ is the rotation of the fibre of L under the covariant parallel translation along the loop γ. The cohomology class $[\chi_\alpha] \in H^1(X, U(1))$ remains unchanged under gauge transformations of the connection α (that is, under the natural action of the group $\mathrm{Map}(S, U(1))$), and the coboundary homomorphism of the long cohomology sequence

$$0 \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathbb{R}) \longrightarrow H^1(X, U(1)) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow \cdots \tag{1.7}$$

associated with the exact triple of constant sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{t \mapsto e^{2\pi i t}} U(1) \longrightarrow 0, \tag{1.8}$$

takes the class $[\chi_a]$ to $0 = c_1(L)$. Thus, gauge classes of flat unitary connections are in one-to-one correspondence with cohomology classes of unitary characters of the fundamental group. They are parametrized by points of the real torus

$$J_S \stackrel{\text{def}}{=} H^1(S, \mathbb{R}) / H^1(S, \mathbb{Z}), \tag{1.9}$$

which is called the *Jacobian* of the manifold X.

1.3.3. *Jacobian bundles.* We denote by $C = S \times \mathbb{C}$ the trivial bundle on S, and by $\mathcal{A}_{\mathrm{FH}}(C)$ the space of flat Hermitian connections on C. The group of all diffeomorphisms $C \xrightarrow{\sim} C$ that are homotopic to the identity and linearly map fibres to fibres is isomorphic to the semidirect product

$$\mathcal{G} \stackrel{\text{def}}{=} \mathrm{Map}(S, U(1)) \rtimes \mathrm{Diff}_0(S),$$

where $\mathrm{Diff}_0(S)$ acts on the normal subgroup $\mathrm{Map}(S, U(1))$ by

$$h\gamma h^{-1} \stackrel{\text{def}}{=} \gamma \circ h^{-1}.$$

The orbit space of the diagonal action of $\mathcal{G}$ on $\mathrm{Map}_{\mathrm{LA}}(S, M) \times \mathcal{A}_{\mathrm{FH}}(C)$,

$$\mathcal{J} \stackrel{\text{def}}{=} \mathrm{Map}_{\mathrm{LA}}(S, M) \times \mathcal{A}_{\mathrm{FH}}(C) / \mathcal{G} \tag{1.10}$$

is called the *Jacobian bundle* over $\mathfrak{L}$. Its fibre over a given smoothly immersed Lagrangian cycle $S \xhookrightarrow{f} M$ is the space of gauge equivalence classes of flat Hermitian connections, that is, the Jacobian (1.9). We note that there is a canonical linear epimorphism which maps the tangent space $T_S\mathfrak{L} = Z^1_{\mathrm{DR}}(S)$ of the base of the Jacobian bundle onto the tangent space $TJ_S = H^1(S, \mathbb{R})$ of the fibre by sending a closed 1-form to its cohomology class. The existence of such a map plays a key (but not always explicit) role in what follows.

§ 2. Kählerian geometry of Lagrangian cycles

2.1. Prequantization data. These are defined by endowing a symplectic manifold (M, ω) with a complex Hermitian line bundle $L \twoheadrightarrow M$ and a Hermitian connection α whose real curvature coincides with ω. (Thus the curvature itself equals $F_\alpha = 2\pi i\, \omega$, see § 1.3.2.) Such data exist if and only if the cohomology class of the symplectic form is integer:

$$[\omega] \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R}).$$

In what follows we always assume that this condition holds and fix some prequantization data (M, ω, L, α). We note that $[\omega] = c_1(L)$, and the 1-form α defines a *contact structure* on the principal $U(1)$-bundle $P \xrightarrow{p} M$ tautologically associated with L, that is, the corresponding top form $\alpha \wedge (d\alpha)^{\wedge n} = \alpha \wedge p^*\omega^{\wedge n}$ is nowhere vanishing [6].

For each $k \in \mathbb{N}$ we denote by $P_k \xrightarrow{p_k} M$ the principal $U(1)$-bundle tautologically associated with $L^{\otimes k}$. The connection α naturally induces a Hermitian connection α_k on P_k with real curvature form $\omega_k = k\omega$. If $k, m \in \mathbb{N}$, then we have commutative diagrams of these principal bundles

$$\begin{tikzcd} P_k \arrow[rr, two heads, "\mu_{m,k}"] \arrow[dr, "p_k"'] & & P_{mk} \arrow[dl, "p_{mk}"] \\ & M & \end{tikzcd} \tag{2.1}$$

where $P_k \xrightarrow{\mu_{m,k}} P_{mk}$ are unbranched covering of degree m determined by taking the m-th power on each fibre: $z \mapsto z^m$. We say that the quadruple $(M, \omega, P_k, \alpha_k)$ determines *prequantization data of level k* on the phase space (M, ω). The Hermitian line bundle $L^{\otimes k} \twoheadrightarrow M$ is called the *prequantization line bundle of level k.*

2.2. Bohr–Sommerfeld cycles. Since the curvature of α_k is proportional to the symplectic form, the restriction $\varphi^* L^{\otimes k}$ of the prequantization bundle to any Lagrangian cycle $S \xrightarrow{\varphi} M$ is a trivial bundle on S with a flat connection, which defines a point of the Jacobian J_S (see (1.9)). Sending each Lagrangian cycle $S \in \mathfrak{L}$ to such a point, we get a section of the Jacobian bundle $\mathfrak{L} \xhookrightarrow{j_k} \mathcal{J}$ (see § 1.3.3). The zero set of this section is called the *space of Bohr–Sommerfeld cycles* of level k (in short: BS_k*-cycles*) and is denoted by $\mathcal{B}_k$. In other words, a Lagrangian cycle $S \xrightarrow{\varphi} M$ is a BS_k-cycle if and only if the holonomy of the connection $\varphi^*\alpha_k$ on the trivial bundle $\varphi^* L^{\otimes k}$ is also trivial, that is, this bundle admits a *covariantly constant* trivialization $\varphi^* L^{\otimes k} \simeq S \times \mathbb{C}$. Since all the powers $\varphi^* L^{\otimes mk}$ become covariantly trivializable together with $\varphi^* L^{\otimes k}$, we have natural inclusions

$$\mathcal{B}_k \xhookrightarrow{\beta_{k,m}} \mathcal{B}_{mk} \quad \forall\, k, m \in \mathbb{Z}. \tag{2.2}$$

[6] For more details on contact structures see [1], Appendix 4.

2.3. Planck cycles and Berry bundles. As mentioned above, the connection 1-form α_k may be regarded as a contact structure on P_k with volume form

$$\alpha_k \wedge (d\alpha_k)^{\wedge n} = k^n \alpha_k \wedge (p^*\omega)^{\wedge n}. \tag{2.3}$$

We denote by $\mathrm{Map}_{\mathrm{LE}}(S, P_k) \overset{\mathrm{def}}{=} \{S \xrightarrow{\widetilde{\varphi}} P_k \mid \widetilde{\varphi}^*(\alpha_k) = 0\}$ the space of smooth *Legendrian* maps. (Their images $\widetilde{\varphi}(S) \subset P$ are *Legendrian* cycles for the contact structure α_k.) The orbit space

$$\mathcal{P}_k \overset{\mathrm{def}}{=} \mathrm{Map}_{\mathrm{LE}}(S, P_k)/\operatorname{Diff}_0(S)$$

is called the *space of Planck cycles* of level k.

We write $\varphi = p_k \circ \widetilde{\varphi} \colon S \longrightarrow M$ for the composite of a Legendrian map $S \xrightarrow{\widetilde{\varphi}} P_k$ and the canonical projection $P_k \xrightarrow{p_k} M$. This composite is a BS_k-cycle since $\varphi^*\omega = d\widetilde{\varphi}^*\alpha_k/k = 0$. Thus, any Planck cycle $S \xrightarrow{\widetilde{\varphi}} P_k \varphi^* P_k$ is just a covariantly constant unitary section of the bundle $\varphi^* L^{\otimes k}$ over S, that is, a BS_k-cycle along with a fixed covariant trivialization of $L^{\otimes k}$. Since such trivializations form a principal homogeneous space over $U(1)$ (see § 1.3.2), the space of Plank cycles of level k is a principal $U(1)$-bundle over BS_k-cycles:

$$\mathcal{P}_k \xrightarrow{\pi_k} \mathcal{B}_k. \tag{2.4}$$

This bundle is called the *Berry principal bundle* of level k. The complex Hermitian line bundle $\mathcal{L}_k \longrightarrow \mathcal{P}_k$ tautologically associated to (2.4) is called the *Berry line bundle*. As follows from (2.1) and (2.2), taking the m-th power on each fibre yields unbranched m-sheet coverings

$$\mathcal{P}_k \xrightarrow{\widetilde{\mu}_{m,k}} \beta^*_{m,k}\mathcal{P}_{mk} \quad \forall\, k, m \in \mathbb{N}, \tag{2.5}$$

that is, $\beta^*_{k,m}\mathcal{L}_{mk} = \mathcal{L}_k^{\otimes m}$.

2.3.1. Proposition. *The tangent space $T_{\widetilde{\varphi}}\mathcal{P}_k$ at a smoothly embedded Planck cycle $S \overset{\widetilde{\varphi}}{\hookrightarrow} P_k$ is naturally identified with the space $C^\infty(S)$ of smooth functions. Moreover, the differentials $d\widetilde{\mu}_{k,m}$ of the coverings (2.5) act on $C^\infty(S)$ by scalar multiplication by m.*

Proof. Arguing as in § 1.1.2, we see that

$$T_{\widetilde{\varphi}}\big(\mathrm{Map}_{\mathrm{LE}}(S, P_k)/\operatorname{Diff}_0(S)\big) = \big\{v \in C^\infty\big(S, N_{P_k/S}\big) \bigm| \widetilde{\varphi}^*(\mathcal{L}ie_v \alpha_k) = 0\big\},$$

that is, a tangent vector to $\mathcal{P}_k$ at $\widetilde{\varphi}$ is a class of local vector fields (modulo fields tangent to $\widetilde{\varphi}(S)$) defined on P_k near $\widetilde{\varphi}(S)$ such that $\mathcal{L}ie_v\alpha_k = 0$. The vertical component of such a field is of the form $f_v \cdot u$, where u is the standard basis invariant vector field on P_k, and $f_v = \alpha_k(v) \in C^\infty(P)$ is the contraction of v and the connection α_k. The correspondence

$$v \longmapsto \widetilde{\varphi}^* f_v \in C^\infty(S) \tag{2.6}$$

yields a well-defined map $T_{\widetilde{\varphi}}\mathcal{P}_k \longrightarrow C^\infty(S)$. (Indeed, α_k annihilates fields tangent to $\widetilde{\varphi}(S)$ since $\widetilde{\varphi}$ is Legendrian.) This map is an isomorphism because the field v is uniquely determined (modulo fields tangent to $\widetilde{\varphi}(S)$) by $\widetilde{\varphi}^* f_v$. Namely, let f_v be any smooth extension of the function $\widetilde{\varphi}^* f_v \in C^\infty(S)$ to a neighbourhood of the image $\widetilde{\varphi}(S) \subset P_k$. Then we can take the field $f_v \cdot u$ for the vertical component of v, and the horizontal component of v is given by the horizontal component of any vector field $\widetilde{v}$ on P_k such that

$$\widetilde{v} \llcorner d\alpha_k = -df_v.$$

(Since $\ker d\alpha_k = \ker p_k^* \omega$ coincides with the vertical direction in TP_k, such a field exists and its horizontal component is well-defined.)[7] Indeed, the horizontal component of v contracted with $d\alpha_k$ and any horizontal vector field w is

$$d\alpha_k\big(v - f_v \cdot u,\, w\big) = d\alpha_k(v, w) = \big[\mathcal{L}ie_v \alpha_k + d(\alpha_k(v))\big](w) = -df_v(w).$$

2.3.2. *The tangent space to $\mathcal{B}_k$.* Suppose that the Planck cycle $S \overset{\widetilde{\varphi}}{\hookrightarrow} P_k$ is given by a covariantly constant section $S \overset{\sigma}{\hookrightarrow} \varphi^* P_k$ over a smoothly embedded BS_k-cycle $S \xrightarrow{\varphi} M$. The previous proof enables us to interpret the differential

$$T_{\widehat{\varphi}}\mathcal{P}_k \xrightarrow{d\pi|_{\widehat{\varphi}}} T_\varphi \mathcal{B}_k$$

of the projection of the Berry bundle $\mathcal{P}_k \xrightarrow{\pi} \mathcal{B}_k$ of level k as the exterior differential on S divided by k:

$$C^\infty(S, \mathbb{R}) \xrightarrow{\frac{1}{k}d} B^1_{\mathrm{DR}}(S).$$

Thus the space $\mathcal{B}_k \subset \mathfrak{L}$ of BS_k-cycles is an *isodrast*, that is, $T_S\mathcal{B}_k \simeq B^1_{\mathrm{DR}}(S)$. In terms of this identification, the differentials $d\beta_{k,m}$ of the embeddings (2.2) act on $B^1_{\mathrm{DR}}(S)$ by scalar multiplication by m.

2.3.3. *Local coordinates on $\mathcal{P}$ near a smooth cycle.* Let $S \xrightarrow{\varphi} M$ be the Bohr–Sommerfeld cycle that underlies the Planck cycle $S \overset{\widetilde{\varphi}}{\hookrightarrow} P_k$. By the Darboux–Weinstein theorem (see § 1.1.4), $S \xrightarrow{\varphi} M$ may be extended to a symplectic map $U \overset{\widehat{\varphi}}{\hookrightarrow} M$, where U is a neighbourhood of the zero section $S \hookrightarrow T^*S$ in the cotangent bundle. We denote by $S_f^{(k)} \subset T^*S$ the graph of the exact 1-form $df/k \in B^1_{\mathrm{DR}}(S)$. Then the map

$$df \longmapsto \widehat{\varphi}\big(S_f^{(k)}\big) \tag{2.7}$$

defines a chart of $\mathcal{B}_k$ in a neighbourhood of φ, thus identifying this neighbourhood with a neighbourhood of zero in the space $B^1_{\mathrm{DR}}(S)$ of exact forms (see § 1.1.4). The proof of Proposition 2.3.1 shows that the map (2.7) may be extended to an identification of some neighbourhood of zero in $C^\infty(S, \mathbb{R}) = T_{\widetilde{\varphi}}\mathcal{P}_k$ with an open

[7] *Translators' Note.* From here till the end of the paragraph, the original text is changed in the course of translation.

neighbourhood of $\widetilde{\varphi}$ in the space $\mathcal{P}_k$ of Planck cycles.[8] Namely, the Hermitian line bundle $\widehat{\varphi}^* L^{\otimes k}$ admits an unitary section $U \overset{\sigma}{\hookrightarrow} \widehat{\varphi}^* L^{\otimes k}$ (perhaps, over a smaller U) such that σ coincides with the initial Planck cycle $\widetilde{\varphi}$ over the zero section $S \subset U$, and the covariant differentiation with respect to the connection a_k is given in the base σ by the canonical action form η on T^*S, that is, $\nabla_\alpha \sigma = 2\pi i \, \eta \otimes \sigma$ over U. We now define the *Darboux–Weinstein coordinatization* of $\mathcal{P}_k$ by a neighbourhood of zero in $C^\infty(S, \mathbb{R})$ by sending each sufficiently small function f to the graph $S_f^{(k)} \subset U$ equipped with the global unitary horizontal section

$$e^{-if} \cdot \sigma|_{S_f^{(k)}}. \tag{2.8}$$

2.3.4. *The complex Darboux–Weinstein coordinates on $T\mathcal{P}_k$.* We have seen in § 1.2.3 that every tangent bundle has a canonical integrable complex structure. Consider the tangent bundle $T\mathcal{P}_k$ to the space of Planck cycles. The corresponding complex coordinates identify a neighbourhood of a point $(f, \widetilde{\varphi})$ (where $S \overset{\widetilde{\varphi}}{\hookrightarrow} P_k$ is a Planck cycle, and $f \in C^\infty(S) = T_{\widetilde{\varphi}}\mathcal{P}_k$ is a tangent vector) with a neighbourhood of zero in the space of smooth complex-valued functions $\psi = \psi_1 + i\psi_2 \in C^\infty(S, \mathbb{C})$. According to the notation of § 1.2.3, we assume that the real parts $\psi_1 \in C^\infty(S, \mathbb{R}) = T_{\widetilde{\varphi}}\mathcal{P}_k$ give linear coordinates on fibres of the tangent bundle, and the imaginary parts $\psi_2 \in C^\infty(S, \mathbb{R})$ are the Darboux–Weinstein coordinates on $\mathcal{P}_k$ near $\widetilde{\varphi}$. The functions $\psi_1 + i\psi_2$ are called the *complex Darboux–Weinstein coordinates* on $T\mathcal{P}_k$.

2.4. Half-weighted Planck cycles. We want to combine the canonical complex structure on $T\mathcal{P}$ and the canonical symplectic structure on $T^*\mathcal{P}$ in order to get an integrable Kähler structure on an appropriate natural bundle over $\mathcal{P}$. This bundle must admit fiberwise identifications with both tangent and cotangent bundles. These properties are possessed by the bundle of half-weights.

2.4.1. *Half-weights.* We denote by $\mathfrak{M} = \mathfrak{M}(S)$ the equivalence class of measures on S containing the measure μ_0 which is induced by the smooth manifold structure (compare with [21], § 2–6). This class contains a subclass $\mathfrak{M}_0 \subset \mathfrak{M}$ of *smooth* measures, obtained by multiplying μ_0 by a positive smooth function. We recall that the equivalence $\lambda \sim \mu$ of measures λ, μ on S means that $\lambda(B) = 0 \Leftrightarrow \mu(B) = 0$ for every Borel subset $B \subset S$. By the Radon–Nikodym theorem, this is equivalent to the existence of positive integrable real functions

$$\varrho_{\mu\lambda} = \frac{d\mu}{d\lambda} \in L^1(S, \lambda), \qquad \varrho_{\lambda\mu} = \frac{d\lambda}{d\mu} \in L^1(S, \mu)$$

such that

$$\int_B d\mu = \int_B \varrho_{\mu\lambda} \, d\lambda, \qquad \int_B d\lambda = \int_B \varrho_{\lambda\mu} \, d\mu$$

for every Borel subset $B \subset S$. These functions are called the *Radon–Nikodym derivatives.* The equivalence of λ and μ is usually expressed by writing $\lambda = \varrho_{\lambda\mu}\mu$,

[8] *Translators' Note.* From here till the end of the paragraph, the original text is changed in the course of translation.

where, of course, $\varrho_{\mu\lambda}\varrho_{\lambda\zeta} = \varrho_{\mu\zeta}$ and $\varrho_{\mu\lambda}\varrho_{\lambda\mu} = 1$ almost everywhere with respect to any measure in $\mathfrak{M}$.

We define a *half-weight* (in other terminology, a *σ-function*) $\vartheta = f\sqrt{\mu}$ on S as a class of equivalent pairs (f,μ) with $\mu \in \mathfrak{M}$, $f \in L^2_{\mathbb{R}}(S,\mu)$, where the equivalence relation $(f_1,\mu_1) \sim (f_2,\mu_2)$ means that $f_1\sqrt{\varrho_{\mu_1\lambda}} = f_2\sqrt{\varrho_{\mu_2\lambda}}$ almost everywhere with respect to any measure $\lambda \in \mathfrak{M}$. A half-weight $f\sqrt{\mu}$ is called *smooth* if $f \in C^\infty(S)$ and $\mu \in \mathfrak{M}_0$. One directly verifies (see for example [21], § 2–6 or [17], § 4.1) that half-weights on S form a complete normed vector space $\mathfrak{H}(S)$ over $\mathbb{R}$. The norm on $\mathfrak{H}(S)$ is induced by the scalar product

$$\left(f_1\sqrt{\mu_1}, f_2\sqrt{\mu_2}\right) \stackrel{\text{def}}{=} \int_S f_1 f_2 \sqrt{\varrho_{\mu_1\lambda}\varrho_{\mu_2\lambda}}\, d\lambda, \tag{2.9}$$

and linear combinations of half-weights are defined by

$$t_1 f_1\sqrt{\mu_1} + t_2 f_2\sqrt{\mu_2} \stackrel{\text{def}}{=} \left(t_1 f_1\sqrt{\varrho_{\mu_1\lambda}} + t_2 f_2\sqrt{\varrho_{\mu_2\lambda}}\right)\sqrt{\lambda}.$$

(In both cases, λ is an arbitrary measure in $\mathfrak{M}$.) Smooth half-weights form a dense subspace $\mathring{\mathfrak{H}}(S) \subset \mathfrak{H}(S)$.

The group $\operatorname{Diff}_0(S)$ acts on $\mathfrak{H}(S)$ by

$$(f,\mu) \stackrel{h}{\longmapsto} \left(f\circ h^{-1}, (h^{-1})^*\mu\right),$$

which maps $\mathring{\mathfrak{H}}$ to itself. The orbit spaces of the diagonal action of $\operatorname{Diff}_0(S)$ on the products

$$\mathfrak{L}^{\text{hw}} \stackrel{\text{def}}{=} \operatorname{Map}_{\text{LA}}(S,M)\times\mathfrak{H}(S)/\operatorname{Diff}_0(S), \qquad \mathcal{P}_k^{\text{hw}} \stackrel{\text{def}}{=} \operatorname{Map}_{\text{LE}}(S,P_k)\times\mathfrak{H}(S)/\operatorname{Diff}_0(S)$$

are called the spaces of *half-weighted* Lagrangian and Planck cycles respectively. Replacing $\mathfrak{H}$ by $\mathring{\mathfrak{H}}$ in these definitions, we get dense subspaces $\mathring{\mathfrak{L}}_k^{\text{hw}} \subset \mathfrak{L}_k^{\text{hw}}$ and $\mathring{\mathcal{P}}_k^{\text{hw}} \subset \mathcal{P}_k^{\text{hw}}$ of *smoothly half-weighted* cycles. The forgetting maps $\mathfrak{L}^{\text{hw}} \xrightarrow{F} \mathfrak{L}$ and $\mathcal{P}_k^{\text{hw}} \xrightarrow{\widetilde{F}} \mathcal{P}_k$ form the following commutative diagram of vector bundles:

$$\begin{tikzcd}
\mathcal{P}_k^{\text{hw}} \arrow[r, two heads] \arrow[d, "\widetilde{F} = \pi^* F|_{\mathcal{B}_k^{\text{hw}}}"'] & \mathcal{B}_k^{\text{hw}} \arrow[r, hook] \arrow[d, "F|_{\mathcal{B}_k^{\text{hw}}}"] & \mathfrak{L}^{\text{hw}} \arrow[d, "F"] \\
\mathcal{P} \arrow[r, two heads, "\pi"] & \mathcal{B}_k \arrow[r, hook] & \mathfrak{L}
\end{tikzcd} \tag{2.10}$$

2.4.2. *The complex structure on* $\mathring{\mathcal{P}}_k^{\text{hw}}$. Each half-weight $f\sqrt{\mu}$ defines an embedding of the tangent space $T_{\widetilde{\varphi}(S)}\mathcal{P}_k = C^\infty(S)$ of the Planck cycle space to the tangent space $T_{f\sqrt{\mu}}\mathfrak{H} \simeq \mathfrak{H}$ of the space of half-weights by

$$f_1 \longmapsto f f_1\sqrt{\mu}. \tag{2.11}$$

All these maps together yield an embedding of bundles $T\mathcal{P}_k \hookrightarrow \mathcal{P}_k^{\mathrm{hw}}$. Its image is dense and coincides with the subbundle of smooth half-weights: $T\mathcal{P}_k \xrightarrow{\sim} \overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}}$. This enables us to transfer the canonical complex structure of $T\mathcal{P}_k$ (see § 2.3.4) to $\overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}}$. Consider a point $(\widetilde{\varphi}, \vartheta) \in \overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}}$, where $S \xhookrightarrow{\widetilde{\varphi}} P$ is a smooth Planck cycle lying over a BS$_k$-cycle $S \xhookrightarrow{\varphi} M$ (as in § 2.3.3), and $\vartheta = f\sqrt{\mu}$ is a half-weight. Then our complex structure identifies a complex coordinate neighbourhood of this point with a neighbourhood of zero in the space $C^\infty(S, \mathbb{C})$ of smooth complex-valued functions on S. A function $\psi = \psi_1 + i\psi_2 \in C^\infty(S, \mathbb{C})$ corresponds to a half-weighted cycle which is constructed as follows. The embedding $S \xrightarrow{\varphi} M$ of our BS$_k$-cycle is extended by the Darboux–Weinstein theorem to an embedding $U \xrightarrow{\widehat{\varphi}} M$ of a neighbourhood $U \subset T^*S$ of the zero section, and the covariantly constant section $S \xhookrightarrow{\sigma} \varphi^* Pk$ is extended to a covariantly constant section $U \xhookrightarrow{\hat{\sigma}} P_k^U$ of the bundle $P_k^U = \widehat{\varphi}^* P_k$ over U (as in § 2.3.3). The graph of the exact 1-form $d\psi_2/k$ is a BS$_k$-cycle $S_{\psi_2}^{(k)} \subset U$. We consider the covariantly constant section

$$\sigma_{\psi_2} \overset{\mathrm{def}}{=} e^{i\psi_2} \hat{\sigma}|_{S_{\psi_2}^{(k)}}$$

of the bundle P_k^U over $S_{\psi_2}^{(k)}$ and take the image of this section in P_k for the Planck cycle that corresponds to the function $\psi = \psi_1 + i\psi_2$. The real part of ψ defines a half-weight $\psi_1\vartheta = f\psi_1\sqrt{\mu}$ on this cycle.

These complex coordinates are called the *complex Darboux–Weinstein coordinates.* They generate the following decomposition of the tangent space of $\overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}}$ into the real and imaginary components:

$$T_{(\widetilde{\varphi},\vartheta)} \overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}} \simeq T_\vartheta \overset{\circ}{\mathfrak{H}}(S) \oplus T_{\widetilde{\varphi}} \overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}} \simeq C^\infty(S) \oplus C^\infty(S). \tag{2.12}$$

2.4.3. *The symplectic and Riemannian structures on* $\mathcal{P}_k^{\mathrm{hw}}$. We regard the product of two half-weights $\vartheta_1 = f_1\sqrt{\mu_1}$, $\vartheta_2 = f_2\sqrt{\mu_2}$ as a measure multiplied by a function (compare with [21], § 2–6):

$$\vartheta_1\vartheta_2 \overset{\mathrm{def}}{=} f_1 f_2 \sqrt{\varrho_{\mu_1\lambda}\varrho_{\mu_2\lambda}}\,\lambda$$

(with an arbitrary $\lambda \in \mathfrak{M}$), and we define the *integration with respect to* $\vartheta_1\vartheta_2$ by

$$\int_S f\, d\vartheta_1\vartheta_2 \overset{\mathrm{def}}{=} \int_S f f_1 f_2 \sqrt{\varrho_{\mu_1\lambda}\varrho_{\mu_2\lambda}}\, d\lambda, \tag{2.13}$$

which is obviously independent of the choice of $\lambda \in \mathfrak{M}$. In particular, the square of a half-weight ϑ may be regarded as a linear functional

$$\varkappa(\vartheta) \colon f \longmapsto \int_S f\, d\vartheta^2 \tag{2.14}$$

on the tangent space $T_{\widetilde{\varphi}(S)}\mathcal{P}_k \simeq C^\infty(S)$. In other words, we have a fiberwise quadratic covering of vector bundles over $\mathcal{P}_k$:

$$\begin{tikzcd} \mathcal{P}_k^{\mathrm{hw}} \arrow[rr, two heads, "\varkappa"] \arrow[dr] & & T^*\mathcal{P}_k \arrow[dl] \\ & \mathcal{P}_k & \end{tikzcd}$$

which is ramified along the zero section. The differential of this covering at a half-weight ϑ maps a half-weight ϑ_1 to the linear functional

$$d\varkappa_\vartheta(\vartheta_1)\colon f \longmapsto 2\int_S f\,d\vartheta\vartheta_1$$

on $C^\infty(S) = T_{\widetilde{\varphi}(S)}\mathcal{P}_k$. Hence $d\varkappa$ identifies $T^*\mathcal{P}_k$ with $\mathcal{P}_k^{\mathrm{hw}}$.

Using this identification, we can lift the canonical action 1-form η (see § 1.1.1) on the cotangent bundle $T^*\mathcal{P}_k$ to a 1-form $\mathrm{H} \overset{\mathrm{def}}{=} \varkappa^*\eta$ on the bundle of half-weights. In terms of the decomposition (2.12), the form H acts on a tangent vector $v = (\vartheta_1, f_1) \in T_{(S,\vartheta_0)}\mathcal{P}^{\mathrm{hw}} \simeq \mathfrak{H}(S) \oplus C^\infty(S)$ by

$$\mathrm{H}(v) = 2\int_S f_1\,d\vartheta_0\vartheta_1, \tag{2.15}$$

and its differential $\Omega = d\mathrm{H}$ acts by

$$\Omega\big((\vartheta_1, f_1), (\vartheta_2, f_2)\big) = \int_S f_2\,d\vartheta_0\vartheta_1 - \int_S f_1\,d\vartheta_0\vartheta_2. \tag{2.16}$$

This is a direct generalization of the corresponding finite-dimensional formulae[9] given in (1.3).

In terms of the complex Darboux–Weinstein coordinates (see § 2.4.2), real tangent vectors $(\psi_1, 0)$ are symplectically paired with imaginary tangent vectors $(0, \psi_2)$ by the formula

$$\Omega\big((\psi_1, 0), (0, \psi_2)\big) = \int_S \psi_1\psi_2\,d\vartheta_0^2$$

whose right-hand side is the L^2-product of ψ_1 and ψ_2 with respect to the measure ϑ_0^2. In particular, Ω is compatible (in the sense of § 1.2) with the complex structure on $\overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}}$ described above. Hence Ω corresponds to an integrable Kähler structure on $\overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}}$, whose third term is the Riemannian metric G on $T_{(S,\vartheta_0)}\mathcal{P}^{\mathrm{hw}} = C^\infty(S) \oplus \mathfrak{H}(S)$ given by

$$G\big((\vartheta_1, f_1), (\vartheta_2, f_2)\big) = \int_S f_1\,d\vartheta_1\vartheta_0 + \int_S f_2\,d\vartheta_2\vartheta_0. \tag{2.17}$$

Thus we get the following theorem.

[9] We note that this infinite-dimensional analogue of the canonical formalism for the simplest case $S = S^1$ is widely used in the analysis of the Korteweg–de Vries equation (see [4] and our § 4). It also arises in the Hamiltonian analysis of vortex motions of an incompressible fluid.

2.4.4. Theorem. *The space $\mathcal{P}_k^{\mathrm{hw}}$ of smoothly half-weighted Planck cycles of level k carries the natural structure of an infinite-dimensional Kähler manifold. The symplectic and Riemannian forms associated with this Kähler structure are given by* (2.16) *and* (2.17). *The complex-analytic chart in a neighbourhood of a smoothly immersed smoothly half-weighted cycle S is given by a neighbourhood of zero in the space of smooth complex-valued functions on S.*

2.5. Half-weighted BS_k-cycles of a fixed volume. The group $U(1)$ naturally acts on $\mathcal{P}_k$ by changes of the covariantly constant unitary trivializing section of L over each BS_k-cycle. This action canonically extends to the tangent bundle $T\mathcal{P}_k$ and, therefore, to the space $\mathcal{P}_k^{\mathrm{hw}} \subset T\mathcal{P}_k$ of smoothly half-weighted cycles. The forms (2.16), (2.17) are obviously preserved by this action (since the identification $T_S\mathcal{P}_k \simeq C^\infty(S)$ does not depend on the choice of trivialization of L over S, see § 2.3.1). There is a natural $U(1)$-invariant *volume function* on $\mathcal{P}_k^{\mathrm{hw}}$:

$$\mathcal{P}_k^{\mathrm{hw}} \xrightarrow{\mathrm{vol}} \mathbb{R} : (S, \vartheta) \longmapsto \int_S \vartheta^2 .$$

Its level sets are the spaces $\mathcal{P}_{k,t}^{\mathrm{hw}}$ of half-weighted Planck cycles of volume t. They are $U(1)$-invariant, and the corresponding orbit spaces

$$\mathcal{B}_{k,t}^{\mathrm{hw}} \stackrel{\mathrm{def}}{=} \mathcal{P}_{k,t}^{\mathrm{hw}} / U(1)$$

are called the spaces of *half-weighted* BS_k*-cycles of volume* t.

2.5.1. Theorem. *The spaces $\mathcal{B}_{k,t}^{\mathrm{hw}}$ are infinite-dimensional Kähler manifolds. Their Kähler structure is induced from $\mathcal{P}_k^{\mathrm{hw}}$. In particular, its symplectic and Riemannian forms are still given by* (2.16) *and* (2.17).

Proof. In terms of decomposition (2.12), the space $T_{(\widetilde{\varphi}(S),\vartheta_0)}\mathcal{P}_{k,t}^{\mathrm{hw}}$ consists of all pairs $v = (f, \vartheta)$ with $\int_S d\vartheta\vartheta_0 = 0$. (Indeed, this integral just determines the infinitesimal change of volume along the tangent vector v in the Darboux–Weinstein coordinates.) Taking the quotient by $U(1)$ transforms such a pair (f, ϑ) to the pair $(df, \vartheta) \in T_{(S,\vartheta_0)}\mathcal{B}_{k,t}^{\mathrm{hw}}$. The forms (2.16), (2.17) remain well-defined since the ambiguity in the lift of df to f is removed by the relations

$$\int_S d\vartheta_1\vartheta_0 = \int_S d\vartheta_2\vartheta_0 = 0.$$

2.5.2. *Remark.* The previous construction actually yields the *symplectic quotient* $\mathcal{P}_k^{\mathrm{hw}}//U(1)$. Indeed, since the group $U(1)$ is commutative, the invariant function vol is a *moment map* for the action of $U(1)$ on $\mathcal{P}_k^{\mathrm{hw}}$. Factorizing its level sets, we follow the standard procedure that constructs the symplectic quotient. (A detailed discussion of this construction for the general case is given in [6] and [8].)

2.6. Prequantization data on the spaces $\mathcal{B}_{k,t}^{\mathrm{hw}}$. The Kähler manifolds $\mathcal{B}_{k,t}^{\mathrm{hw}}$ have canonical prequantization data. Indeed, restricting the bundle $\mathcal{P}_k^{\mathrm{hw}}$ =

$\widetilde{F}^*\mathcal{P} \twoheadrightarrow \mathcal{B}_k^{\mathrm{hw}}$ (the left horizontal arrow in the upper row of (2.10)) to $\mathcal{B}_{k,t}^{\mathrm{hw}}$, we get a principal $U(1)$-bundle

$$\mathcal{P}_{k,t}^{\mathrm{hw}} \xrightarrow{\pi_t^{\mathrm{hw}}} \mathcal{B}_{k,t}^{\mathrm{hw}} \tag{2.18}$$

and a Hermitian connection $\mathfrak{a} \stackrel{\mathrm{def}}{=} \mathrm{H}|_{\mathcal{P}_{k,t}^{\mathrm{hw}}}$, which is the restriction of the action form (2.15) to the space of Planck cycles of volume t. Its real curvature 2-form $\omega_{\mathfrak{a}} = \Omega$ is tautologically compatible with the symplectic structure of $\mathcal{B}_k^{\mathrm{hw}}$. Moreover, these prequantization data possess a Kählerian polarization and are amenable to the *holomorphic quantization*, which is discussed in the next section.

2.6.1. *Remark on weightings.* The prequantization data $(\mathcal{B}_{k,t}^{\mathrm{hw}}, \Omega, \mathcal{P}_{k,t}^{\mathrm{hw}}, \mathfrak{a})$ enable us to construct prequantization data on the (non-half-weighted) Berry bundle $\mathcal{P}_{k,t} \xrightarrow{\pi_t} \mathcal{B}_{k,t}$. This bundle has no *a priori* distinguished Hermitian connection, and we may fix such a connection by lifting the universal connection $\mathfrak{a}$ on the bundle $\mathcal{P}_{k,t}^{\mathrm{hw}}$ along some section $\mathcal{P}_k \hookrightarrow \mathcal{P}_k^{\mathrm{hw}}$. A choice of this section is called a *half-weighting* of Planck cycles. The half-weighting of Planck cycles may in turn be lifted from a half-weighting of BS_k-cycles, which is by definition a section

$$\mathcal{B}_k \hookrightarrow \mathcal{B}_k^{\mathrm{hw}}. \tag{2.19}$$

For example, let the manifold M be pseudo-Kählerian. Restricting its Riemannian metric g to Lagrangian cycles, we endow each cycle S with the canonical volume form $\mu_{S,g}$ induced by the metric. The map that sends a cycle S to the half-weight $\vartheta_{S,g} = 1 \cdot \sqrt{\mu_{S,g}}$ is called the *Kähler half-weighting.*

§ 3. Holomorphic quantization

3.1. Kähler polarization. To quantize prequantization data $(M, \omega, L^{\otimes k}, \alpha_k)$ (see § 2.1), we must *polarize* them (see [24]). By definition, a *complex* (or *Kähler*) polarization is an (integrable) complex structure I on M that makes M_I a complex Kähler manifold with Kähler form ω.

We fix such a polarization. Then the curvature form of the connection α_k on $L^{\otimes k}$ has Hodge type $(1,1)$, whence α_k endows $L^{\otimes k}$ with the structure of a *holomorphic* vector bundle over M_I (see [3], § 3). The space of global holomorphic sections of this bundle,

$$Q_k(M_I) = H^0(M_I, L^{\otimes k}) \tag{3.1}$$

is called the *wave function space of level k* (see the survey [5]). In what follows we always assume for simplicity that M is compact and, in particular, all spaces (3.1) are finite-dimensional. However, we do this mainly because of the algebro-geometric tradition. All constructions that follow are equally applicable to the non-compact case, in particular, to $M = T^*X$, with infinite-dimensional spaces (3.1).

3.1.1. *Hardy spaces, Szegö projectors and Toeplitz operators.* We denote by $P_{-1} \xrightarrow{p_{-1}} M$ the principal $U(1)$-bundle tautologically associated with L^*, the dual line bundle to the Hermitian prequantization bundle L on M_I. Its total space P_{-1} is the boundary of the unit disk bundle around the zero section of L^*:

$$P_{-1} = \partial D_{-1}, \qquad D_{-1} \subset L^* \twoheadrightarrow M_I. \tag{3.2}$$

The total space D_{-1} is a strictly pseudoconvex complex domain. (See [2], Ch. IX. We stress again that the complex structure on D strongly depends on the choice of the Kähler complex structure I on M.) As usual, we denote by $L^2(P_{-1})$ the Hilbert space obtained by completing the space $C^\infty(P_{-1}, \mathbb{C})$ of smooth complex-valued functions on P_{-1} in the norm associated with the Hermitian product

$$(f,g) \stackrel{\text{def}}{=} \int_{P_{-1}} f\bar{g}\alpha_{-1} \wedge (d\alpha_{-1})^{\wedge n}, \tag{3.3}$$

where α_{-1} is the connection on L^* dual to $\alpha = \alpha_1$, and $\varrho_{-1} = \alpha_{-1} \wedge (d\alpha_{-1})^{\wedge n}$ is the volume form of type (2.3) induced by the contact structure α_{-1} on P_{-1}.

The closed subspace $\mathcal{H} \subset L^2(P_{-1})$ of all functions admitting a holomorphic continuation to D_{-1} is called the *Hardy space*, and the orthogonal projector

$$\Pi\colon L^2(P_{-1}) \longrightarrow\!\!\!\!\rightarrow \mathcal{H} \tag{3.4}$$

onto this space is called the *Szegö projector*. The Hardy space may be decomposed into a topological direct sum of weight subspaces for the natural action of the structure group $U(1)$ of the bundle $L^* \supset D_{-1} \supset P_{-1}$:

$$\mathcal{H} = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_k.$$

The k-th weight subspace $\mathcal{H}_k$ of this decomposition consists of all functions on L^* that are fiberwise homogeneous polynomials of degree k. In other words, there is a natural identification $\mathcal{H}_k = H^0(M, L^{*\otimes k})$, whence the weight decomposition takes the form

$$\mathcal{H} = \bigoplus_k H^0(L^{\otimes k}). \tag{3.5}$$

Moreover, the Hermitian structure on $\mathcal{H}$ described above is compatible with the natural Hermitian structures on $H^0(M, L^{\otimes k})$ that are defined by integrating the scalar product of sections of $L^{\otimes k}$ with respect to the Liouville form on M:

$$(f,g)_k \stackrel{\text{def}}{=} \int_M (f,g)_{L^{\otimes k}} \omega^{\wedge n}. \tag{3.6}$$

The orthogonal projectors $L^2(P_{-1}) \xrightarrow{\Pi_k} H^0(L^{\otimes k})$ are called the *Szegö projectors of level k*. They are used to define a series of *Toeplitz operators* $H^0(L^{\otimes k}) \xrightarrow{Q_f^{(k)}} H^0(L^{\otimes k})$ for every smooth function f on M. These operators act on a section $\psi \in H^0(L^{\otimes k})$ as follows: regard ψ as an element of the space $\mathcal{H} \subset L^2(P_{-1})$, multiply it by $p_{-1}^* f$ in this space, and project the result onto $H^0(L^{\otimes k})$ by Π_k. Thus,

$$Q_f^{(k)}(\psi) = \Pi_k(p_{-1}^* f\psi) \in H^0(L^{\otimes k}). \tag{3.7}$$

We recall that here $P_{-1} \xrightarrow{p_{-1}} M$ is the principal $U(1)$-bundle tautologically associated with L^*. Toeplitz operators on $\mathcal{H}$ form a Poisson algebra, which is a *quantum deformation* of the Poisson algebra of smooth functions on the symplectic manifold M.

3.1.2. *State spaces.* In the standard way, the holomorphic line bundle $L^{\otimes k}$ defines a rational map $M_I \longrightarrow \mathbb{P}\big(H^0(L^{\otimes k})^*\big)$ that sends a point $x \in M_I$ to the hyperplane

$$\operatorname{Ann}(x) = \big\{\psi \in H^0(L^{\otimes k}) \bigm| \psi(x) = 0\big\}.$$

Consider the long cohomology sequence

$$\cdots \longrightarrow H^0(\mathcal{I}_x \otimes L^{\otimes k}) \longrightarrow H^0(L^{\otimes k}) \longrightarrow \mathbb{C} \longrightarrow H^1(\mathcal{I}_x \otimes L^{\otimes k}) \longrightarrow \cdots$$

associated with the following exact triple of sheaves on M:

$$0 \longrightarrow \mathcal{I}_x \otimes L^{\otimes k} \longrightarrow L^{\otimes k} \xrightarrow{\text{res}} \mathcal{O}_M|_x \longrightarrow 0.$$

(Here $\mathcal{I}_x$ is the ideal sheaf of the point x, and res is the restriction epimorphism.) It shows that $H^0(\mathcal{I}_x \otimes L^{\otimes k})$ is a hyperplane if $H^1(\mathcal{I}_x \otimes L^{\otimes k}) = 0$. By Serre's theorems we have $H^1(\mathcal{I}_x \otimes L^{\otimes k}) = 0$ for $k \gg 0$, whence the maps

$$\varphi_k \colon M_I \xrightarrow{x \mapsto \operatorname{Ann}(x)} \mathbb{P}\big(H^0(L^{\otimes k})^*\big) \tag{3.8}$$

are holomorphic embeddings for $k \gg 0$. Combining them with the isomorphisms

$$H^0(L^{\otimes k}) \xrightarrow{\sim} \overline{H^0(L^{\otimes k})}^*, \tag{3.9}$$

which are defined by the Hermitian structure (3.6), we get antiholomorphic embeddings of M_I into the projectivizations of the wave function spaces:

$$\overline{\varphi}_k \colon M_I \hookrightarrow \mathbb{P}H^0(L^{\otimes k}). \tag{3.10}$$

The spaces $\mathbb{P}H^0(L^{\otimes k})$ are called the *state spaces*, and states lying in the image of $\overline{\varphi}_k$ are called *coherent states.*

3.1.3. *Important warning.* Although the principal prequantization bundle $P_k \xrightarrow{p_k} M$ coincides with the pull back under (3.10) of the Hopf circle bundle of degree k over $\mathbb{P}H^0(M_I, L^{\otimes k})$, the connection 1-form α_k on P_k need not in general coincide with the pull back of the invariant Fubini–Study 1-form on the Hopf bundle.

3.2. Restriction of holomorphic sections onto Planck cycles. Let $S \overset{\widetilde{\varphi}}{\hookrightarrow} P_k$ be a Planck cycle, that is, a BS_k-cycle $S \overset{\varphi}{\hookrightarrow} M$ together with a covariantly constant unitary section $S \overset{\sigma}{\hookrightarrow} \varphi^* L^{\otimes k}$. Then the restriction of a holomorphic section $s \in H^0(M_I, L^{\otimes k})$ onto $S \overset{\widetilde{\varphi}}{\hookrightarrow} P_k$ is represented by a smooth complex-valued function $\psi_{\widetilde{\varphi}} \in C^\infty(S, \mathbb{C})$ such that $\varphi^* s = \psi_{\widetilde{\varphi}} \cdot \sigma$. Moreover, the linear map

$$H^0(M, L^{\otimes k}) \overset{s \mapsto \psi_{\widetilde{\varphi}}}{\hookrightarrow} C^\infty(S, \mathbb{C}) \tag{3.11}$$

is *injective* by Corollary 1.2.2 and $\mathbb{C}$-*linear* with respect to the standard complex structures on these spaces. We recall that the complex vector space $C^\infty(S, \mathbb{C})$ is the fibre $T_{(\widetilde{\varphi}, \vartheta)} \overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}}$ of the *holomorphic* tangent bundle to the infinite-dimensional

Kähler manifold $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$ of half-weighted Planck cycles, and the complex structure on $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$ corresponds to the standard complex structure on $C^\infty(S,\mathbb{C})$ (see § 2.4.2). Thus we have the following embedding of coherent sheaves over $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$:

$$0 \longrightarrow \mathcal{H}_k \otimes \mathcal{O}_{\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k} \longrightarrow \mathcal{T}_{\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k}. \tag{3.12}$$

Here the first term is the sheaf of holomorphic sections of the trivial vector bundle with fibre $\mathcal{H}_k = H^0(M_I, L^{\otimes k})$, and the second term is the sheaf of holomorphic vector fields on $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$.

3.2.1. *The maps* $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}} \xrightarrow{\gamma_k} H^0(M, L^{\otimes k})^*$. For each half-weighted Planck cycle $(\widetilde{\varphi}, \vartheta)$, we consider a complex-linear form on $H^0(M, L^{\otimes k})$ whose value on a holomorphic section $s \in H^0(M, L^{\otimes k})$ equals

$$\int_S \psi_{\widetilde{\varphi}}\, d\vartheta^2 = \int_S \psi_{\widetilde{\varphi}} f^2\, d\mu \in \mathbb{C}.$$

This yields a map

$$\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}} \xrightarrow{\bar{\gamma}_k} H^0(M, L^{\otimes k})^*, \tag{3.13}$$

which is smooth in a neighbourhood of each smooth Planck cycle. In the complex Darboux–Weinstein coordinates on $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$ near $(\widetilde{\varphi}, \vartheta)$, the action of the linear functional $\bar{\gamma}_k(\psi_1 + i\psi_2)$ on $H^0(M, L^{\otimes k})$ is given by

$$s \longmapsto \int_S \psi_{\widetilde{\varphi}} \psi_1^2 e^{-i\psi_2}\, d\vartheta^2$$

since the section s is holomorphic (that is, $\alpha_k(s) = 0$), and P_k is trivialized over $S^{(k)}_{\psi_2}$ (that is, over the cycle with coordinate $\psi_1 + i\psi_2$, see § 2.4.2) by the section $e^{i\psi_2}\hat{\sigma}$. The differential $d\bar{\gamma}_k$ maps $\psi_1 + i\psi_2$ to the functional on $H^0(M, L^{\otimes k})$ acting by

$$s \longmapsto \int_S \psi_{\widetilde{\varphi}} (2\psi_1 - i\psi_2)\, d\vartheta^2. \tag{3.14}$$

Here the right-hand side is the standard Hermitian scalar product on $L^2(S,\mathbb{C})$ with respect to the measure ϑ^2. The space $H^0(M, L^{\otimes k}) \subset L^2(S,\mathbb{C})$ is finite-dimensional, whence any $\mathbb{C}$-linear functional on this space has the form (3.14) for an appropriate *smooth* function $2\psi_1 - i\psi_2$. (Indeed, a linear operator with values in a finite-dimensional space is epimorphic if its image is dense.) Thus the differential $d\bar{\gamma}_k$ is surjective.

Consider the composite of the map (3.14) and the isomorphism (3.9) induced by the Hermitian structure. We denote it by

$$\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k \xrightarrow{\gamma_k} H^0(M, L^{\otimes k}). \tag{3.15}$$

This map sends a smoothly half-weighted Planck cycle $(\widetilde{\varphi}, \vartheta)$ to a holomorphic section $\gamma_k(\widetilde{\varphi}, \vartheta) \in H^0(M, L^{\otimes k})$ which satisfies the following equation for any $s \in H^0(M, L^{\otimes k})$:

$$\int_M \big(s, \gamma_k(\widetilde{\varphi}, \vartheta)\big)_{L^{\otimes k}}\, d\varrho = \int_S \psi_{\widetilde{\varphi}}\, d\vartheta^2,$$

and its differential $T_{(\widetilde{\varphi},\vartheta)} \xrightarrow{d\gamma_k} H^0(M, L^{\otimes k})$ is injective and sends a tangent vector $\psi = \psi_1 + i\psi_2 \in C^\infty(S, \mathbb{C}) = T_{(\widetilde{\varphi},\vartheta)} \overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$ to the holomorphic section $s_\psi \in H^0(M, L^{\otimes k})$ defined by

$$\int_M (s, s_\psi)_{L^{\otimes k}}\, d\varrho = \int_S \psi_{\widetilde{\varphi}} \overline{\psi'}\, d\vartheta^2,$$

where $\psi' = 2\psi_1 + i\psi_2$. Hence this differential differs only by a constant $\mathbb{R}$-linear factor from the $\mathbb{C}$-linear embedding (3.11), which was defined by restricting holomorphic sections to the Planck cycle. That is, although the maps (3.15) are not holomorphic, they have a *constant Kähler angle.*

3.2.2. *Symplectic reduction of the maps γ_k.* The map (3.15) is equivariant with respect to the natural actions of $U(1)$ by changes of the trivializing unitary section in the left-hand side and by scalar homothety in the right-hand side. This is easily seen from the formulae for the differential of (3.15). These actions have compatible moment maps (that is, $U(1)$-invariant functions): the volume

$$(\widetilde{\varphi}, \vartheta) \mapsto \operatorname{vol}(\widetilde{\varphi}, \vartheta) = \int_S d\vartheta^2$$

on the left-hand space and the Hermitian length

$$s \longmapsto \|s\|^2 = \int_M (s, s)_{L^{\otimes k}}\, d\varrho$$

on the right-hand space. Thus the map γ_k factors to a map of symplectic quotients

$$\big(\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k // U(1)\big) \longrightarrow \big(H^0(M, L^{\otimes k}) // U(1)\big).$$

We recall that the symplectic quotient $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k // U(1)$ (that is, the space of $U(1)$-orbits on the level t set of the moment map) was described in § 2.5 as the space $\overset{\circ}{\mathcal{B}}{}^{\mathrm{hw}}_{k,t}$ of half-weighted BS_k-cycles of volume t. On the other hand, it is known that the result of symplectic reduction of a Hermitian vector space V by the homothety $U(1)$-action [10] is the projectivization of V (see [6]), that is, $V//U(1) \simeq V/\mathbb{C}^* = \mathbb{P}V$. In particular, the symplectic quotient $\mathcal{H}_k//U(1)$ is the state space $\mathbb{P}\mathcal{H}_k$. Thus the symplectic reduction of γ_k is a map

$$\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_{k,t} \xrightarrow{\gamma_{k,t}} \mathbb{P}\mathcal{H}_k$$

[10] With respect to the moment map $v \longmapsto \|v\|^2$.

with epimorphic differential. It differs only by a constant Kähler angle twisting from the standard *holomorphic* map

$$\mathring{\mathcal{P}}^{\mathrm{hw}}_{k,t} \xrightarrow{\beta_{k,t}} \mathbb{P}\mathcal{H}_k$$

of the Kähler manifold $\mathring{\mathcal{B}}^{\mathrm{hw}}_{k,t}$ to the projective space $\mathbb{P}\big(H^0(\mathring{\mathcal{B}}^{\mathrm{hw}}_{k,t}, \mathcal{L}_{k,t})^*\big)$. This projective space corresponds to the holomorphic line bundle $\mathcal{L}_{k,t}$ tautologically associated with the Berry principal $U(1)$-bundle $\mathring{\mathcal{P}}^{\mathrm{hw}}_{k,t} \xrightarrow{\pi_t} \mathring{\mathcal{B}}^{\mathrm{hw}}_{k,t}$. We call the last map the *holomorphic Borthwick–Paul–Uribe map* and denote it by β_k. We note that γ_k and β_k associate *states* with each half-weighted BS_k-cycle of fixed volume t.

3.3. Analytic description of the Borthwick–Paul–Uribe maps. The maps BPU_k were discovered in the framework of a purely analytic theory (see [13]). We briefly outline the main points of the original construction of BPU_k to enable the reader to compare it with our geometric definition. The main technical tool in the construction of BPU_k is the extension of the Szegö projector

$$L^2(P, \mathbb{C}) \xrightarrow{\Pi_k} H^0(M, L^{\otimes k})$$

to an appropriate space of distributions $\mathcal{D}(\mathcal{H}) \supset L^2(P, \mathbb{C})$, which in particular contains the δ-functions δ_p for all points $p \in P$. (This uses the apparatus of *Toeplitz structures* developed in [11]. We discuss it in §§ 3.3.1, 3.3.2 below.) Applying the extended Szegö projector to all δ_p with $p \in \widetilde{\varphi}(S) \subset P$, we get a family of holomorphic sections $\Pi_k(\delta_{\widetilde{\varphi}(x)})$ parametrized by points $x \in S$. The map $\mathring{\mathcal{P}}^{\mathrm{hw}}_{k} \xrightarrow{\beta_k} H^0(M_I, L^{\otimes k})$ sends a half-weighted cycle $(\widetilde{\varphi}(S), \vartheta) \in \mathring{\mathcal{P}}^{\mathrm{hw}}_{k}$ to the holomorphic section

$$\beta_k(S, \vartheta) = \int_S \Pi_k(\delta_{\widetilde{\varphi}(x)})\, d\vartheta$$

obtained by averaging this family over S. Denote by

$$K_k(p_1, p_2) = \langle \Pi_k(\delta_{p_2}), \Pi_k(\delta_{p_1})\rangle$$

the Schwartz kernel of the extended Szegö projector $\mathcal{D}(\mathcal{H}) \xrightarrow{\Pi_k} H^0(M, L^{\otimes k})$. On the diagonal, we have $K_k(p,p) = \|\Pi_k(\delta_p)\|^2 \in L^2(P_k)$ and, by the Riemann–Roch theorem, the L^2-norm of $K_k(p,p)$ as a function of $p \in P$ has the following asymptotic behaviour for $k \gg 0$:

$$\|K_k(p,p)\|^2 = \int_{P_k} K_k(p,p)\alpha_k \wedge (d\alpha_k)^{\wedge n} = \operatorname{rk} H^0(M_I, L^{\otimes k}) \sim \left(\frac{k}{2\pi}\right)^n.$$

3.3.1. *Generalized Toeplitz operators.* We remove the zero section from the cotangent bundle T^*P_k. The resulting space contains the positive real cone

$$\Sigma \overset{\mathrm{def}}{=} \{t\alpha_k|_p \subset T^*P_k \setminus \{0\} \mid p \in P_k,\ t \in \mathbb{R},\ t > 0\} \tag{3.16}$$

over the graph of the connection form $P_k \xhookrightarrow{\alpha_k} T^*P_k \setminus \{0\}$. This cone is a *symplectic submanifold* of the cotangent bundle with the standard symplectic structure. According to [11], the pair (Π, Σ), where Π is the usual Szegö projector (3.4), is called a *Toeplitz structure*. Each pseudodifferential operator A of order n on P_k generates a *generalized Toeplitz operator*

$$T_A = \Pi \circ A \circ \Pi \colon L^2(P) \longrightarrow \mathcal{H}.$$

The restriction $\sigma(T_A) \overset{\text{def}}{=} p(A)|_\Sigma$ of the principal symbol $p(A)$ to the symplectic submanifold (3.16) is called the *symbol* of the generalized Toeplitz operator T_A. These symbols have the usual properties:

$$\sigma(T_{A_1}T_{A_2}) = \sigma(T_{A_1})\sigma(T_{A_2}),$$
$$\sigma([T_{A_1}, T_{A_2}]) = i\{\sigma(T_{A_1}), \sigma(T_{A_2})\},$$

where the last Poisson bracket is induced by the standard symplectic structure on Σ. In particular the usual (non-generalized) Toeplitz operator T_f corresponds to the multiplication by the function $f \in L^2(P_k)$, and its symbol is $\sigma(T_f) = p_\Sigma^*(f)$, where p_Σ is the composite of the projections $\Sigma \twoheadrightarrow P_k \xrightarrow{p_k} M$.

3.3.2. *Extension of the Szegö projector.* The conormal bundle $N^*\widetilde{\varphi}(S) = \operatorname{Ann}(T\widetilde{\varphi}(S)) \subset T^*P_k$ of a smooth Planck cycle is a Lagrangian submanifold in the cotangent bundle T^*P_k. According to [11], it defines a series of subspaces $I^m(P_k, N^*S) \subset D(P_k)$ of *Hermitian* distributions of semi-integer order $m \in \frac{1}{2}\mathbb{Z}$. We have $L^2(P_k) \subset I^m(P_k, N^*S)$ as a dense subspace. Each Hermitian distribution also has a *symbol*, which is a symplectic spinor on N^*S. Theorems of [11] on compositions of Hermitian distributions enable one to extend the Szegö projector (3.4) to an operator

$$I^m(P_k, N^*\widetilde{\varphi}(S)) \xrightarrow{\Pi} \mathcal{D}(\mathcal{H})$$

in the following way. The Schwartz kernel of the initial Szegö projector Π is an element of the space $I^{\frac{1}{2}}(P_k \times P_k, \Sigma_2)$, where

$$\Sigma_2 = \big\{(p_1, t\alpha(p_1)); (p_2, -t\alpha(p_2))\big\} \subset T^*P_k \times P_k$$

is an isotropic manifold which is *composable* with N^*S in the sense of [11], Theorem 9.4. This enables us to apply the Fourier integral operator associated with Σ_2 to Hermitian distributions on N^*S. Thus we get a map

$$I^m(P_k, N^*\widetilde{\varphi}(S)) \hookrightarrow \mathcal{D}(\mathcal{H}),$$

whose image is denoted by $J^{m-\frac{n}{2}}(P_k, S) \subset \mathcal{D}(\mathcal{H})$ and is called *the space of Legendrian distributions of order* $m - \frac{1}{4}\dim M$. The space $J^{\frac{1}{2}}(P_k, S)$ is naturally identified with the space of half-weights on S. Indeed, the symbol of each element of $J^{\frac{1}{2}}(P_k, S)$, being a symplectic spinor on the Legendrian submanifold $\widetilde{\varphi}(S) \subset P_k$, naturally determines a half-weight (see [11] and [13]). Conversely, each half-weight

ϑ on S enables us to define the δ-function $\delta_{(S,\vartheta)}$ supported on the cycle and then use it to construct the Legendrian distribution

$$s(\vartheta) = T^{m-\frac{1}{2}}_{\mathcal{L}ie_{u_k}} \Pi(\delta_S).$$

Here $T_{\mathcal{L}ie_{u_k}}$ is the first-order generalized Toeplitz operator corresponding to the Lie derivative along the standard basis $U(1)$-invariant vector field u_k on P_k. (The symbol of this operator is the function t, the coordinate along the rulers on the conic submanifold (3.16).)

§ 4. Example: Legendrian knots in S^3

4.1. Prequantization data of the Riemann sphere. We shall visually illustrate our constructions by the simplest non-trivial example. It appears when the initial symplectic manifold (M,ω) is the Riemann sphere $M = S^2 = \mathbb{CP}_1$ (the projectivization of the two-dimensional Hermitian space $\mathbb{C}^2$) whose Kähler structure is induced by the standard Hermitian form on $\mathbb{C}^2$. The corresponding Riemannian metric and symplectic form are given in the standard affine chart $z = z_0/z_1$ on $\mathbb{CP}_1$ by

$$g = \frac{4dzd\overline{z}}{(1+|z|^2)^2}, \qquad \omega = \frac{4dz \wedge d\overline{z}}{(1+|z|^2)^2}.$$

For the prequantization line bundle on $\mathbb{CP}_1$, we take the sheaf $L = \mathcal{O}(1)$ of homogeneous form of degree 1 with the standard connection obtained by restriction to $S^3 \subset \mathbb{C}^2$ of the $U(2)$-invariant 1-form

$$2\pi i \alpha_1 \stackrel{\text{def}}{=} (\partial - \bar{\partial}) \log |z|.$$

Its real curvature form is ω. The unit circle bundle of the dual bundle $L^* \supset P_{-1} \twoheadrightarrow \mathbb{CP}_1$ is just the Hopf bundle

$$S^3 \twoheadrightarrow \mathbb{CP}_1,$$

whose fibre over a point representing a line $\ell \subset \mathbb{C}^2$ is the unit circle equal to the intersection of this line and the unit sphere $S^3 \subset \mathbb{C}^2$. (The class of this bundle is the standard "negative" generator of the group $\pi_3(S^2) = \mathbb{Z}$.)

4.2. Lagrangian cycles, or unicursal curves on S^2. We take the circle $S = S^1$ for the manifold S that parametrizes Lagrangian cycles in $M = S^2$. Every smooth map $S^1 \xrightarrow{\varphi} S^2$ is Lagrangian since there are no non-zero 2-forms on the circle. Therefore the space $\mathrm{Map}_{\mathrm{LA}}(S^1, S^2)$ of Lagrangian maps is just the loop space LS^2, and the space $\mathfrak{L} = \mathrm{Map}_{\mathrm{LA}}(S^1,S^2)/\operatorname{Diff}_0(S^1)$ of Lagrangian cycles is the space of *unicursal curves* (non-parametrized loops) $\varphi(S^1) \subset S^2$. We note that each tangent space $T_\varphi(LS^2) = C^\infty(S^1, \varphi^*TS^2)$ of the loop space carries a non-degenerate skew form $\varphi^*\omega$, which induces a skew 2-form

$$\Omega_{LS^2}(\sigma_1,\sigma_2) = \int_{S^1} \varphi^*\omega^{-1}(\sigma_1) \wedge \varphi^*\omega^{-1}(\sigma_2) \qquad (4.1)$$

on $T(LS^2)$. This form is $\operatorname{Diff}_0(S^1)$-invariant and closed, but it is only weakly non-degenerate: it associates Hamiltonian vector fields with only *supersmooth* functions on $\mathfrak{L}$ (see [14], Definition 3.1.5).

4.2.1. *Darboux–Weinstein coordinates.* As explained in § 1.1.2, we may identify a neighbourhood of the zero section in the normal bundle of S^1 under a smooth immersion $S^1 \xhookrightarrow{\varphi} S^2$ and a neighbourhood of the zero section in the *trivial* cotangent bundle of S^1:

$$N_\varphi S^1 = \varphi^* TS^2/TS^1 \simeq \varphi^* T^* S^2/\operatorname{Ann}(TS^1) = T^* S^1 = S^1 \times \mathbb{R}. \tag{4.2}$$

We fix the last trivialization by choosing the following rotation-invariant 1-form for the global basis section of the cotangent bundle:

$$\xi \overset{\text{def}}{=} \frac{1}{2\pi} dt.$$

Here t is the natural parameter on the circle (the oriented angle), whence

$$\int_{S^1} \xi = 1.$$

This form defines a smooth measure μ_0 on the circle, and we shall use μ_0 in the study of half-weights below. By § 1.1.4, each smooth immersion $S^1 \longrightarrow S^2$ may be extended to a smooth map $U \xrightarrow{\widehat{\varphi}} S^2$ (where $U \subset T^* S^1$ is some neighbourhood of the zero section in the cotangent bundle) such that $\widehat{\varphi}^* \omega = d\eta$, where η is the canonical action form on $T^* S^1$. Images of graphs (lying in U) of closed 1-forms constitute an open neighbourhood of φ in $\mathfrak{L}$. We use this neighbourhood to introduce local coordinates on $\mathfrak{L}$ near φ via a neighbourhood of zero in the space of closed 1-forms on S^1.

4.2.4 *Bohr–Sommerfeld cycles.* For each integer $k \geqslant 0$, the lift $\varphi^* L^{\otimes k}$ along an arbitrary loop $S^1 \xrightarrow{\varphi} S^2$ is a trivial complex 1-bundle over S^1 with the flat $U(1)$-connection $\varphi^* \alpha_k$. Its holonomy is a character $\pi_1(S^1) \xrightarrow{\chi_k} U(1)$ of the fundamental group. The value of χ_k on the generator $1 \in \pi_1(S^1) = \mathbb{Z}$ is equal to the rotation angle of the fibre of $\varphi^* L^{\otimes k}$ under its parallel translation along S^1 once in the positive direction. Then

$$\chi_k(1) = k \cdot \chi_1(1)$$

is an integer multiple of the oriented area of the domain in S^2 bounded by the unicursal curve $\varphi(S^1)$. The map

$$j_k \colon LS^2 \xhookrightarrow{\varphi \mapsto k \cdot \chi_k(1)} \mathcal{J}$$

is a section of the Jacobian bundle (see § 1.3.3). Since

$$H^1(S^1, \mathbb{R})/H^1(S^1, \mathbb{Z}) = U(1),$$

the Jacobian bundle in our case is a principal $U(1)$-bundle over LS^2. The zero set $\mathcal{B}_k \subset \mathfrak{L}$ of the section j_k has codimension 1 and is called the *space of Bohr–Sommerfeld cycles of level k*. It consists of unicursal curves whose area is a rational (with denominator k) multiple of the sphere area. In particular, for all $k, m \in \mathbb{Z}$ we have inclusions

$$\mathcal{B}_k \xhookrightarrow{\beta_{k,m}} \mathcal{B}_{mk}$$

(compare with (2.2)).

4.2.3. *Remark on special properties of the example studied.* Our example has a number of features that are absent in the general case. Firstly, the cotangent bundle of $S = S^1$ is trivial. Secondly, $S = S^1$ is a principal homogeneous space over $U(1)$ and, in particular, $U(1) \subset \mathrm{Diff}_0(S^1)$. Finally, $U(1)$ is naturally identified with the Jacobian $J(S^1) = H^1(S^1, \mathbb{R})/H^1(S^1, \mathbb{R})$.

4.3. Planck cycles, or Legendrian knots in S^3. A Planck cycle of level k is a class of $\mathrm{Diff}_0(S^1)$-equivalent smooth embeddings $S^1 \xrightarrow{\widetilde{\varphi}} S^3$ that are Legendrian with respect to the contact form α_k on $S^3 \simeq P_k$. These cycles are usually called *Legendrian knots* on S^3. From our viewpoint, every such knot is a covariantly constant global section $S^1 \xhookrightarrow{\sigma} \varphi^* P_k$ of the principal prequantization bundle over the Bohr–Sommerfeld cycle $S^1 \xrightarrow{\varphi} S^2$ or, equivalently, it is a covariantly constant global unitary section of the trivial Hermitian line bundle $\varphi^* L^{\otimes k}$ over S^1. Thus, Planck cycles of level k form a principal $U(1)$-bundle $\mathcal{P}_k \xrightarrow{\pi_k} \mathcal{B}_k$ over the space of Bohr–Sommerfeld cycles. This bundle is called the *Berry principal bundle* of level k. The corresponding Hermitian line bundles $\mathcal{L}_k$ on $\mathcal{B}_k$ satisfy

$$\beta_{k,m}^* \mathcal{L}_{mk} = \mathcal{L}_k^{\otimes m}.$$

The tangent space $T_{\widetilde{\varphi}} \mathcal{P}_k$ at a smooth knot $S^1 \xhookrightarrow{\widetilde{\varphi}} P_k \simeq S^3$ is identified with the space $C^\infty(S^1)$ of smooth functions. This is almost obvious from the special features listed in § 4.2.3. Indeed, $T_{\widetilde{\varphi}} \mathcal{P}_k$ is the space of smooth sections of the normal bundle

$$N_{\widetilde{\varphi}} = \widetilde{\varphi}^* T^* P_k / T S^1 = T^* S^1 \oplus T^* U(1) = S^1 \times \mathbb{R} \times \mathbb{R}.$$

(Here the last equation corresponds to (4.2), and the penultimate equation is the global splitting into the vertical and horizontal components with respect to the connection[11] α_k.) Each function $f \in C^\infty(S^1)$ determines a section $\tau_f \in C^\infty(S^1, N_{\widetilde{\varphi}})$ whose vertical (resp. horizontal) component is equal to $k f \xi$ (resp. $k\,df$), where $\xi \in C^\infty(T^* S^1)$ is the basis 1-form on $S^1 = U(1)$ (as specified in § 4.2.1 and (4.2)) such that $k\,df = 2\pi k \frac{df}{dt} \xi$.

4.4. Half-weighted cycles. The invariant form ξ determines a smooth measure μ_0 on S^1 and the space $\mathring{\mathfrak{H}} = \mathring{\mathfrak{H}}(S^1)$ of half-weights. Consider the corresponding orbit spaces for the natural diagonal action of diffeomorphisms of the circle:

$$\mathring{\mathfrak{L}}^{\mathrm{hw}} \stackrel{\mathrm{def}}{=} \mathrm{Map}_{\mathrm{LA}}(S^1, S^2) \times \mathring{\mathfrak{H}}(S^1) / \mathrm{Diff}_0(S^1),$$

$$\mathring{\mathcal{P}}_k^{\mathrm{hw}} \stackrel{\mathrm{def}}{=} \mathrm{Map}_{\mathrm{LE}}(S^1, S^3) \times \mathring{\mathfrak{H}}(S^1) / \mathrm{Diff}_0(S^1).$$

These are the spaces of *half-weighted* unicursal curves and Legendrian knots respectively. Fixing a half-weight $f\sqrt{\mu}$ on a Legendrian knot, we identify the tangent space $T_{\widetilde{\varphi}} \mathcal{P}_k = C^\infty(S^1)$ with $T_{f\sqrt{\mu}} \mathring{\mathfrak{H}} \simeq \mathring{\mathfrak{H}}$ by sending a smooth function f_1 to the half-weight $f f_1 \sqrt{\mu}$. As a result, we have decomposition (2.12) on every smooth knot $S^1 \xhookrightarrow{\widetilde{\varphi}} S^3$ of half-weight $\vartheta = f\sqrt{\mu}$ lying over the unicursal curve $S^1 \xhookrightarrow{\varphi} S^3$:

$$T_{(\widetilde{\varphi}, \vartheta)} \mathring{\mathcal{P}}_k^{\mathrm{hw}} \simeq T_\vartheta \mathring{\mathfrak{H}}(S^1) \oplus T_{\widetilde{\varphi}} \mathring{\mathcal{P}}_k^{\mathrm{hw}} \simeq C^\infty(S^1) \oplus C^\infty(S^1).$$

[11] In the general case, such a splitting exists only locally.

The Darboux–Weinstein construction identifies a neighbourhood of $(\widetilde{\varphi}, \vartheta)$ in $\overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}}$ with a neighbourhood of zero in the space of smooth complex-valued functions $\psi = \psi_1 + i\psi_2 \in C^\infty(S^1, \mathbb{C})$. Here the imaginary part ψ_2 determines the knot itself as the graph of the section $(d\psi_2, e^{i\pi\psi_2})$ of the bundle

$$\widehat{\varphi}^* P_k|_U \subset T^* S^1 \times U(1),$$

and the real part ψ_1 determines the half-weight $\psi_1 \vartheta = \psi_1 f \sqrt{\mu}$. This yields an integrable complex structure on the space of half-weighted Planck cycles. We also get a compatible symplectic form, which acts on each fibre $T_{(\widetilde{\varphi},\vartheta_0)} \overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}}$ of the tangent bundle by

$$\Omega\big((\vartheta_1, f_1), (\vartheta_2, f_2)\big) = \int_{S^1} f_2 \, d\vartheta_0 \vartheta_1 - \int_{S^1} f_1 \, d\vartheta_0 \vartheta_2$$

and which is the differential of the 1-form $\mathrm{H}\big((\vartheta_1, f_1)\big) = 2 \int_{S^1} f_1 \, d\vartheta_0 \vartheta_1$. The following Riemannian metric complements this complex structure and symplectic form to a Kähler triple:

$$G\big((\vartheta_1, f_1), (\vartheta_2, f_2)\big) = \int_{S^1} f_1 \, d\vartheta_1 \vartheta_0 + \int_{S^1} f_2 \, d\vartheta_2 \vartheta_0.$$

Using symplectic reduction with respect to the $U(1)$-action on the fibres of the Hopf bundle, we can pull this Kähler structure down to a Kähler structure on the space $\overset{\circ}{\mathcal{B}}{}_{k,t}^{\mathrm{hw}}$ of half-weighted Bohr–Sommerfeld cycles of fixed length

$$t = \int_{S^1} d\vartheta^2.$$

We note that, in our example, fibres of the forgetting map

$$\overset{\circ}{\mathcal{B}}{}_{k,t}^{\mathrm{hw}} \twoheadrightarrow \mathcal{B}_{k,t} \tag{4.3}$$

are naturally identified with the quotient $\mathrm{Diff}_0(S^1)/U(1)$ since the choice of a half-weight is equivalent to the choice of a natural parameter on the circle modulo rotations. The space $\mathrm{Diff}_0(S^1)/U(1)$ carries a well-known complex structure (see [5]). On the other hand, the space $\mathcal{B}_{k,t}$ also has the following almost complex structure. If we identify S^1 with $U(1)$ (as mentioned in § 4.2.3), then the normal bundle of any loop $S^1 \xrightarrow{\widetilde{\varphi}} S^3$ is decomposed into a direct sum: $N_{\widetilde{\varphi}} = T^*S^1 \oplus T^*U(1)$. This yields an almost complex structure on the whole of the loop space LS^3, whose tangent space $T_{\widetilde{\varphi}} LS^3$ is the space $C^\infty(S^1, N_{\widetilde{\varphi}})$ of smooth sections. This almost complex structure may be restricted to the space of Legendrian knots and pulled down to an almost complex structure on $\mathcal{B}_{k,t} = \mathcal{P}_{k,t}/U(1)$. With respect to these structures, the forgetting map (4.3) is easily seen to be pseudoholomorphic and to have pseudoholomorphic fibres.

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Received 15/AUG/2000
Translated by THE AUTHORS

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

* *Translators' Note.* Reference [29] is added in translation.